Electromagnetic Field Theory 1

(fundamental relations and definitions)

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Fundamental Question of Classical Electrodynamics

A specified distribution of elementary charges is in a state of arbitrary (but known) motion. At certain time we pick one of them and ask what is the force acting on it.

Rather difficult question – will not be fully answered

Elementary Charge

Coulomb

$$e = 1.602176634 \cdot 10^{-19}\ \mathrm{C}$$

smallest known
amount of charge

As far as we know, all charges in nature have values $\pm Ne, N \in \mathbb{Z}$

Charge conservation

Amount of charge is conserved in every frame (even non-inertial).

Neutrality of atoms has been verified to 20 digits

Continuous approximation of charge distribution

Volumetric density of charge $\left[\mathrm{C}\cdot\mathrm{m}^{-3}\right]$

Surface density of charge $\left[\mathrm{C}\cdot\mathrm{m}^{-2}\right]$

Line density of charge $\left[\mathrm{C}\cdot\mathrm{m}^{-1}\right]$

$$Q = \int\limits_V \rho\left(\boldsymbol{r}\right)\mathrm{d}V = \int\limits_S \sigma\left(\boldsymbol{r}\right)\mathrm{d}S = \int\limits_l \tau\left(\boldsymbol{r}\right)\mathrm{d}l$$

Net charge $\left[\mathrm{C}\right]$

Continuous approximation allows for using powerful mathematics

Fundamental Question of Electrostatics

There exist a specified distribution of static elementary charges. We pick one of them and ask what is the force acting on it.

This will be answered in full details

Coulomb('s) Law

Measuring charge $[\mathrm{C}]$

Source charge $[\mathrm{C}]$

Radius vector of the measuring charge $[\mathrm{m}]$

Radius vector of the source charge $[\mathrm{m}]$

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{qq'(\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 \left|\boldsymbol{r} - \boldsymbol{r}'\right|^3}$$

Force on measuring charge $[\mathrm{N}]$

Permittivity of vacuum

Farad

Speed of light

$$\varepsilon_0 = \frac{1}{\mu_0 c_0^2} = 8.8541878128 \cdot 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1}$$

$$c_0 = 299792458\ \mathrm{m} \cdot \mathrm{s}^{-1}$$

$$\mu_0 = 1.25663706212 \cdot 10^{-6}\ \mathrm{H} \cdot \mathrm{m}^{-1}$$

Permeability of vacuum

Henry

Coulomb(’s) Law + Superposition Principle

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = \frac{q}{4\pi\varepsilon_0}\sum_n \frac{q_n'\left(\boldsymbol{r} - \boldsymbol{r}_n'\right)}{\left|\boldsymbol{r} - \boldsymbol{r}_n'\right|^3}$$

Entire electrostatics can be deduced from this formula

Electric Field

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = q\boldsymbol{E}\left(\boldsymbol{r}\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0}\sum_n \frac{q'_n\left(\boldsymbol{r} - \boldsymbol{r}'_n\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'_n\right|^3}$$

Intensity of electric field

$$\left[\mathrm{V}\cdot\mathrm{m}^{-1}\right]$$

Force is represented by field – entity generated by charges and permeating the space

Continuous Distribution of Charge

$$\boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0}\sum_n \frac{q_n'\left(\boldsymbol{r}-\boldsymbol{r}_n'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}_n'\right|^3} \Rightarrow \boldsymbol{E}\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho\left(\boldsymbol{r}'\right)\left(\boldsymbol{r}-\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|^3}\,\mathrm{d}V'$$

Continuous description of charge allows for using powerful mathematics

Continuous Description of a Point Charge

Dirac's delta "function"

Defining property of Dirac's delta "function"

$$\rho\left(\boldsymbol{r}\right)=\sum_{n} q_{n}\delta\left(\boldsymbol{r}-\boldsymbol{r}_{n}\right)$$

$$\boldsymbol{F}\left(\boldsymbol{r}_{n}\right)=\int_{V}\boldsymbol{F}\left(\boldsymbol{r}\right)\delta\left(\boldsymbol{r}-\boldsymbol{r}_{n}\right)\mathrm{d}V$$

Gauss(') Law

Mind the orientation of the surface

Total charge enclosed by the surface

$[\mathrm{C}]$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\varepsilon_0} \quad \Longleftrightarrow \quad \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \int_V \rho(\boldsymbol{r}) \, \mathrm{d}V = \frac{Q}{\varepsilon_0}$$

Differential law (local)

Integral law (global)

Rotation of Electric Field

$$\nabla \times \boldsymbol{E} = 0 \quad \Longleftrightarrow \quad \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Differential law
(local)

Integral law
(global)

Various Views on Electrostatics

Integral laws of electrostatics		Differential laws of electrostatics		Coulomb's law
$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$ $$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$	⟷	$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\varepsilon_0}$$ $$\nabla \times \boldsymbol{E} = 0$$	⟷	$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\boldsymbol{r}')(\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} \mathrm{d}V'$$

The physics content is the same, the formalism is different.

