Maxwell’s-Lorentz’s Equations

\[ \nabla \times \mathbf{H}(r,t) = \mathbf{J}(r,t) + \frac{\partial \mathbf{D}(r,t)}{\partial t} \]
\[ \nabla \times \mathbf{E}(r,t) = -\frac{\partial \mathbf{B}(r,t)}{\partial t} \]
\[ \nabla \cdot \mathbf{B}(r,t) = 0 \]
\[ \nabla \cdot \mathbf{D}(r,t) = \rho(r,t) \]

Equations of motion for fields

Equation of motion for particles

\[ \mathbf{f}(r,t) = \rho(r,t) \mathbf{E}(r,t) + \mathbf{J}(r,t) \times \mathbf{B}(r,t) \]

Interaction with materials

\[ \mathbf{D}(r,t) = \varepsilon_0 \mathbf{E}(r,t) + \mathbf{P}(r,t) \]
\[ \mathbf{B}(r,t) = \mu_0 \left( \mathbf{H}(r,t) + \mathbf{M}(r,t) \right) \]

Absolute majority of things happening around us is described by these equations
Boundary Conditions

\[ n(r) \times [E_1(r, t) - E_2(r, t)] = 0 \]

\[ n(r) \times [H_1(r, t) - H_2(r, t)] = K(r, t) \]

\[ n(r) \cdot [B_1(r, t) - B_2(r, t)] = 0 \]

\[ n(r) \cdot [D_1(r, t) - D_2(r, t)] = \sigma(r, t) \]
Electromagnetic Potentials

Lorentz('s) calibration

\[ \nabla \cdot A(r, t) = -\sigma \mu \varphi(r, t) - \varepsilon \mu \frac{\partial \varphi(r, t)}{\partial t} \]

\[ B(r, t) = \nabla \times A(r, t) \]

\[ E(r, t) = -\nabla \varphi(r, t) - \frac{\partial A(r, t)}{\partial t} \]
Wave Equation

\[ \Delta A(r, t) - \sigma \mu \frac{\partial A(r, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 A(r, t)}{\partial t^2} = -\mu J_{\text{source}}(r, t) \]

Material parameters are assumed independent of coordinates.
Energy balance in an electromagnetic system

\[- \int_{V} E \cdot J_{source} \, dV = \oint_{S} (E \times H) \cdot dS + \int_{V} \sigma |E|^2 \, dV + \frac{1}{2} \frac{\partial}{\partial t} \int_{V} \left( \varepsilon |E|^2 + \mu |H|^2 \right) \, dV\]
**Frequency Domain**

\[
F(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(r, \omega) e^{j\omega t} d\omega \quad \leftrightarrow \quad \hat{F}(r, \omega) = \int_{-\infty}^{\infty} F(r, t) e^{-j\omega t} dt
\]

- Time derivatives reduce to algebraic multiplication.
- Spatial derivatives are untouched.

**Frequency domain helps us to remove explicit time derivatives.**
Phasors

\[ \hat{F}(r, -\omega) = \hat{F}^*(r, \omega) \]

\[ F(r, t) = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \hat{F}(r, \omega) e^{i\omega t} \right] d\omega \]

Reduced frequency domain representation
Maxwell(’s) Equations – Frequency Domain

\[ \nabla \times \hat{H}(r, \omega) = \hat{J}(r, \omega) + j\omega \varepsilon \hat{E}(r, \omega) \]

\[ \nabla \times \hat{E}(r, \omega) = -j\omega \mu \hat{H}(r, \omega) \]

\[ \nabla \cdot \hat{H}(r, \omega) = 0 \]

\[ \nabla \cdot \hat{E}(r, \omega) = \frac{\hat{\rho}(r, \omega)}{\varepsilon} \]

We assume linearity of material relations
Helmholtz(‘s) equation

\[ \Delta \hat{A}(r, \omega) - j\omega\mu (\sigma + j\omega\varepsilon) \hat{A}(r, \omega) = -\mu \hat{J}_{\text{source}}(r, \omega) \]
Heat Balance in Time-Harmonic Steady State

\[-\int_V \langle E \cdot J_{\text{source}} \rangle \, dV = \oint_S \langle E \times H \rangle \cdot dS + \int_V \langle \sigma |E|^2 \rangle \, dV\]

\[-\frac{1}{2} \int_V \text{Re} \left[ \hat{E} \cdot \hat{J}^*_{\text{source}} \right] \, dV = \frac{1}{2} \oint_S \text{Re} \left[ \hat{E} \times \hat{H}^* \right] \cdot dS + \frac{1}{2} \int_V \sigma |\hat{E}|^2 \, dV\]

Valid for general periodic steady state

Cycle mean

Valid for time-harmonic steady state
**Plane Wave**

The simplest wave solution of Maxwell’s equations is given by:

\[
\hat{E}(r, \omega) = E_0(\omega) e^{-jkr} e^{-j\omega t} \\
\hat{H}(r, \omega) = \frac{k}{\omega \mu} \left[ n \times E_0(\omega) \right] e^{-jkr} e^{-j\omega t}
\]

- **Unitary vector representing the direction of propagation**
- **Electric and magnetic fields are mutually orthogonal**
- **Electric and magnetic fields are orthogonal to propagation direction**
- **Wave-number**

\[
n \cdot E_0(\omega) = 0 \\
n \cdot H_0(\omega) = 0 \\
k^2 = -j\omega \mu \left( \sigma + j\omega \varepsilon \right)
\]

