Electroencephalography and Magnetoencephalography

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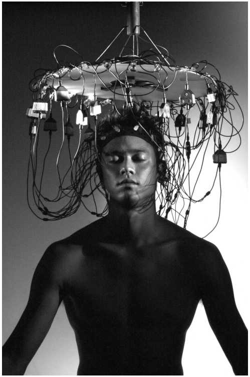
Overview

- EEG
- MEG
- Physiological background
- Localization. Direct and inverse problems
- Differential methods (FDM, FEM)
- Integral methods (3 flavours BEM)
- Fast Multipole Method
- Inverse Problem

Some images courtesy of CTF, INRIA, Arye Nehorai, and others.

Electroencephalography (EEG)



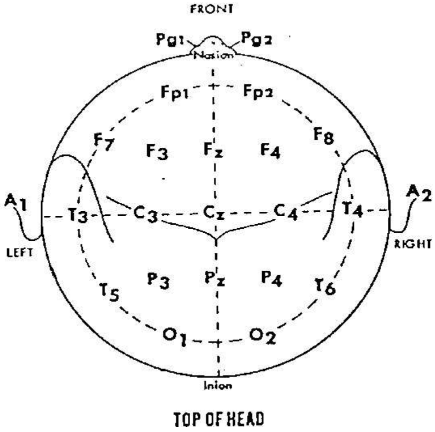
- Hans Berger, 1920s
- $16 \sim 64$ electrodes
- $100\,\mu$V on the scalp ($1 \sim 2\,$mV on the cortex)

Reference

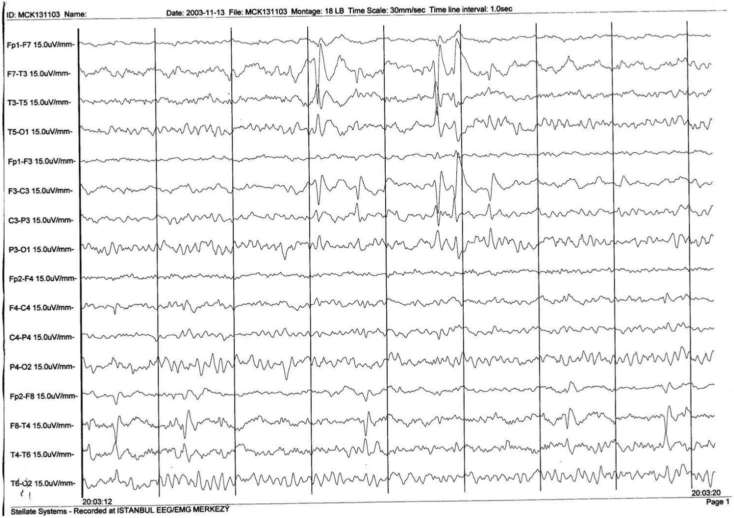
- Common reference (Wilson terminal)
- Averaged reference
- Bipolar derivation (differences)

Reference can be changed a posteriori.

Standard electrode layout



EEG output


ID: MCK131103 Name:
Date: 2003-11-13 File: MCK131103 Montage: 18 LB Time Scale: 30mm/sec Time line interval: 1.0sec
Fp1-F7 15.0uV/mm-
F7-T3 15.0uV/mm-
T3-T5 15.0uV/mm-
T5-O1 15.0uV/mm-
Fp1-F3 15.0uV/mm-
F3-C3 15.0uV/mm-
C3-P3 15.0uV/mm-
P3-O1 15.0uV/mm-
Fp2-F4 15.0uV/mm-
F4-C4 15.0uV/mm-
C4-P4 15.0uV/mm-
P4-O2 15.0uV/mm-
Fp2-F8 15.0uV/mm-
F8-T4 15.0uV/mm-
T4-T6 15.0uV/mm-
T6-O2 15.0uV/mm-
20:03:12
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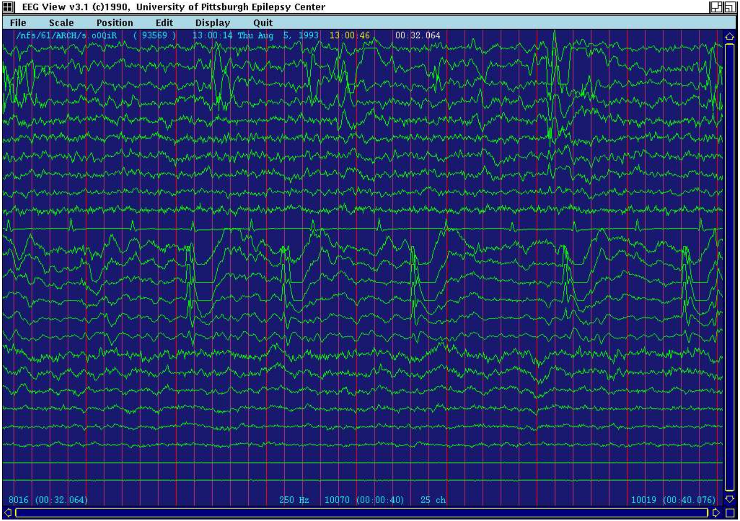
Page 1
Stellate Systems - Recorded at ISTANBUL EEG/EMG MERKEZÝ