Electric potential

Defined up to arbitrary constant

Electric potential

$$\nabla \times \boldsymbol{E} = 0 \quad \Rightarrow \quad \boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r}) \quad \Rightarrow \quad \varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{V'} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' + K$$

Scalar description of electrostatic field

Voltage

Potential difference is a unique number

Voltage

$\left[\mathrm{V}\right]$

Work necessary to take charge *q* from point A to point B

$$-\int_A^B \mathbf{E}\cdot d\mathbf{l} = \varphi(B) - \varphi(A) = U$$

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = qU$$

Voltage represents connection of abstract field theory with experiments

Electrostatic Energy

$$W = \frac{1}{2}\int_V \rho\left(\boldsymbol{r}\right)\varphi\left(\boldsymbol{r}\right)\mathrm{d}\,V$$

Energy is carried by charges and fields

Energy is carried by charges

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j\\ j\neq i}}\frac{q_i q_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|}$$

$$W = \frac{1}{8\pi\varepsilon_0}\int_V\int_{V'}\frac{\rho\left(\boldsymbol{r}\right)\rho\left(\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}\mathrm{d}\,V'\mathrm{d}\,V$$

$$W = \frac{1}{2}\varepsilon_0\int_V\left|\boldsymbol{E}\left(\boldsymbol{r}\right)\right|^2\mathrm{d}\,V$$

Energy is carried by fields

Be careful with point charges (self-energy)

Electrostatic Energy vs Force

Energy of a system of point charges

Coulomb's law

$$W = \frac{1}{8\pi\varepsilon_0}\sum_{\substack{i,j\\ j\neq i}}\frac{q_i q_j}{\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|} \quad \Rightarrow \quad \boldsymbol{F}\left(\boldsymbol{r}_\xi\right) = -\nabla_\xi W = \frac{q_\xi}{4\pi\varepsilon_0}\sum_{\substack{j\\ j\neq \xi}} q_j \frac{\left(\boldsymbol{r}_\xi - \boldsymbol{r}_j\right)}{\left|\boldsymbol{r}_\xi - \boldsymbol{r}_j\right|^3}$$

Electrostatic forces are always acting so as to minimize energy of the system

Electric Stress Tensor

Total electric force acting in a volume

Stress tensor

$$\boldsymbol{F} = \int\limits_V \rho\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right)\mathrm{d}V = \varepsilon_0 \oint\limits_S \overline{\overline{\boldsymbol{T}}} \cdot \mathrm{d}\boldsymbol{S} \quad \Longrightarrow \quad \overline{\overline{\boldsymbol{T}}} = \boldsymbol{E}\boldsymbol{E} - \frac{1}{2}\overline{\overline{\boldsymbol{I}}}\left|\boldsymbol{E}\right|^2$$

All the information on the volumetric Coulomb's force is contained at the boundary

Ideal Conductor – classical description

Ideal conductor contains unlimited amount of free charges which under action of external electric field rearrange so as to annihilate electric field inside the conductor.

In 3D, the free charge always resides on the external bounding surface of the conductor.

In 1D and 2D it is not so

Generally, free charges in conductors move so as to minimize the energy

Ideal Conductor – quantum description

In an ideal conductor, wave functions of electrons in outer shells perceive flat potential background. In reaction to an external electric field, these wave functions are slightly modified so as to provide zero average charge density inside the conductor. Due to flat potential background, there is no counter interaction.

Long-range transport of charge does not truly happen in a solid conductor

Boundary Conditions on Ideal Conductor

- Inside conductor
  - $\boldsymbol{E}(\boldsymbol{r}) = 0 \quad \Leftrightarrow \quad \varphi(\boldsymbol{r}) = \text{const.}$
- Just outside conductor
  - $\boldsymbol{n}(\boldsymbol{r}) \times \boldsymbol{E}(\boldsymbol{r}) = 0 \quad \Leftrightarrow \quad \varphi(\boldsymbol{r}) = \text{const.}$
  - $\boldsymbol{n}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r}) = \dfrac{\sigma}{\varepsilon_0} \quad \Leftrightarrow \quad \dfrac{\partial \varphi(\boldsymbol{r})}{\partial n} = -\dfrac{\sigma}{\varepsilon_0}$

Potential is continuous across the boundary

Surface charge residing on the outer surface of the conductor

Outward normal to the conductor

Normal derivative

Capacitance of a System of *N* conductors

Self and mutual capacitances

$[\mathrm{F}]$

Electrostatic energy

Capacitances depend solely on geometry and position of conductors

$$Q_i = \sum_j C_{ij} \varphi_j \quad \Rightarrow \quad W = \frac{1}{2} \sum_{i,j} C_{ij} \varphi_j \varphi_i$$

Electrostatic system is fully characterized by capacitances (we know the energy)

Capacitance of a System of two conductors

Potential difference between conductors

Capacitance

$$Q = CU \quad \Rightarrow \quad W = \frac{1}{2}CU^2$$

Charge on positively charged conductor

Poisson('s) equation

$$\Delta\varphi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon_0}$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the charge density is known throughout the volume.

Laplace('s) equation

$$\Delta\varphi\left(\boldsymbol{r}\right)=0$$

The solution to Laplace's equation is unique in a given volume once the potential is known on its bounding surface.

Mean Value Theorem

Only for spheres containing no charge

Center of the sphere

$$\varphi\left(\boldsymbol{r}_{\text{center}}\right) = \frac{1}{4\pi R^2} \oint\limits_{\text{sphere}} \varphi\left(\boldsymbol{r}\right) \mathrm{d}S$$

Radius of the sphere

The solution to Laplace's equation posses neither maxima nor minima inside the solved volume.

Earnshaw('s) Theorem

Consequence of
mean value theorem

A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

Mind that the solution to Laplace's equation posses neither maxima nor minima inside the solved volume. This means that charged particle will always travel towards the boundary.

Image Method

When solving field generated by charges in the presence of conductors, it is sometimes possible to remove the conductor and mimic its boundary conditions by adding extra charges to the exterior of the solution volume. The uniqueness theorem claims that this is a correct solution.

Image method always works with planes and spheres.