*The simplest wave solution of Maxwell’s equations*
Plane Wave Characteristics

\[ k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)} \]

- \( \text{Re}[k] > 0; \ \text{Im}[k] < 0 \)

\[ \lambda = \frac{2\pi}{\text{Re}[k]} \]

\[ v_f = \frac{\omega}{\text{Re}[k]} \]

\[ Z = \frac{\omega\mu}{k} \]

\[ \delta = -\frac{1}{\text{Im}[k]} \]

Vacuum

\[ k = \frac{\omega}{c_0} \]

- \( \text{Re}[k] > 0; \ \text{Im}[k] = 0 \)

\[ \lambda = \frac{c_0}{f} \]

\[ v_f = c_0 \]

\[ Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \ \Omega \]

\[ \delta \to \infty \]

General isotropic material
Cycle Mean Power Density of a Plane Wave

\[
\left\langle E(r, t) \times H(r, t) \right\rangle = \frac{1}{2} \frac{\text{Re}[k]}{\omega \mu} |E_0(\omega)|^2 \ e^{2\text{Im}[k]n \cdot r}
\]
Source Free Maxwell(’s) Equations in Free Space

\[ \nabla \times \mathbf{H}(\mathbf{r}, t) = \sigma(\mathbf{r}, t) \times \mathbf{E}(\mathbf{r}, t) + \varepsilon(\mathbf{r}, t) \times \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \]
\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu(\mathbf{r}, t) \times \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \]
\[ \nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0 \]
\[ \nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0 \]

\[ |k|^2 = k^2 = -j\omega \mu(\omega)\sigma(\omega) + j\omega \varepsilon(\omega) \]
\[ \hat{\mathbf{E}}(k, \omega) = \frac{k}{\omega \varepsilon(\omega) - j\sigma(\omega)} \times \hat{\mathbf{H}}(k, \omega) \]
\[ \hat{\mathbf{H}}(k, \omega) = -\frac{k}{\omega \mu(\omega)} \times \hat{\mathbf{E}}(k, \omega) \]
\[ k \cdot \hat{\mathbf{E}}(k, \omega) = 0 \]
\[ k \cdot \hat{\mathbf{H}}(k, \omega) = 0 \]

\[ \mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{k,t} \hat{\mathbf{F}}(k, \omega) e^{j(k \cdot r - \omega t)} \, dk \, d\omega \]

Fourier’s transform leads to simple algebraic equations
Spatial Wave Packet

\[ |k|^2 = k^2 = -j\omega\mu(\omega)(\hat{\sigma}(\omega) + j\omega\epsilon(\omega)) \quad \rightarrow \quad \omega = \omega(|k|) \]

This can be electric or magnetic intensity

\[ \mathbf{F}(r, t) = \frac{1}{(2\pi)^3} \int k \hat{\mathbf{F}}_0(k) e^{j(k \cdot r + \omega(|k|) t)} \, dk \]

\[ k \cdot \hat{\mathbf{F}}_0(k) = 0 \]

General solution to free-space Maxwell’s equations
Spatial Wave Packet in Vacuum

\[ \omega(|k|) = \pm c_0 |k| \]

\[ k \cdot \hat{F}_0^+ (k) = k \cdot \hat{F}_0^- (k) = 0 \]

\[ \hat{F}_0^- (k) = [\hat{F}_0^+ (-k)]^* \]

\[ \hat{F}_0^+ (k) = [\hat{F}_0^- (-k)]^* \]

\[ F(r, t) = \frac{1}{(2\pi)^3} \int_k e^{jk \cdot r} \left[ \hat{F}_0^+ (k)e^{jc_0 |k| t} + \hat{F}_0^- (k)e^{-jc_0 |k| t} \right] dk \]

\[ \hat{F}_0^+ (k) = \frac{1}{2} \int_r F(r, 0) + \frac{1}{jc_0 |k|} \frac{\partial F(r, t)}{\partial t} \bigg|_{t=0} \right] e^{-jk \cdot r} dr \]

\[ \hat{F}_0^- (k) = \frac{1}{2} \int_r F(r, 0) - \frac{1}{jc_0 |k|} \frac{\partial F(r, t)}{\partial t} \bigg|_{t=0} \right] e^{-jk \cdot r} dr \]

The field is uniquely given by initial conditions
Spatial Wave Packet in Vacuum

$$\omega(\lvert \mathbf{k} \rvert) = \pm c_0 \lvert \mathbf{k} \rvert$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{(2\pi)^3} \left[ \hat{\mathbf{E}}^+ (\mathbf{k}) e^{i c_0 t \lvert \mathbf{k} \rvert} + \hat{\mathbf{E}}^- (\mathbf{k}) e^{-i c_0 t \lvert \mathbf{k} \rvert} \right] d\mathbf{k}$$

$$\mathbf{H}(\mathbf{r}, t) = -\frac{1}{(2\pi)^3} \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{(2\pi)^3} \frac{\mathbf{k}}{Z_0 \lvert \mathbf{k} \rvert} \times \left[ \hat{\mathbf{E}}^+ (\mathbf{k}) e^{i c_0 t \lvert \mathbf{k} \rvert} - \hat{\mathbf{E}}^- (\mathbf{k}) e^{-i c_0 t \lvert \mathbf{k} \rvert} \right] d\mathbf{k}$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}^+ (\mathbf{k}) = \mathbf{k} \cdot \hat{\mathbf{E}}^- (\mathbf{k}) = 0$$