EEG output


EEG View v3.1 (c)1990, University of Pittsburgh Epilepsy Center
File Scale Position Edit Display Quit
/nfs/61/ARCH/s.o0QiR (93569) 13:00:14 Thu Aug 5, 1993 13:00:46 00:32.064
8016 (00:32.064) 250 Hz 10070 (00:00:40) 25 ch 10019 (00:40.076)

EEG waves

Alpha $8 \sim 12\,\text{Hz}$ — relaxed, alert consciousness state, eyes closed. From 2 years of age. (Berger's wave)

Beta $> 12\,\text{Hz}$ — active, busy, anxious thinking, concentration. Dominant frequencies — pathologies, drug effects.

Gamma $26 \sim 80\,\text{Hz}$ — higher mental activity, perception, problem solving, fear, consciousness.

Delta $< 4\,\text{Hz}$ — young children, encephalopathies, lesions. Deep sleep.

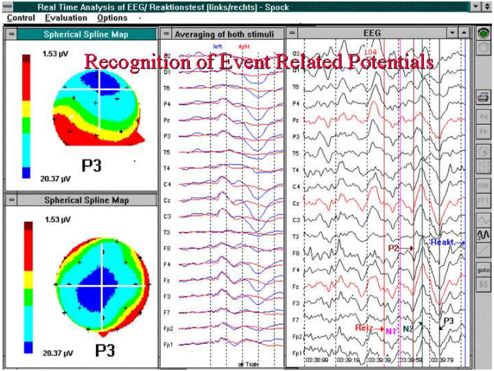
Theta $4 \sim 8\,\text{Hz}$ — drowsiness, childhood$\sim$young adulthood, hyperventilation, hypnosis, lucid dreams, light sleep

SMR (sensorimotor rhytm) $12 \sim 16\,\text{Hz}$ — physical stillness, body presence.

EEG analysis, diagnostics

- Very high time resolution
- No or crude localization
- Awake/sleep, sleep phases
- Epilepsy
- Pathologies, lesions
- Effects of opiates, anesthetics

Event related potential

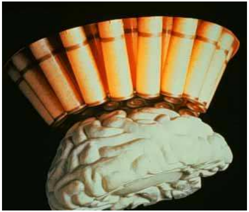


- Repetitive action
- Averaging out noise
- Visual/acoustic/somatosensory evoked potential (electric stimulation, median nerve)

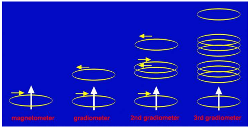
Magnetoencephalography

- Measuring (very weak) magnetic fields ($\sim 10^{-14}\,\mathrm{T} = 100\,\mathrm{fT}$)
- Shielded room, superconductive sensors

MEG sensors



SQUID array



Gradiometers

- Superconducting Quantum Interference Device
- Josephson junction
- flux transformer feedback loop

MEG & EEG measurements

EEG: Voltage

$$v_i = \langle V, \mu_i^{\mathrm{e}} \rangle \approx V(\mathbf{x}_i)$$

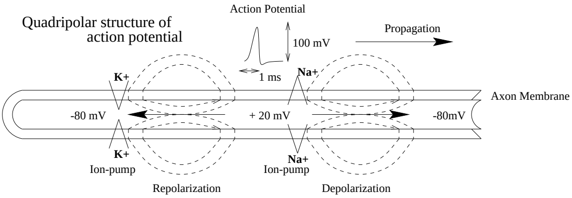
MEG: Projection of the magnetic field intensity