Separation of Variables

Constants
determined by
boundary conditions

$$\Delta\varphi\left(\boldsymbol{r}\right)=0 \quad \Rightarrow \quad \varphi_{ijk}\left(\boldsymbol{r}\right)=X_i\left(x\right)Y_j\left(y\right)Z_k\left(z\right) \quad \Rightarrow \quad \varphi\left(\boldsymbol{r}\right)=\sum_{ijk}C_{ijk}\varphi_{ijk}\left(\boldsymbol{r}\right)$$

Semi-analytical method for canonical problems

Finite Differences

$$\varphi(x+h,y,z) \rightarrow \varphi_{(i+1)jk}$$

$$\Delta\varphi(\boldsymbol{r}) \approx \frac{\varphi_{(i+1)jk} - 2\varphi_{ijk} + \varphi_{(i-1)jk}}{h^2} + \frac{\varphi_{i(j+1)k} - 2\varphi_{ijk} + \varphi_{i(j-1)k}}{h^2} + \frac{\varphi_{ij(k+1)} - 2\varphi_{ijk} + \varphi_{ij(k-1)}}{h^2}$$

$$\Delta\varphi(\boldsymbol{r}) = 0 \quad \Rightarrow \quad \varphi_{ijk} = \frac{\varphi_{(i+1)jk} + \varphi_{(i-1)jk} + \varphi_{i(j+1)k} + \varphi_{i(j-1)k} + \varphi_{ij(k+1)} + \varphi_{ij(k-1)}}{6}$$

Approximation by a system of linear algebraic equations

Mind the mean value theorem

Powerful numerical method for closed problems

Integral Equation & Method of Moments

Assumed to be known in volume where the charge resides

Distribution of charge is unknown

Simple functions for which the potential integral can be easily evaluated

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{V'} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}V'$$

$$\rho(\boldsymbol{r}) \approx \sum_n \alpha_n \rho_n(\boldsymbol{r})$$

$$\int\limits_V \rho_m(\boldsymbol{r}) \varphi(\boldsymbol{r}) \mathrm{d}V = \sum_n \alpha_n \frac{1}{4\pi\varepsilon_0} \int\limits_V \int\limits_{V'} \frac{\rho_m(\boldsymbol{r}) \rho_n(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}V' \mathrm{d}V$$

Known

Approximation by a system of linear algebraic equations

Known

Powerful numerical method for open problems

Dielectrics

- Material in which charges cannot move freely
- Charges are forming clusters (atoms, molecules)
- Under influence of electric field the clusters change shape or rotate
- Electric field induces electric dipoles with density $\boldsymbol{P}(\boldsymbol{r})$ $\left[\mathrm{C}\cdot\mathrm{m}^{-2}\right]$

Clusters are electrically neutral

Number of dipoles in unitary volume

Electric Field of a Dipole

Two opposite charges very close to each other

$$\left|\boldsymbol{r}-\boldsymbol{r}_{\text{center}}\right| \gg \left|\boldsymbol{r}_1-\boldsymbol{r}_2\right|$$

$$\varphi\left(\boldsymbol{r}\right)=\frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\boldsymbol{r}-\boldsymbol{r}_1\right|}-\frac{1}{4\pi\varepsilon_0}\frac{q}{\left|\boldsymbol{r}-\boldsymbol{r}_2\right|}\approx\frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\left(\boldsymbol{r}-\boldsymbol{r}_{\text{center}}\right)}{\left|\boldsymbol{r}-\boldsymbol{r}_{\text{center}}\right|^3}$$

$$\boldsymbol{p}=q\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)=\int_V \boldsymbol{r}\rho\left(\boldsymbol{r}\right)\mathrm{d}V$$

Electric dipole moment $\left[\mathrm{C}\cdot\mathrm{m}\right]$

Formula for two opposite charges

General formula

Field Produced by Polarized Matter

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int\limits_{V'}\frac{\boldsymbol{P}(\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}V' = \frac{1}{4\pi\varepsilon_0}\oint\limits_{S'}\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\mathrm{d}\boldsymbol{S}' - \frac{1}{4\pi\varepsilon_0}\int\limits_{V'}\frac{\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

Only apply at infinitely sharp boundary (unrealistic)

Potential of volumetric charge density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside

Electric Displacement

Electric displacement $\left[\mathrm{C\cdot m^{-2}}\right]$

$$\nabla\cdot\boldsymbol{D}\left(\boldsymbol{r}\right)=\rho\left(\boldsymbol{r}\right)$$

$$\boldsymbol{D}\left(\boldsymbol{r}\right)=\varepsilon_0\boldsymbol{E}\left(\boldsymbol{r}\right)+\boldsymbol{P}\left(\boldsymbol{r}\right)$$

$$\oint_S \boldsymbol{D}\cdot\mathrm{d}\boldsymbol{S}=\int_V \rho\left(\boldsymbol{r}\right)\mathrm{d}V=Q$$

Only free charge
(compare to divergence
of electric field)

Linear Isotropic Dielectrics

Relative permittivity

$$[\;]\quad \varepsilon_{\mathrm{r}}(\boldsymbol{r}) = 1 + \chi_{\mathrm{e}}(\boldsymbol{r})$$

$$\boldsymbol{P}(\boldsymbol{r}) = \varepsilon_0 \chi_{\mathrm{e}}(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) \qquad \boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \varepsilon_{\mathrm{r}}(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r}) = \varepsilon(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r})$$

Electric susceptibility

$$[\;]$$

Permittivity

$$\left[\mathrm{F}\cdot\mathrm{m}^{-1}\right]$$

All the complicated structure of matter reduces to a simple scalar quantity

Fields in Presence of Dielectrics 1/2

Analogy with electric field in vacuum can only be used when entire space is homogeneously filled with dielectric.

$$\nabla \times \boldsymbol{D}\left(\boldsymbol{r}\right) = \nabla \times \left[\varepsilon\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right)\right] \neq 0$$

Inequality is due to boundaries

Analogy with vacuum can only be used when space is homogeneously filled with dielectric