Electric and magnetic field are not independent
Vacuum Dispersion

1D waves in vacuum propagate without dispersion

\[ \mathbf{E}(z, t) = \mathbf{E}^+(z + c_0 t) + \mathbf{E}^-(z - c_0 t) \]

\[ \mathbf{H}(z, t) = -\frac{1}{Z_0} \mathbf{z}_0 \times \left[ \mathbf{E}^+ (z + c_0 t) - \mathbf{E}^- (z - c_0 t) \right] \]
Vacuum Dispersion

In general this term does not represent translation

\[ \left( [x, y, z] \pm c_0 t \right) \]

\[ E(r, t) = \frac{1}{(2\pi)^3} \int \frac{e^{ikr}}{k^3} \left[ E^+ (k) e^{j\omega t|k|} + E^- (k) e^{-j\omega t|k|} \right] dk \]

Waves propagating in all directions

2D and 3D waves in vacuum always disperse = change shape in time
Angular Spectrum Representation

\[ |\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)\left(\hat{\sigma}(\omega) + j\omega\hat{\varepsilon}(\omega)\right) \Rightarrow k_z = \pm \sqrt{k_x^2 - k_y^2 - k_z^2} \]

\[ \hat{\mathbf{H}}_0(k_x, k_y, \omega) = -\frac{k}{Z|\mathbf{k}|} \times \hat{\mathbf{E}}_0(k_x, k_y, \omega) \quad \hat{\mathbf{E}}_0(k_x, k_y, \omega) = \mathcal{F}_{x,y,t}\{\mathbf{E}(x, y, 0, t)\} \]

\[ \mathbf{E}(x, y, z < 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{j\sqrt{k_x^2 - k_y^2 - k_z^2}} dk_x dk_y d\omega \]

\[ \mathbf{E}(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{-j\sqrt{k_x^2 - k_y^2 - k_z^2}} dk_x dk_y d\omega \]

\[ k \cdot \hat{\mathbf{E}}_0 = 0 \]

General solution to free-space Maxwell's equations
Propagating vs Evanescent Waves

These waves propagate and can carry information to far distances

\[ k_x^2 + k_y^2 < k^2 \]

These waves exponentially decay in amplitude and cannot carry information to far distances

\[ k_x^2 + k_y^2 > k^2 \]

Field picture losses it resolution with distance from the source plane
Paraxial Waves

\[ \hat{E}_0(k_x, k_y, \omega) \quad \Rightarrow \quad k_x^2 + k_y^2 \ll k^2 \quad \Rightarrow \quad \sqrt{k_x^2 - k_x^2 - k_y^2} \approx k - \frac{1}{2k} (k_x^2 + k_y^2) \]

\[ E(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{i(k_x x + k_y y - k_z z - \omega t)} \hat{E}_0(k_x, k_y, \omega) e^{i \frac{1}{2k}(k_x^2 + k_y^2)} dk_x \, dk_y \, d\omega \]

\[ k \cdot \hat{E}_0 = 0 \]

Propagates almost as a planewave

\[ z_0 \cdot \hat{E}_0 \approx 0 \]
Gaussian Beam

\[
\hat{E}_{0\perp}(k_x, k_y, \omega) = A_{0\perp} \pi w_0^2 e^{-\frac{1}{4}w_0^2(k_x^2 + k_y^2)}
\]

\[
E_{\perp}(x, y, z > 0, t) = \frac{1}{2\pi} \int_{\omega} A_{0\perp} \frac{w_0}{w(z)} e^{\frac{-(x^2+y^2)}{w^2(z)}} e^{\frac{2\pi j}{\lambda_0} \left[ x \frac{z}{w(z)} + \frac{x^2+y^2}{w^2(z)} \frac{z}{w(z)} \right]} e^{\frac{2\pi j t}{\lambda_0}} d\omega
\]

Approximates radiation of sources large in comparison to wavelength
Gaussian Beam

Half-width of the beam

\[ w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \]

Beam divergence

\[ \Theta = \frac{2\lambda}{\pi w_0} \]

Rayleigh's distance

\[ z_R = \frac{1}{2} kw_0^2 = \frac{\pi w_0^2}{\lambda} \]
Gaussian Beam – Time-Harmonic Case

\[ \langle S \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E}(x, y, z, \omega) \times \mathbf{H}^*(x, y, z, \omega) \right] = z_0 S_0 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}} \]

86.5 % of power flows through the beam width

\[ w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \]

Power density at origin
Material Dispersion

Causality requirement
\[ \varepsilon(\tau) = 0, \tau < 0 \]

Stability requirement
\[ \varepsilon(\tau) \to 0, \tau \to \infty \]

\[ D(r,t) = \int_{-\infty}^{\infty} \varepsilon(\tau) E(r,t-\tau) d\tau \]

\[ B(r,t) = \int_{-\infty}^{\infty} \mu(\tau) H(r,t-\tau) d\tau \]

\[ J(r,t) = \int_{-\infty}^{\infty} \sigma(\tau) E(r,t-\tau) d\tau \]

\[ \hat{D}(r,\omega) = \hat{\varepsilon}(\omega) \hat{E}(r,\omega) \]

\[ \hat{B}(r,\omega) = \hat{\mu}(\omega) \hat{H}(r,\omega) \]

\[ \hat{J}(r,\omega) = \hat{\sigma}(\omega) \hat{E}(r,\omega) \]