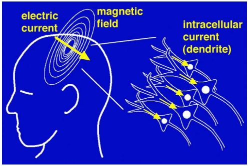
$$b_j = \langle \mathbf{B}, \boldsymbol{\mu}_j^{\mathrm{m}} \rangle \approx \left(\mathbf{B} \cdot \mathbf{n}_j\right)(\mathbf{x}_i)$$

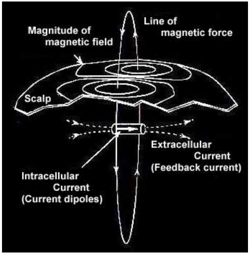
Gradiometers (real and virtual):

$$\mathbf{b} = \mathrm{W}\, \mathbf{b}_{\mathrm{raw}}$$

Currents in the brain



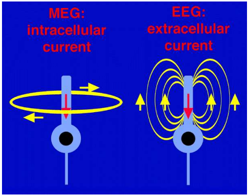


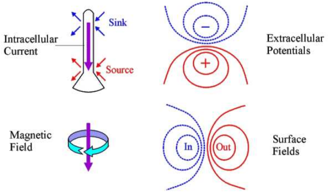


Primary currents $\mathbf{J}^{\mathsf{p}}[\mathrm{A/m^2}]$

MEG versus EEG

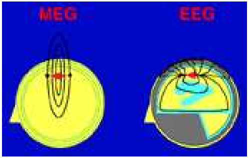
Complementarity



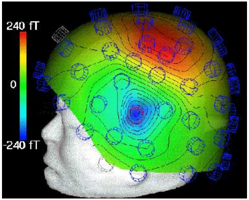


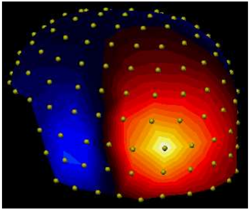
EEG Distortion

EEG much cheaper



EEG/MEG topography





→ Better reconstruction needed.

Task – Inverse problem

From measurements $\mathbf{m} = \left[v_1, \ldots, v_{N_\mathrm{e}}, b_1, \ldots, b_{N_\mathrm{m}}\right]$
estimate primary currents $\mathbf{J}^\mathrm{p}$.

→ First we need to solve the forward problem.

Forward problem

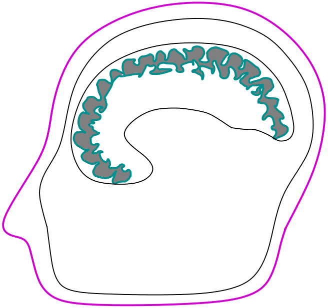
Find V and $\mathbf{B}$ from $\mathbf{J}^{\mathrm{p}}$

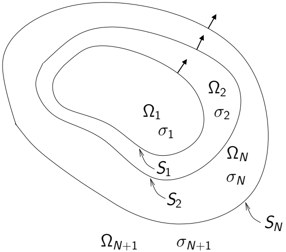
- Head modeling.
- Solving field equations.

Necessary prerequisite for solving the inverse problem.

Classical head model

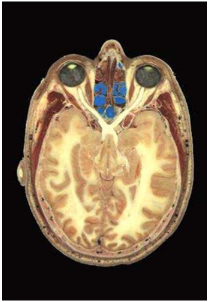
Nested surfaces:

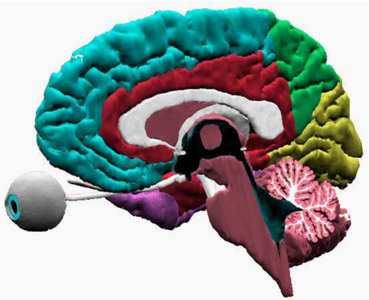




Nested volumes Ω_i with conductivities σ_i, separated by surfaces S_j.

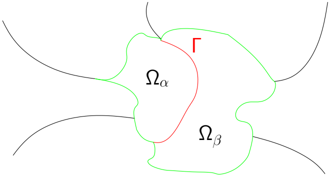
Real head





Images from the Visible Human and Digital Anatomist projects.

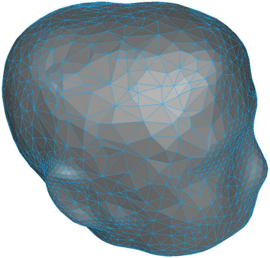
Generalized topology head model

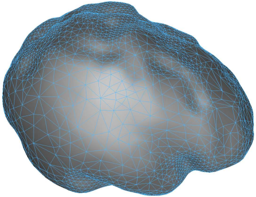


- $N + 1$ disjoint open sets $\Omega_1, \ldots, \Omega_{N+1}$ partitioning $\mathbb{R}^3$.
- Boundaries $S_{\alpha\beta} = \partial\Omega_\alpha \cap \partial\Omega_\beta$ are empty or decomposable into a finite number of connected regular surfaces.
- $\rightarrow$ Boundaries $S_{\alpha\beta}$ are regular (a.e.).