Fields in Presence of Dielectrics 2/2

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \Leftrightarrow \boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r}) \quad \Rightarrow \quad \nabla \cdot \left[\varepsilon(\boldsymbol{r}) \nabla \varphi(\boldsymbol{r})\right] = -\rho(\boldsymbol{r})$$

$$\Delta \varphi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon}$$

Not a function of coordinates

Poisson's equation holds only when permittivity does not depend on coordinates

Dielectric Boundaries

$$\boldsymbol{n}(\boldsymbol{r})\times\left[\boldsymbol{E}_1(\boldsymbol{r})-\boldsymbol{E}_2(\boldsymbol{r})\right]=0 \quad\Leftrightarrow\quad \varphi_1(\boldsymbol{r})-\varphi_2(\boldsymbol{r})=0$$

$$\boldsymbol{n}(\boldsymbol{r})\cdot\left[\varepsilon_1\boldsymbol{E}_1(\boldsymbol{r})-\varepsilon_2\boldsymbol{E}_2(\boldsymbol{r})\right]=\sigma(\boldsymbol{r}) \quad\Leftrightarrow\quad \varepsilon_1\frac{\partial\varphi_1(\boldsymbol{r})}{\partial n}-\varepsilon_2\frac{\partial\varphi_2(\boldsymbol{r})}{\partial n}=-\sigma(\boldsymbol{r})$$

Normal pointing to region (1)

Both conditions are needed for unique solution

Electrostatic Energy in Dielectrics

$$W = \frac{1}{2}\varepsilon_0 \int_V \left|\boldsymbol{E}\left(\boldsymbol{r}\right)\right|^2 \mathrm{d}\,V \quad \Longrightarrow \quad W = \frac{1}{2}\int_V \boldsymbol{E}\left(\boldsymbol{r}\right)\cdot\boldsymbol{D}\left(\boldsymbol{r}\right)\mathrm{d}\,V$$

Forces on Dielectrics

This only holds when charge is held constant

$$W = \frac{1}{2} C U^2 = \frac{1}{2} \frac{Q^2}{C}$$

$$W = \frac{1}{2} \int_V \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{D}(\boldsymbol{r}) \, \mathrm{d}V$$

$$\Rightarrow \quad \boldsymbol{F}(\boldsymbol{r}_\xi) = -\nabla_\xi W$$

Electric Current

Current density

$\left[\mathrm{A} \cdot \mathrm{m}^{-2}\right]$

Charge

$\left[\mathrm{C}\right]$

Velocity of charge

$\left[\mathrm{m} \cdot \mathrm{s}^{-1}\right]$

$$\boldsymbol{J}\left(\boldsymbol{r}, t\right) = \sum_{k=1}^{N} q_k \delta\left(\boldsymbol{r} - \boldsymbol{r}_k\left(t\right)\right) \boldsymbol{v}_k\left(t\right)$$

Volumetric density represented by Dirac delta

$\left[\mathrm{m}^{-3}\right]$

Charges in motion are represented by current density

Local Charge Conservation

$$\nabla \cdot \boldsymbol{J}\left(\boldsymbol{r}, t\right) = -\frac{\partial}{\partial t}\sum_{k=1}^{N} q_k \delta\left(\boldsymbol{r} - \boldsymbol{r}_k\left(t\right)\right) = -\frac{\partial \rho\left(\boldsymbol{r}, t\right)}{\partial t}$$

Charge is conserved locally at every space-time point

Global Charge Conservation

When charge leaves a given volume, it is always accompanied by a current through the bounding envelope

$$\oint_S \boldsymbol{J}\left(\boldsymbol{r}, t\right) \cdot d\boldsymbol{S} = -\frac{\partial Q\left(t\right)}{\partial t}$$

Charge can neither be created nor destroyed. It can only be displaced.

Stationary Current

When charge enters a volume, another must leave it without any delay

$$\nabla \cdot \boldsymbol{J}\left(\boldsymbol{r}\right) = 0 \quad \Leftrightarrow \quad \oint\limits_{S} \boldsymbol{J}\left(\boldsymbol{r}\right) \cdot d\boldsymbol{S} = 0$$

There is no charge accumulation in stationary flow

Ohm('s) Law

Conductivity

$$\left[\mathrm{S}\cdot\mathrm{m}^{-1}\right]$$

$$\boldsymbol{J}\left(\boldsymbol{r}\right)=\sigma\left(\boldsymbol{r}\right)\boldsymbol{E}\left(\boldsymbol{r}\right)$$

This simple linear relation holds for enormous interval of electric field strengths

Electromotive Force

Stationary flow of charges cannot be caused by electrostatic field. The motion forces are non-conservative, are called electromotive forces, and are commonly of chemical, magnetic or photoelectric origin.

$$\oint_l \boldsymbol{E}(\boldsymbol{r}) \cdot d\boldsymbol{l} \neq 0$$

For curves passing through sources of electromotive force

$$\oint_l \boldsymbol{E}(\boldsymbol{r}) \cdot d\boldsymbol{l} = 0$$

For curves not crossing sources of electromotive force

Boundary Conditions for Stationary Current

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{E}_1(\boldsymbol{r}) - \boldsymbol{E}_2(\boldsymbol{r})\right] = 0 \quad \Leftrightarrow \quad \varphi_1(\boldsymbol{r}) - \varphi_2(\boldsymbol{r}) = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\varepsilon_1 \boldsymbol{E}_1(\boldsymbol{r}) - \varepsilon_2 \boldsymbol{E}_2(\boldsymbol{r})\right] = \sigma(\boldsymbol{r}) \quad \Leftrightarrow \quad \varepsilon_1 \frac{\partial \varphi_1(\boldsymbol{r})}{\partial n} - \varepsilon_2 \frac{\partial \varphi_2(\boldsymbol{r})}{\partial n} = -\sigma(\boldsymbol{r})$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\sigma_1 \boldsymbol{E}_1(\boldsymbol{r}) - \sigma_2 \boldsymbol{E}_2(\boldsymbol{r})\right] = 0 \quad \Leftrightarrow \quad \sigma_1 \frac{\partial \varphi_1(\boldsymbol{r})}{\partial n} - \sigma_2 \frac{\partial \varphi_2(\boldsymbol{r})}{\partial n} = 0$$