Even single planewave undergoes time dispersion when materials are present
Lorentz’s Dispersion Model

\[
\frac{\partial^2 P(t)}{\partial t^2} + \Gamma \frac{\partial P(t)}{\partial t} + \omega_0^2 P(t) = \varepsilon_0 \omega_p^2 E(t)
\]

\[
\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}\right)
\]

\[
\varepsilon(\omega) = \varepsilon_0 \left(1 + \sum_i \frac{\omega_{p,i}^2}{\omega_{0,i}^2 - \omega^2 + j\omega\Gamma_i}\right)
\]

Dispersion model able to describe vast amount of natural materials
Drude’s Dispersion Model

Special case of Lorentz's dispersion

\[ \omega_0 = 0 \]
\[ \omega_p^2 = \frac{\sigma_0 \Gamma}{\varepsilon_0} \]

Permittivity model

\[ \varepsilon(\omega) = \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega(\omega - j\Gamma)} \right) \]

Conductivity model

\[ \sigma(\omega) = \frac{\sigma_0}{1 + j\frac{\omega}{\Gamma}} \]

Collisionless plasma

\[ \frac{\Gamma}{\omega} \ll 1 \quad \Rightarrow \quad \varepsilon(\omega) \approx \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \]

Dispersion model describing neutral plasma
Appleton’s Dispersion Model

\[ \frac{\partial^2 P(t)}{\partial t^2} + \omega_c \frac{\partial P(t)}{\partial t} \times z_0 = \varepsilon_0 \omega_p^2 E(t) \]

- Cyclotron frequency
- Plasma frequency
- Direction of magnetization

\[ \hat{\varepsilon} \neq \hat{\varepsilon}^T \]
Propagation in opposite directions is not the same

\[ \hat{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & -j\omega_c \frac{\omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 0 \\ \frac{j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{bmatrix} \]

Dispersion model describing magnetized neutral plasma
Propagating waves in Appleton’s Dispersion Model

Planewave propagation along magnetization:

\[ \mathbf{E} = E_0 e^{j k_z z} \]
\[ k \cdot E_0 = 0 \]

Fundamental modes are circularly polarized waves:

\[ \frac{k_z^2}{k_0^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)} \]
\[ \hat{E}_x = \mp j \hat{E}_y \]

Dispersion model describing magnetized neutral plasma.
Radiation

Microscopic charge velocity

\[ \frac{\partial v(t)}{\partial t} \neq 0 \]

Macroscopic current density

\[ \frac{\partial J(r,t)}{\partial t} \neq 0 \]

Any surface circumscribing the sources

\[ \int_{-\infty}^{\infty} \oint S \left( E(r,t) \times H(r,t) \right) \cdot dS \, dt \neq 0 \]

Only accelerating charges can radiate
**Time-Harmonic Electric Dipole**

\[ \hat{P}(r, \omega) = z_0 p_z(\omega) \delta(x) \delta(y) \delta(z) \]

\[ \rho(r, \omega) \approx 0 \]

\[ \hat{A}(r, \omega) = j Z_0 k^2 \left( r_0 \cos \theta - \theta_0 \sin \theta \right) p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \]

\[ \hat{H}(r, \omega) = c_0 k^3 \varphi_0 \sin \theta \left( -1 + \frac{j}{kr} \right) p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \]

\[ \hat{E}(r, \omega) = Z_0 c_0 k^3 \left[ 2r_0 \cos \theta \left( \frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left( -1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \]

Elementary source of radiation
Time-Harmonic Electric Dipole - Field Zones

\[ \hat{E}(r, \omega) = Z_0 c_0 k^3 \left[ 2r_0 \cos \theta \left( \frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left( -1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \]

Static, quasi-static and fully dynamic terms all appear in the formula.
Time-Harmonic Electric Dipole - Radiation Zone

\[ \hat{P}(r, \omega) = z_0 p_z(\omega) \delta(x) \delta(y) \delta(z) \]

Farfield has a planewave-like geometry

\[ \hat{E}_\infty(r, \omega) \approx -Z_0 c_0 k^3 \theta_0 p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \sin \theta \]

\[ \hat{H}_\infty(r, \omega) \approx \frac{1}{Z_0} r_0 \times \hat{E}_\infty(r, \omega) \]

\[ \langle S_\infty \rangle = \frac{1}{2} \text{Re}[\hat{E}_\infty \times \hat{H}_\infty^*] = \frac{1}{2Z_0} \left|\hat{E}_\infty(r, \omega)\right|^2 r_0 \]

Radiated power [W]

\[ P_{\text{rad}} = \frac{c_0^2 Z_0 k^4}{12\pi} \left|p_z(\omega)\right|^2 \]
Time-Harmonic Electric Dipole – General Case

\[ \hat{P}(r, \omega) = \hat{p}(\omega) \delta (r - r') \]

\[ \hat{H}(r, \omega) = c_0 k^3 \left( \frac{R}{R} \times \hat{p} \right) \left(1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{4\pi kR} \]

\[ \hat{E}(r, \omega) = Z_0 c_0 k^3 \left[ -\frac{R}{R} \times \left( \frac{R}{R} \times \hat{p} \right) + \left( 3 \frac{R}{R} \left( \hat{p} \cdot \frac{R}{R} \right) - \hat{p} \right) \left( \frac{1}{k^2 R^2} + \frac{j}{kR} \right) \right] \frac{e^{-jkR}}{4\pi kR} \]

Elementary source of radiation

\[ R = \left| r - r' \right| \]

\[ R = r - r' \]
General Radiator

\[ \hat{\mathbf{J}}(\mathbf{r}, \omega), \mathbf{J}(\mathbf{r}, t) \]

\[ \hat{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) \frac{e^{-jk_0 R}}{R} dV' \]

\[ \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \mathbf{J} \left[ \mathbf{r}', t - \frac{R}{c_0} \right] \frac{dV'}{R} \]

\[ R = \left| \mathbf{r} - \mathbf{r}' \right| \]