High-resolution realistic model

Head meshes:





with about 13000 points and 26000 faces
$\rightarrow$ 34000 unknowns.

Field equations

Maxwell equations (in void):

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$\varepsilon_0 \approx 8.85 \cdot 10^{-12}\,[\mathrm{Fm}^{-1}]$, $\mu_0 = 4\pi \cdot 10^{-7}\,[\mathrm{Hm}^{-1}]$, $\varepsilon_0 \mu_0 c^2 = 1$

Quasistatic approximation:

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = 0 \qquad \rightarrow \qquad \mathbf{E} = -\nabla V$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \rightarrow \qquad \nabla \cdot \mathbf{J} = 0$$

Field equations

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Conductive medium: (conductivity $\sigma\,[\Omega^{-1}\mathrm{m}^{-1}]$)

$$\mathbf{J} = \mathbf{J}^{\mathsf{p}} + \sigma \mathbf{E} = \mathbf{J}^{\mathsf{p}} - \sigma \nabla V$$

$$\boxed{\nabla \cdot \left(\sigma \nabla V\right) = \nabla \cdot \mathbf{J}^{\mathsf{p}}}$$

Field equations

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$$\boxed{\nabla \cdot \left(\sigma \nabla V \right) = \nabla \cdot \mathbf{J}^{\mathsf{p}}}$$

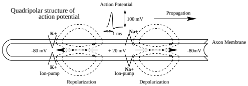
Magnetic field:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{\Omega} \mathbf{J}(\mathbf{r}') \times \nabla_{\mathbf{r}'} \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \, \mathrm{d}\mathbf{r}'$$

Current sources

- Current "dipole" $\mathbf{q}\,[\mathrm{Am}]$:

$$\mathbf{q} = \mathbf{J}\,\Delta l\,\Delta S$$



$$\mathbf{J}_{\mathrm{dip}}(\mathbf{x}) = \mathbf{q}\,\delta_{\mathbf{p}} = \mathbf{q}\,\delta(\mathbf{x} - \mathbf{p})$$

Dipole field

$$V_{\mathrm{dip}}(\mathbf{x}) = \frac{1}{4\pi\sigma}\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{x})}{\|\mathbf{p}-\mathbf{x}\|^3}$$

Surface versus volume methods

Solve $\boxed{\nabla\left(\sigma\nabla V\right) = f = \nabla \mathbf{J}^{\mathrm{p}}}$ for V.

- Volume methods
  - unknowns ($V(\mathbf{x})$) in the volume $\mathbf{x} \in \Omega \subset \mathbb{R}^3$
  - differential equations
- Surface methods
  - conductivity piecewise constant
  - unknowns ($V(\mathbf{x})$) on a set of surfaces $\mathbf{x} \in S_i(\mathbb{R}^2)$
  - integral equations

Finite difference method

$$\boxed{\nabla\big(\sigma\nabla V\big) = f}$$

- Regular grid:

$$v_{ij} = V(hi, hj)$$

- Finite difference approximation

$$\text{for example} \quad \Delta V \approx \frac{1}{4h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} * \ v$$

- Linear system of equations: $A\mathbf{v} = \mathbf{b}$
- Where $\sigma = \text{const}$ $\rightarrow$ stationarity $\rightarrow$ FFT solution

Finite element method

$$\boxed{\nabla\big(\sigma\nabla V\big) = f}$$

- Test functions ϕ_j

$$\big\langle\nabla\big(\sigma\nabla V\big), \phi_j\big\rangle = \big\langle f, \phi_j\big\rangle$$

- Discretize $V = \sum_i v_i\phi_i$

$$\sum_i v_i\big\langle\nabla(\sigma\nabla\phi_i), \phi_j\big\rangle = \big\langle f, \phi_j\big\rangle$$

$$-\sum_i v_i\,\sigma_i\big\langle\nabla\phi_i, \nabla\phi_j\big\rangle = \big\langle f, \phi_j\big\rangle$$