Charge conservation forces the continuity of current across the boundary

Electric Current

Current

$\left[\mathrm{A}\right]$

$$I = \int_S \boldsymbol{J}\left(\boldsymbol{r}\right) \cdot d\boldsymbol{S}$$

Cross-section of current path

$\left[\mathrm{m}^2\right]$

Existence of high contrast in conductivity between conductors and dielectrics allows for well defined current paths.

Resistance (Conductance)

Potential difference (voltage) $[\mathrm{V}]$

Resistance $[\Omega]$

Current $[\mathrm{A}]$

Conductance $[\mathrm{S}]$

$$U = RI \quad \Leftrightarrow \quad I = GU$$

$$R = \frac{1}{G} = \frac{L}{\sigma S}$$

Resistance of a cylinder homogeneous cylinder of conductive material

Cross-section of current path $[\mathrm{m}^2]$

Length along current path $[\mathrm{m}]$

Resistive Circuits and Kirchhoff('s) Laws

In a loop

On a resistor

At a junction

$$\sum_i U_i = U_{\text{electromotive}}$$

$$U_i = R_i I_i$$

$$\sum_i I_i = 0$$

Kirchhoff's laws are a consequence of electrostatics and law's of stationary current flow

Joule('s) Heat

Power lost via conduction

$[\mathrm{W}]$

Power lost on resistor

$[\mathrm{W}]$

$$P = \int_V \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r}) \, \mathrm{d}V = \int_V \sigma(\boldsymbol{r}) \left|\boldsymbol{E}(\boldsymbol{r})\right|^2 \mathrm{d}V \quad \Leftrightarrow \quad P = UI = RI^2 = \frac{U^2}{R}$$

Electric field within conducting material produces heat

Fundamental Question of Magnetostatics

There exist a specified distribution of stationary current. We pick a differential volume of it and ask what is the force acting on it.

Biot-Savart('s) Law

Permeability of vacuum

$$\mu_0 = 1.25663706212 \cdot 10^{-6}\ \mathrm{H} \cdot \mathrm{m}^{-1}$$

Measuring current element $\left[\mathrm{A}\right]$

Source current element $\left[\mathrm{A}\right]$

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = \frac{\mu_0}{4\pi} \cdot \frac{\boldsymbol{J}\left(\boldsymbol{r}\right)\mathrm{d}V \times \left[\boldsymbol{J}\left(\boldsymbol{r}'\right)\mathrm{d}V' \times \left(\boldsymbol{r} - \boldsymbol{r}'\right)\right]}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3}$$

Force on measuring current $\left[\mathrm{N}\right]$

Radius vector of the measuring current $\left[\mathrm{m}\right]$

Radius vector of the source current $\left[\mathrm{m}\right]$

Biot-Savart('s) Law + Superposition Principle

$$\boldsymbol{F}\left(\boldsymbol{r}\right) = \boldsymbol{J}\left(\boldsymbol{r}\right)\mathrm{d}\,V \times \frac{\mu_0}{4\pi}\int\limits_{V'} \frac{\boldsymbol{J}\left(\boldsymbol{r}'\right)\times\left(\boldsymbol{r}-\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|^3}\mathrm{d}\,V'$$

Entire magnetostatics can be deduced from this formula

Magnetic Field

$$\boldsymbol{F}\left(\boldsymbol{r}\right)=\boldsymbol{J}\left(\boldsymbol{r}\right)\mathrm{d}\,V\times\boldsymbol{B}\left(\boldsymbol{r}\right) \quad \Longleftrightarrow \quad \boldsymbol{B}\left(\boldsymbol{r}\right)=\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\boldsymbol{J}\left(\boldsymbol{r}'\right)\times\left(\boldsymbol{r}-\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|^3}\mathrm{d}\,V'$$

Magnetic field
(Magnetic induction)
$\left[\mathrm{T}\right]$

Divergence of Magnetic Field

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \quad \Longleftrightarrow \quad \oint\limits_S \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = 0$$

There are no point sources of magnetostatic field

Curl of Magnetic Field – Ampere('s) Law

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \quad \Longleftrightarrow \quad \oint_l \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I$$

Total current captured within the curve

$[\mathrm{A}]$

Magnetic Vector Potential

Magnetic vector potential

Defined up to arbitrary scalar function

$$\nabla \cdot \boldsymbol{B} = 0 \quad \Rightarrow \quad \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \quad \Rightarrow \quad \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' + \nabla \psi(\boldsymbol{r})$$

Reduced description of magnetostatic field

Poisson('s) equation

$$\Delta \boldsymbol{A}\left(\boldsymbol{r}\right) = -\mu_0 \boldsymbol{J}\left(\boldsymbol{r}\right)$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the current density is known through out the volume.