Superposition of dipole fields
**Field in Radiation Zone – General Case FD**

\[ kR \gg 1 \land r \gg r' \]

\[ \hat{A}_\infty (r, \omega) \approx \frac{\mu_0}{4\pi r} e^{\frac{-jkr}{r}} \int_{V'} \hat{J} (r', \omega) e^{\frac{jkr_0}{r} \cdot r'} dV' \]

\[ \hat{H}_\infty (r, \omega) \approx -\frac{j\omega}{Z_0} r_0 \times \hat{A}_\infty (r, \omega) \]

\[ \hat{E}_\infty (r, \omega) \approx j\omega r_0 \times (r_0 \times \hat{A}_\infty (r, \omega)) \]

\[ \langle S_\infty \rangle = \frac{1}{2Z_0} \omega^2 \left| r_0 \times \hat{A}_\infty (r, \omega) \right|^2 r_0 \]

Farfield has a planewave-like geometry
Field in Radiation Zone – General Case TD

\[ E_\infty \approx -Z_0 \left( r_0 \times H_\infty \right) \]

\[ A_\infty (r, t) \approx \frac{\mu}{4\pi r} \int_{V'} J \left( r', t - \frac{r'}{c_0} + \frac{r_0 \cdot r'}{c_0} \right) dV' \]

\[ H_\infty (r, t) \approx -\frac{1}{Z_0} r_0 \times \dot{A}_\infty (r, t) \]

\[ E_\infty (r, t) \approx r_0 \times \left( r_0 \times \dot{A}_\infty (r, t) \right) \]

\[ S_\infty \approx \frac{1}{Z_0} \left| r_0 \times \dot{A}_\infty (r, t) \right|^2 r_0 \]

Farfield has a planewave-like geometry
Radiation Zone = Rays

\[ \hat{A}_\infty (\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{-jk_0r}}{r} \int_{V'} \hat{J} (\mathbf{r}', \omega) e^{jk_0r_0r'} \, dV' \]

4D Fourier(’s) transform

\[ \hat{J} (k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ J (x, y, z, t) \} \]

\[ A_\infty (\mathbf{r}, t) \approx \mathcal{F}^{-1}_\omega \left[ \frac{\mu_0}{4\pi} \hat{J} (-k_0 \mathbf{r}_0, \omega) e^{-jk_0r} \right] \]

Radiation diagram is formed by Fourier(’s) transform of sources
Angular Spectrum Representation (Sources)

4D Fourier(´s) transform

\[
\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ \mathbf{J}(x, y, z, t) \}
\]

\[
k_z = \sqrt{k_x^2 - k_y^2 - k_z^2}
\]

\[
\text{Im}[k_z] < 0
\]

\[
\hat{\mathbf{G}} = \frac{\hat{\mathbf{J}}(k_x, k_y, \mp k_z, \omega)}{2k_z} e^{\pm jk_z z}
\]

Valid only outside the source region

\[
\mathbf{H} \left( x, y, z > \max \left( z' \right), t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left[ \left[ k_x, k_y, \mp k_z \right] \times \hat{\mathbf{G}} \right]
\]

\[
\mathbf{E} \left( x, y, z > \max \left( z' \right), t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left[ \frac{Z_0}{k} \left[ k_x, k_y, \mp k_z \right] \times \left( \left[ k_x, k_y, \mp k_z \right] \times \hat{\mathbf{G}} \right) \right]
\]

General solution to free-space Maxwell's equations
Angular Spectrum in Radiation Zone

\[ F(\mathbi{x}, \mathbi{y}, \mathbi{z} > 0, \omega) = \mathcal{F}^{-1}_{k_x, k_y} \{ \hat{G}(k_x, k_y) e^{-jk_z z} \} \]

As \( k_0 r \to \infty \): 
- \( k_z = \sqrt{k_0^2 - k_x^2 - k_y^2} \)
- \( \text{Im}[k_z] < 0 \)

\[ r_0 = \frac{[x, y, z]}{r} \]

\[ F_\infty (\mathbi{r}, \omega) = \frac{1}{(2\pi)^2} \int_{k^2 > k_x^2 + k_y^2} \hat{G}(k_x, k_y) e^{jk_0 r \left[\frac{k_x r_{0x} + k_y r_{0y} - k_z r_{0z}}{k_x k_y k_z} \right]} dk_x dk_y \]

As \( k_0 r \to \infty \): 
- Stationary phase method

\[ F_\infty (\mathbi{r}, \omega) = \frac{jk_0 r_{0z}}{2\pi} \hat{G}\left( -k_0 r_{0x}, -k_0 r_{0y}, \frac{e^{-jk_0 r}}{r} \right) \]

Farfield is made of propagating planewaves
Angular Spectrum in Radiation Zone

4D Fourier(´s) transform

\[ \hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t}\{\mathbf{J}(x, y, z, t)\} \]

\[ H_{\infty}(r, t) \approx \mathcal{F}_\omega^{-1}\left\{ \frac{jk_0}{4\pi} r_0 \times \hat{\mathbf{J}}(-k_0 r_0, \omega) \frac{e^{-jk_0 r}}{r} \right\} \]

\[ E_{\infty}(r, t) \approx \mathcal{F}_\omega^{-1}\left\{ \frac{jk_0 Z_0}{4\pi} r_0 \times [r_0 \times \hat{\mathbf{J}}(-k_0 r_0, \omega)] \frac{e^{-jk_0 r}}{r} \right\} \]