Boundary Conditions

Surface current on the boundary

$$\boldsymbol{n}(\boldsymbol{r})\times\left[\boldsymbol{B}_1(\boldsymbol{r})-\boldsymbol{B}_2(\boldsymbol{r})\right]=\mu_0\boldsymbol{K}(\boldsymbol{r})$$

$$\boldsymbol{n}(\boldsymbol{r})\cdot\left[\boldsymbol{B}_1(\boldsymbol{r})-\boldsymbol{B}_2(\boldsymbol{r})\right]=0$$

$$\boldsymbol{A}_1(\boldsymbol{r})-\boldsymbol{A}_2(\boldsymbol{r})=0$$

Normal pointing to region (1)

Magnetostatic Energy

$$W = \frac{1}{2}\int_V \boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}V \quad \Longleftrightarrow \quad W = \frac{1}{2\mu_0}\int_V \left|\boldsymbol{B}(\boldsymbol{r})\right|^2\,\mathrm{d}V$$

For now it is just a formula that works – it must be derived with the help of time varying fields

Magnetostatic Energy – Current Circuits

$$M_{ij} = M_{ji} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_j} \int_{V_i'} \frac{\boldsymbol{J}_j\left(\boldsymbol{r}_j\right) \cdot \boldsymbol{J}_i\left(\boldsymbol{r}_i'\right)}{\left|\boldsymbol{r}_j - \boldsymbol{r}_i'\right|} \mathrm{d}V_i' \mathrm{d}V_j$$

Mutual-Inductance $\left[\mathrm{H}\right]$

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2}\sum_{i \neq j} M_{ij} I_i I_j$$

Self-Inductance $\left[\mathrm{H}\right]$

$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} \int_{V_i'} \frac{\boldsymbol{J}_i\left(\boldsymbol{r}_i\right) \cdot \boldsymbol{J}_i\left(\boldsymbol{r}_i'\right)}{\left|\boldsymbol{r}_i - \boldsymbol{r}_i'\right|} \mathrm{d}V_i' \mathrm{d}V_i$$

Mutual Inductance – Thin Current Loop

$$\Phi_{ji} = \int_{S_j} \boldsymbol{B}_i\left(\boldsymbol{r}_j\right)\cdot \mathrm{d}\boldsymbol{S}_j$$

Magnetic flux induced by *i*-th current through *j*-th current

$$\left[\mathrm{Wb}\right]$$

$$M_{ij} = \frac{\Phi_{ji}}{I_i}$$

Magnetic Materials

- Material response is due to magnetic dipole moments
- Magnetic moment comes from spin or orbital motion of an electron
- Magnetic field tends to align magnetic moments
- Magnetic field induces magnetic dipoles with density $\boldsymbol{M}(\boldsymbol{r}) \quad [\mathrm{A} \cdot \mathrm{m}^{-1}]$

Number of dipoles
in unitary volume

Magnetic Field of a Dipole

Dipole is assumed at the origin

$\boldsymbol{r} \neq 0$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \cdot \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \quad \Leftrightarrow \quad \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{m})}{r^5} - \frac{\boldsymbol{m}}{r^3}\right]$$

$$\boldsymbol{m} = \frac{1}{2}\int_V \boldsymbol{r} \times \boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}V$$

Magnetic dipole moment

$\left[\mathrm{A} \cdot \mathrm{m}^2\right]$

Magnetic dipole approximates infinitesimally small current loop

Field Produced by Magnetized Matter

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\limits_{V'} \frac{\boldsymbol{M}(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}V' = \frac{\mu_0}{4\pi}\oint\limits_{S'} \frac{\boldsymbol{M}(\boldsymbol{r}') \times \mathrm{d}\boldsymbol{S}'}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{\mu_0}{4\pi}\int\limits_{V'} \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}V'$$

Only applies at infinitely sharp boundary (unrealistic)

Potential of volumetric current density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside

Magnetic Intensity

Magnetic Intensity $\left[\mathrm{A} \cdot \mathrm{m}^{-1}\right]$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r})$$

$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0 \left(\boldsymbol{H}(\boldsymbol{r}) + \boldsymbol{M}(\boldsymbol{r})\right)$$

$$\oint_l \boldsymbol{H}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = I$$

Only free current

Linear Isotropic Magnetic Materials

Relative permeability

$$[\ \ ] \quad \mu_{\mathrm{r}}(\boldsymbol{r}) = 1 + \chi_{\mathrm{m}}(\boldsymbol{r})$$

$$\boldsymbol{M}(\boldsymbol{r}) = \chi_{\mathrm{m}}(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r}) \qquad \boldsymbol{B}(\boldsymbol{r}) = \mu_0\mu_{\mathrm{r}}(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r}) = \mu(\boldsymbol{r})\boldsymbol{H}(\boldsymbol{r})$$

Magnetic susceptibility

$$[\ \ ]$$

Permeability

$$\left[\mathrm{H}\cdot\mathrm{m}^{-1}\right]$$

All the complicated structure of matter reduces to a simple scalar quantity

Fields in Presence of Magnetic Material

$$\nabla \cdot \boldsymbol{B}\left(\boldsymbol{r}\right) = 0 \Leftrightarrow \boldsymbol{B}\left(\boldsymbol{r}\right) = \nabla \times \boldsymbol{A}\left(\boldsymbol{r}\right) \quad \Longrightarrow \quad \nabla \times \left[\frac{1}{\mu\left(\boldsymbol{r}\right)} \nabla \times \boldsymbol{A}\left(\boldsymbol{r}\right)\right] = \boldsymbol{J}\left(\boldsymbol{r}\right)$$

$$\Delta \boldsymbol{A}\left(\boldsymbol{r}\right) = -\mu \boldsymbol{J}\left(\boldsymbol{r}\right)$$

$$\nabla \cdot \boldsymbol{A}\left(\boldsymbol{r}\right) = 0$$

Coulomb('s) gauge

Not a function of coordinates

Poisson's equation holds only when permittivity does not depend on coordinates

Magnetic Material Boundaries

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{H}_1(\boldsymbol{r}) - \boldsymbol{H}_2(\boldsymbol{r})\right] = \boldsymbol{K}(\boldsymbol{r})$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\mu_1 \boldsymbol{H}_1(\boldsymbol{r}) - \mu_2 \boldsymbol{H}_2(\boldsymbol{r})\right] = 0$$