Farfield is made of propagating planewaves

\[ r_0 = \frac{[x, y, z]}{r} \]
Planar Material Boundary

Field is composed of incident, reflected and transmitted waves

\[ k_z = \sqrt{k_x^2 - k_y^2 - k_z^2} \]

Incident wave

Reflected (1 → 1) / Transmitted (2 → 1) wave

\[
H(z < 0) = \mathcal{F}_{k_x,k_y}^{-1} \left\{ \frac{H_1^+(k_x,k_y,\omega)}{k_1} e^{-j k_1 z} + \frac{H_1^-(k_x,k_y,\omega)}{k_1} e^{j k_1 z} \right\}
\]

\[
E(z < 0) = \mathcal{F}_{k_x,k_y}^{-1} \left\{ \left[ \frac{k_x k_y - k_{z1}}{k_1} \right] \times \frac{H_1^+(k_x,k_y,\omega)}{k_1} e^{-j k_1 z} + \left[ \frac{k_x k_y - k_{z1}}{k_1} \right] \times \frac{H_1^-(k_x,k_y,\omega)}{k_1} e^{j k_1 z} \right\}
\]

Reflected (2 → 2) / Transmitted (1 → 2) wave

\[
H(z > 0) = \mathcal{F}_{k_x,k_y}^{-1} \left\{ \frac{H_2^+(k_x,k_y,\omega)}{k_2} e^{-j k_2 z} + \frac{H_2^-(k_x,k_y,\omega)}{k_2} e^{j k_2 z} \right\}
\]

\[
E(z > 0) = \mathcal{F}_{k_x,k_y}^{-1} \left\{ \left[ \frac{k_x k_y - k_{z2}}{k_2} \right] \times \frac{H_2^+(k_x,k_y,\omega)}{k_2} e^{-j k_2 z} + \left[ \frac{k_x k_y - k_{z2}}{k_2} \right] \times \frac{H_2^-(k_x,k_y,\omega)}{k_2} e^{j k_2 z} \right\}
\]

Boundary is at \( z = 0 \)

Incident wave

\(~\)
Planar Material Boundary – Boundary Conditions

\[
\begin{align*}
[k_x, k_y, \mp k_z] \cdot H_1^\pm &= 0 \\
[k_x, k_y, \mp k_z] \cdot H_2^\pm &= 0
\end{align*}
\]

\[
\begin{align*}
\mathbf{z}_0 \times H_1^+ + \mathbf{z}_0 \times H_1^- &= \mathbf{z}_0 \times H_2^+ + \mathbf{z}_0 \times H_2^- \\
\frac{\mathbf{z}_0 \times \left([k_x, k_y, -k_z] \times H_1^+\right)}{Z_1} + \frac{\mathbf{z}_0 \times \left([k_x, k_y, k_z] \times H_1^-\right)}{k_1} &= \\
\frac{\mathbf{z}_0 \times \left([k_x, k_y, -k_z] \times H_2^+\right)}{Z_2} + \frac{\mathbf{z}_0 \times \left([k_x, k_y, k_z] \times H_2^-\right)}{k_2}
\end{align*}
\]

\[k_x, k_y\] are equal on both sides

\[k_z = \sqrt{k_x^2 - k_y^2 - k_z^2}, \quad \text{Im}[k_z] < 0\]

Relations valid for both propagative and evanescent waves
Perpendicular Incidence – Matrix Form

\[ k_x = k_y = 0 \]

Transmission matrix
(multilayer cascade)

\[
\begin{bmatrix}
E_2^+ \\
E_2^-
\end{bmatrix} = [T] \begin{bmatrix}
E_1^+ \\
E_1^-
\end{bmatrix}
\]

\[
[T] = \frac{1}{T_{22}} \begin{bmatrix}
-T_{21} & 1 \\
\det T & T_{12}
\end{bmatrix}
\]

\[
[S] = \frac{1}{S_{12}} \begin{bmatrix}
-S_{11} & S_{22} \\
\det S & S_{22}
\end{bmatrix}
\]

Scattering matrix
(experiments)

\[
\begin{bmatrix}
E_1^- \\
E_2^+
\end{bmatrix} = [S] \begin{bmatrix}
E_1^+ \\
E_2^-
\end{bmatrix}
\]

\[
[S] = \frac{1}{Z_2 + Z_1} \begin{bmatrix}
Z_2 - Z_1 & 2Z_1 \\
2Z_2 & Z_1 - Z_2
\end{bmatrix}
\]

Matrices with the use across the electrical engineering
Perpendicular Incidence – Interesting Cases

Wavelength inside the slab

\[ d = \frac{\lambda}{2N} \]

Transparent dielectric layer

\[ T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \]

Bragg's mirror
(dielectric mirror)

Alternating dielectric layers

\[ k_0 \left( n_1 d_1 + n_2 d_2 \right) = N \pi \]

Technically important special cases
Oblique Incidence – TM / TE Case

Snell’s law is a property of k-vectors

\[ k_x = k_x \]
\[ k_y = 0 \]

Snell’s law of refraction

\[ k_1 \sin \alpha_{\text{inc}} = k_2 \sin \alpha_{\text{trans}} \]

\[ k_1 \sin \alpha_{\text{inc}} = k_1 \sin \alpha_{\text{refl}} \]

The angle of incidence equals to the angle of reflection

Snell’s law is a property of k-vectors
Oblique Incidence – TM Case

\[
R_{1\rightarrow 1}^{\text{TM}} = \frac{E_{1x}^-}{E_{1x}^+} = \frac{\sqrt{1 - \frac{k_x^2}{k_2^2} Z_2} - \sqrt{1 - \frac{k_x^2}{k_1^2} Z_1}}{\sqrt{1 - \frac{k_x^2}{k_2^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_1^2} Z_1}}
\]

\[
T_{1\rightarrow 2}^{\text{TM}} = \frac{E_{2x}^+}{E_{1x}^+} = \frac{2Z_2 \sqrt{1 - \frac{k_x^2}{k_2^2}}}{\sqrt{1 - \frac{k_x^2}{k_2^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_1^2} Z_1}}
\]

Generalization of reflection and transmission to oblique incidence
Oblique Incidence – TM Case

Vanishing reflection on a boundary

\[ R_{1 \rightarrow 1}^{\text{TM}} = 0 \]

\[ \frac{k_x'}{k_1} = \frac{\sqrt{\varepsilon_2 \left( \mu_2 \varepsilon_1 - \mu_1 \varepsilon_2 \right)}}{\mu_1 \left( \varepsilon_1^2 - \varepsilon_2^2 \right)} \]

Brewster’s angle

Simplification for pure dielectrics

\[ \frac{k_x'}{k_1} = \left(1 + \frac{\varepsilon_1}{\varepsilon_2} \right)^{-\frac{1}{2}} \]

Can be used for polarizing unpolarized light beams
Oblique Incidence – TE Case

\[ R_{1 \rightarrow 1}^{\text{TE}} = \frac{E_{1y}^-}{E_{1y}^+} = \frac{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} - \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}}{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}} \]

\[ T_{1 \rightarrow 2}^{\text{TE}} = \frac{E_{2y}^+}{E_{1y}^+} = \sqrt{2Z_2} \sqrt{1 - \frac{k_x^2}{k_1^2}} \frac{1}{\sqrt{1 - \frac{k_x^2}{k_1^2} Z_2} + \sqrt{1 - \frac{k_x^2}{k_2^2} Z_1}} \]

Generalization of reflection and transmission to oblique incidence
Oblique Incidence – TE Case

\[ R_{1 \rightarrow 1}^{\text{TE}} = 0 \]

\[ \frac{k_x}{k_1} = \sqrt{\frac{\mu_2 (\varepsilon_2 \mu_1 - \varepsilon_1 \mu_2)}{\varepsilon_1 (\mu_1^2 - \mu_2^2)}} \]

Brewster's angle

Vanishing reflection on a boundary

Simplification for pure magnetics

\[ \frac{k_x}{k_1} = \left(1 + \frac{\mu_1}{\mu_2}\right)^{-\frac{1}{2}} \]

Unrealistic scenario for natural materials

\[ k_y = 0 \]
\[ E_x = 0 \]
\[ E_z = 0 \]
Oblique Incidence – Total Reflection

\[
\frac{k_2}{k_1} > \frac{n_2}{n_1} = 1
\]

\[
|R_{1\rightarrow 1}^{TM}| = |R_{1\rightarrow 1}^{TE}| = 1
\]

Valid for both, the TM and the TE case
Guided TEM Wave

Wave propagation identical to a planewave

\[ k^2 = -j\omega \mu (\sigma + j\omega \varepsilon) \]

Generalization of a planewave

\[ \hat{E}(r, \omega) = E_\perp(x, y, \omega) e^{-jkz} \]

\[ \hat{H}(r, \omega) = H_\perp(x, y, \omega) e^{-jkz} \]

Boundary condition on the conductor

\[ \Delta_\perp E_\perp = 0 \]
\[ \Delta_\perp H_\perp = 0 \]

Geometry of a planewave

\[ \hat{H} = \frac{k}{\omega \mu} (z_0 \times \hat{E}) \]
Circuit Parameters of the TEM Wave

\[ \hat{U}(z, \omega) = \hat{U}_0(\omega) e^{-jkz} \]
\[ \hat{I}(z, \omega) = \hat{I}_0(\omega) e^{-jkz} \]

Between conductors

\[ \hat{I}_0(\omega) = \int_A^B \vec{E}_\perp \cdot dl = \frac{k}{\omega \mu} \cdot \frac{Q_{\text{pul}}}{\varepsilon} \]
\[ \hat{U}_0(\omega) = -\int_A^B \vec{H}_\perp \cdot dl = \frac{\omega \mu}{k} \cdot \frac{\Phi_{\text{pul}}}{\mu} \]

Per unit length

\[ Z_{\text{TRL}} = \frac{\hat{U}_0(\omega)}{\hat{I}_0(\omega)} = \frac{\omega \mu}{k} \cdot \frac{\varepsilon}{C_{\text{pul}}} = \frac{\omega \mu}{k} \cdot \frac{L_{\text{pul}}}{\mu} = \frac{L_{\text{pul}}}{\sqrt{C_{\text{pul}}}} \]

Velocity of phase propagation

\[ v_{\text{phase}} = \frac{1}{\sqrt{\varepsilon \mu}} = \frac{1}{\sqrt{C_{\text{pul}} L_{\text{pul}}}} \]
The Telegraph Equations

\[
\frac{\partial U(z,t)}{\partial z} = -L_{\text{pul}} \frac{\partial I(z,t)}{\partial t}
\]

\[
\frac{\partial I(z,t)}{\partial z} = -C_{\text{pul}} \frac{\partial U(z,t)}{\partial t}
\]