Normal pointing to region (1)

Both conditions are needed for unique solution

Magnetostatic Energy in Magnetic Material

$$W = \frac{1}{2\mu_0} \int\limits_V \left| \boldsymbol{B}\left(\boldsymbol{r}\right) \right|^2 \mathrm{d}\,V \quad \Longrightarrow \quad W = \frac{1}{2} \int\limits_V \boldsymbol{H}\left(\boldsymbol{r}\right) \cdot \boldsymbol{B}\left(\boldsymbol{r}\right) \mathrm{d}\,V$$

Magnetic Materials

- Paramagnetic – small positive susceptibility (small attraction – linear)
- Diamagnetic – small negative susceptibility (small repulsion – linear)
- Ferromagnetic – "*large positive susceptibility*" (large attraction – nonlinear)

Ferromagnetic Materials

- Spins are ordered within domains
- Magnetization is a non-linear function of field intensity
- Magnetization curve – Hysteresis, Remanence
- Susceptibility can only be defined as local approximation
- Above Curie('s) temperature ferromagnetism disappears

Exact calculations are very difficult – use simplified models (soft material, permanent magnet)

Faraday('s) Law

Minus sign is called Lenz('s) law

$$-\frac{\partial \Phi}{\partial t}$$

Time variation of magnetic flux

$$\oint_l \boldsymbol{E}\left(\boldsymbol{r},t\right)\cdot \mathrm{d}\boldsymbol{l} = -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\left(\boldsymbol{r},t\right)\cdot \mathrm{d}\boldsymbol{S} \quad \Longleftrightarrow \quad \nabla\times\boldsymbol{E}\left(\boldsymbol{r},t\right) = -\frac{\partial \boldsymbol{B}\left(\boldsymbol{r},t\right)}{\partial t}$$

Time variation in magnetic field produces electric field that tries to counter the change in magnetic flux (electromotive force)

Lenz('s) Law

The current created by time variation of magnetic flux is directed so as to oppose the flux creating it.

Time Varying RL Circuits

In a loop

$$\sum_i U_i(t) = U_{\text{electromotive}}(t)$$

At a junction

$$\sum_i I_i(t) = 0$$

$$U_i(t) = R_i I_i(t)$$

On a resistor

$$U_1(t) = L_{11}\frac{\partial I_1(t)}{\partial t} + M_{12}\frac{\partial I_2(t)}{\partial t}$$

On an inductor

Circuit laws are valid as long as time variations are not too fast

Time Varying Potentials

Potential calibration

$$\nabla \cdot \boldsymbol{A}\left(\boldsymbol{r},t\right) = -\sigma\mu\varphi\left(\boldsymbol{r},t\right)$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right) = \nabla \times \boldsymbol{A}\left(\boldsymbol{r},t\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = -\nabla\varphi\left(\boldsymbol{r},t\right) - \frac{\partial \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t}$$

In time varying fields scalar potential becomes redundant

Source and Induced Currents

Those are fixed, not reacting to fields

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t) + \boldsymbol{J}_{\text{induced}}(\boldsymbol{r},t) = \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t) + \sigma \boldsymbol{E}(\boldsymbol{r},t)$$

Diffusion Equation

$$\Delta \boldsymbol{A}\left(\boldsymbol{r},t\right) - \sigma\mu \frac{\partial \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t} = -\mu \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)$$

$$\Delta \boldsymbol{H}\left(\boldsymbol{r},t\right) - \sigma\mu \frac{\partial \boldsymbol{H}\left(\boldsymbol{r},t\right)}{\partial t} = -\nabla \times \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)$$

$$\Delta \boldsymbol{E}\left(\boldsymbol{r},t\right) - \sigma\mu \frac{\partial \boldsymbol{E}\left(\boldsymbol{r},t\right)}{\partial t} = \frac{1}{\varepsilon}\nabla \rho_{\text{source}}\left(\boldsymbol{r},t\right) + \mu \frac{\partial \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)}{\partial t}$$

Material parameters are assumed independent of coordinates

Maxwell('s)-Lorentz('s) Equations

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)$$

Equations of motion for fields

Equation of motion for particles

$$\boldsymbol{f}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{J}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)$$

Interaction with materials

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \left(\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)\right)$$

Absolute majority of things happening around you is described by these equations

Boundary Conditions

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{E}_1(\boldsymbol{r},t) - \boldsymbol{E}_2(\boldsymbol{r},t)\right] = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \times \left[\boldsymbol{H}_1(\boldsymbol{r},t) - \boldsymbol{H}_2(\boldsymbol{r},t)\right] = \boldsymbol{K}(\boldsymbol{r},t)$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\boldsymbol{B}_1(\boldsymbol{r},t) - \boldsymbol{B}_2(\boldsymbol{r},t)\right] = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \cdot \left[\boldsymbol{D}_1(\boldsymbol{r},t) - \boldsymbol{D}_2(\boldsymbol{r},t)\right] = \sigma(\boldsymbol{r},t)$$

Normal pointing to region (1)

Electromagnetic Potentials

Lorentz(‘s) calibration

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) = -\sigma\mu\varphi(\boldsymbol{r},t) - \varepsilon\mu\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla\varphi(\boldsymbol{r},t) - \frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$

Wave Equation

$$\Delta \boldsymbol{A}(\boldsymbol{r},t) - \sigma\mu \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} = -\mu \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)$$

Material parameters are assumed independent of coordinates

Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \boldsymbol{E}\cdot\boldsymbol{J}_{\text{source}}\,\mathrm{d}V = \oint_S \left(\boldsymbol{E}\times\boldsymbol{H}\right)\cdot\mathrm{d}\boldsymbol{S} + \int_V \sigma\left|\boldsymbol{E}\right|^2\mathrm{d}V + \frac{1}{2}\frac{\partial}{\partial t}\int_V \left(\varepsilon\left|\boldsymbol{E}\right|^2 + \mu\left|\boldsymbol{H}\right|^2\right)\mathrm{d}V$$

Power supplied by sources

Heat losses

Energy balance in an electromagnetic system

Linear Momentum Carried by Fields

Volume integration considerably change
the meaning of Poynting('s) vector

$$\boldsymbol{p} = \frac{1}{c_0^2} \int_V \left(\boldsymbol{E} \times \boldsymbol{H} \right) \mathrm{d}\, V$$

This formula is only valid in vacuum. In material media things are more tricky.