Circuit analog of Maxwell’s equations
Guided TE and TM Waves

Wave propagation differs from a planewave

\[ k_z^2 = k^2 - k_{\perp}^2 \]

\[ \hat{E}(r, \omega) = \left[ E_{\perp}(r_{\perp}, \omega) + z_0 E_z(r_{\perp}, \omega) \right] e^{-j k_z z} \]

\[ \hat{H}(r, \omega) = \left[ H_{\perp}(r_{\perp}, \omega) + z_0 H_z(r_{\perp}, \omega) \right] e^{-j k_z z} \]

\[ E_{\perp} = -\frac{1}{k_{\perp}^2} \left( j k_z \nabla_{\perp} E_z - j \omega \mu z_0 \times \nabla_{\perp} H_z \right) \]

\[ H_{\perp} = -\frac{1}{k_{\perp}^2} \left( j k_z \nabla_{\perp} H_z + (\sigma + j \omega \varepsilon) z_0 \times \nabla_{\perp} E_z \right) \]

\[ \Delta_{\perp} E_z + k_{\perp}^2 E_z = 0 \]

\[ \Delta_{\perp} H_z + k_{\perp}^2 H_z = 0 \]

TEM mode must be completed with TE and TM modes to form a complete set
PEC Waveguides – pure TE, TM modes

Boundary condition on the conductor:

\[ \mathbf{n} \times \hat{\mathbf{E}} = 0 \]

**Impedances differ from those of a planewave**

**Modes are orthogonal in waveguide cross-section**

**Modes form a complete set in waveguide cross-section**

**TEM mode must be completed with TE and TM modes to form a complete set**
PEC Waveguides – modal orthogonality

\[ \int_S E_{\perp \alpha} \cdot E^*_{\perp \beta} dS = C \delta_{\alpha\beta} \]

cross-section of the waveguide

\[ \int_S \left( E_{\perp \alpha} \times H^*_{\perp \beta} \right) \cdot z_0 dS = \frac{1}{Z_\beta} \int_S E_{\perp \alpha} \cdot E^*_{\perp \beta} dS \]

\[ \int_S H_{\perp \alpha} \cdot H^*_{\perp \beta} dS = \frac{1}{Z_\alpha Z^*_\beta} \int_S E_{\perp \alpha} \cdot E^*_{\perp \beta} dS \]

Waveguide modes form an orthogonal set
PEC Waveguides – modal decomposition

**positive direction**

\[
\hat{E}^+ (r, \omega) = \sum_{\alpha} C^+_{\alpha} \left[ E_{\perp \alpha} (r_{\perp}, \omega) + z_0 E_{z\alpha} (r_{\perp}, \omega) \right] e^{-j k_{z\alpha} z}
\]

\[
\hat{H}^+ (r, \omega) = \sum_{\alpha} C^+_{\alpha} \left[ H_{\perp \alpha} (r_{\perp}, \omega) + z_0 H_{z\alpha} (r_{\perp}, \omega) \right] e^{-j k_{z\alpha} z}
\]

**negative direction**

\[
\hat{E}^- (r, \omega) = \sum_{\alpha} C^-_{\alpha} \left[ E_{\perp \alpha} (r_{\perp}, \omega) - z_0 E_{z\alpha} (r_{\perp}, \omega) \right] e^{j k_{z\alpha} z}
\]

\[
\hat{H}^- (r, \omega) = \sum_{\alpha} C^-_{\alpha} \left[ -H_{\perp \alpha} (r_{\perp}, \omega) + z_0 H_{z\alpha} (r_{\perp}, \omega) \right] e^{j k_{z\alpha} z}
\]

Any field within a waveguide can be composed of its modes
PEC Waveguides – Field Sources

Tangential fields within the cross-section fully define the field everywhere

\[ C_{\beta}^{\pm} = \frac{1}{2} \int_S \left[ E_{\perp,\beta}^*(r,\omega) \cdot \hat{E}(r,\omega) \pm |Z_{\beta}|^2 H_{\perp,\beta}^*(r,\omega) \cdot \hat{H}(r,\omega) \right] dS \]

\[ = e^{\pm jk_{\perp} z} \int_S E_{\perp,\beta}^*(r,\omega) \cdot E_{\perp,\beta}(r,\omega) dS \]
PEC Waveguides – Field Sources

\[ C_{\beta}^{\pm} = -\frac{Z_{\beta}}{2} \int_{V} \hat{E}_{\beta}^{\mp}(r, \omega) \cdot \hat{J}_s(r, \omega) \, dV \]

\[ = \frac{1}{2} \int_{S} \frac{E_{\perp \beta}(r_{\perp}, \omega)}{E_{\perp \beta}(r_{\perp}, \omega)} \cdot E_{\perp \beta}(r_{\perp}, \omega) \, dS \]

Valid to the right (+) or to the left (-) of the source region

Full field of the waveguide mode

Source current density existing within the waveguide

transversal field of the waveguide mode

This is how waveguide modes are excited
### Dielectric Waveguides – mixed TE + TM modes

#### Boundary condition on the conductor

\[
\mathbf{n} \times [\mathbf{\hat{E}}_1 - \mathbf{\hat{E}}_2] = 0
\]

\[
\mathbf{n} \times [\mathbf{\hat{H}}_1 - \mathbf{\hat{H}}_2] = 0
\]

- **Finite** number of guided modes
- **Continuum** of radiating modes
- **Only combination** of guided and radiating modes forms a complete set in the waveguide cross-section

*General field is not guided by a dielectric waveguide*