Angular Momentum Carried by Fields

$$\boldsymbol{L} = \frac{1}{c_0^2} \int\limits_V \boldsymbol{r} \times \left(\boldsymbol{E} \times \boldsymbol{H} \right) \mathrm{d}\,V$$

This formula is only valid in vacuum. In material media things are more tricky.

Frequency Domain

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) \in \mathbb{R} \qquad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \in \mathbb{C}$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \mathrm{e}^{\mathrm{j}\omega t} d\omega \quad \Longleftrightarrow \quad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) = \int_{-\infty}^{\infty} \boldsymbol{F}\left(\boldsymbol{r},t\right) \mathrm{e}^{-\mathrm{j}\omega t} dt$$

$$\frac{\partial \boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial t} \leftrightarrow \mathrm{j}\omega \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)$$

$$\frac{\partial \boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial r_\xi} \leftrightarrow \frac{\partial \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)}{\partial r_\xi}$$

Time derivatives reduce to algebraic multiplication

Spatial derivatives are untouched

Frequency domain helps us to remove explicit time derivatives

Phasors

$$\hat{\boldsymbol{F}}\left(\boldsymbol{r},-\omega\right)=\hat{\boldsymbol{F}}^{*}\left(\boldsymbol{r},\omega\right) \quad \Rightarrow \quad \boldsymbol{F}\left(\boldsymbol{r},t\right)=\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\right]d\omega$$

Reduced frequency domain representation

Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\boldsymbol{H}}\left(\boldsymbol{r}, \omega\right) = \hat{\boldsymbol{J}}\left(\boldsymbol{r}, \omega\right) + \mathrm{j}\omega\varepsilon\hat{\boldsymbol{E}}\left(\boldsymbol{r}, \omega\right)$$

$$\nabla \times \hat{\boldsymbol{E}}\left(\boldsymbol{r}, \omega\right) = -\mathrm{j}\omega\mu\hat{\boldsymbol{H}}\left(\boldsymbol{r}, \omega\right)$$

$$\nabla \cdot \hat{\boldsymbol{H}}\left(\boldsymbol{r}, \omega\right) = 0$$

$$\nabla \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{r}, \omega\right) = \frac{\hat{\rho}\left(\boldsymbol{r}, \omega\right)}{\varepsilon}$$

We assume linearity of material relations

Wave Equation – Frequency Domain

$$\Delta \hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) - \mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)\hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) = -\mu\hat{\boldsymbol{J}}_{\text{source}}\left(\boldsymbol{r},\omega\right)$$

Helmholtz(‘s) equation

Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \left\langle \boldsymbol{E} \cdot \boldsymbol{J}_{\text{source}} \right\rangle \mathrm{d}V = \oint_S \left\langle \boldsymbol{E} \times \boldsymbol{H} \right\rangle \cdot \mathrm{d}\boldsymbol{S} + \int_V \left\langle \sigma \left| \boldsymbol{E} \right|^2 \right\rangle \mathrm{d}V$$

$$-\frac{1}{2}\int_V \mathrm{Re}\left[\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{J}}^*_{\text{source}} \right] \mathrm{d}V = \frac{1}{2}\oint_S \mathrm{Re}\left[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^* \right] \cdot \mathrm{d}\boldsymbol{S} + \frac{1}{2}\int_V \sigma \left| \hat{\boldsymbol{E}} \right|^2 \mathrm{d}V$$

Valid for time-harmonic steady state

Plane Wave

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_0\left(\omega\right)\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \frac{k}{\omega\mu}\left[\boldsymbol{n}\times\boldsymbol{E}_0\left(\omega\right)\right]\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\boldsymbol{n}\cdot\boldsymbol{E}_0\left(\omega\right) = 0$$

$$\boldsymbol{n}\cdot\boldsymbol{H}_0\left(\omega\right) = 0$$

$$k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$$

Wave-number

The simplest wave solution of Maxwell(‘s) equations

Plane Wave Characteristics

General isotropic material

$$k = \sqrt{-\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] < 0$$

$$\lambda = \frac{2\pi}{\mathrm{Re}\left[k\right]}$$

$$v_{\mathrm{f}} = \frac{\omega}{\mathrm{Re}\left[k\right]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\mathrm{Im}\left[k\right]}$$

Vacuum

$$k = \frac{\omega}{c_0}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_{\mathrm{f}} = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega$$

$$\delta \to \infty$$

Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\left\langle \boldsymbol{E}\left(\boldsymbol{r},t\right)\times\boldsymbol{H}\left(\boldsymbol{r},t\right)\right\rangle = \frac{1}{2}\frac{\mathrm{Re}\left[k\right]}{\omega\mu}\left|\boldsymbol{E}_0\left(\omega\right)\right|^2 \mathrm{e}^{2\,\mathrm{Im}\left[k\right]\boldsymbol{n}\cdot\boldsymbol{r}}\boldsymbol{n}$$

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