- Linear **symmetric** system $A\mathbf{v} = \mathbf{b}$

Surface metehod / Boundary element method

- Mathematical basis
- Representation theorem
- Integral representations
  - Single layer formulation
  - Double layer formulation
  - Symmetric formulation

The Green connection

Suppose constant σ: $\boxed{\Delta u = f}$

- Green function (Nedelec's convention)

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\|\mathbf{r}-\mathbf{r}'\|} \qquad -\Delta_{\mathbf{r}} G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') = \delta_{\mathbf{r}'}$$

- Stokes theorem

$$\int_{\partial\Omega} \mathbf{g}(\mathbf{r}) \cdot \mathrm{d}\mathbf{s}(\mathbf{r}) = \int_{\Omega} \nabla \cdot \mathbf{g}(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

The Green connection

- Stokes theorem

$$\int\limits_{\partial\Omega} \mathbf{g}(\mathbf{r}) \cdot \mathrm{d}\mathbf{s}(\mathbf{r}) = \int\limits_{\Omega} \nabla \cdot \mathbf{g}(\mathbf{r})\,\mathrm{d}\mathbf{r}$$

- First Green identity

$$\mathbf{g} = u\nabla v \quad \rightarrow \quad \int\limits_{\partial\Omega} u\nabla v\,\mathrm{d}\mathbf{s}(\mathbf{r}) = \int\limits_{\Omega} \nabla u\nabla v + u\Delta v\,\mathrm{d}\mathbf{r}$$

The Green connection

- First Green identity

$$\mathbf{g} = u\nabla v \quad \rightarrow \quad \int\limits_{\partial\Omega} u\nabla v \, \mathrm{d}\mathbf{s}(\mathbf{r}) = \int\limits_{\Omega} \nabla u \nabla v + u\Delta v \, \mathrm{d}\mathbf{r}$$

- Second Green identity

$$\int\limits_{\partial\Omega} u\nabla v - v\nabla u \, \mathrm{d}\mathbf{s}(\mathbf{r}) = \int\limits_{\Omega} u\Delta v - v\Delta u \, \mathrm{d}\mathbf{r}$$

$$\int\limits_{\partial\Omega} u \, \partial_{\mathbf{n}'} v - v \, \partial_{\mathbf{n}'} u \, \mathrm{d}s(\mathbf{r}) = \int\limits_{\Omega} u\Delta v - v\Delta u \, \mathrm{d}\mathbf{r}$$

The Green connection

- Second Green identity

$$\int_{\partial\Omega} u\,\partial_{\mathbf{n}'} v - v\,\partial_{\mathbf{n}'} u \,\mathrm{d}s(\mathbf{r}) = \int_{\Omega} u\Delta v - v\Delta u \,\mathrm{d}\mathbf{r}$$

- Third Green identity

$$v = -G(\mathbf{r},\mathbf{r}'), \quad \Delta u = 0 \quad \rightarrow$$

$$\nu u(\mathbf{r}) = \int_{\partial\Omega} G(\mathbf{r},\mathbf{r}')\,\partial_{\mathbf{n}'} u(\mathbf{r}') - u\,\partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}') \,\mathrm{d}s(\mathbf{r}')$$

$$\nu = \begin{cases} 1, & \mathbf{r} \in \Omega \\ 1/2, & \mathbf{r} \in \partial\Omega \\ 0, & \mathbf{r} \in \mathbb{R}^3 \backslash \overline{\Omega} \end{cases}$$

Representation theorem

Third Green identity:

$$\int_{\partial\Omega} G(\mathbf{r},\mathbf{r}')\,\partial_{\mathbf{n}'} u(\mathbf{r}') - u(\mathbf{r}')\,\partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\,\mathrm{d}s(\mathbf{r}') = \begin{cases} u(\mathbf{r}), & \mathbf{r}\in\Omega \\ u(\mathbf{r})/2, & \mathbf{r}\in\partial\Omega \\ 0, & \mathbf{r}\in\mathbb{R}^3\backslash\overline{\Omega} \end{cases}$$

$$\Delta u^{\text{int}} = 0 \quad \text{in } \Omega, \qquad \Delta u^{\text{ext}} = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \qquad u^{\text{ext}} \overset{\|\mathbf{r}\|\to\infty}{\longrightarrow} O(\|\mathbf{r}\|^{-1})$$

$$\boxed{\int_{\partial\Omega} G(\mathbf{r},\mathbf{r}')\,\big[\partial_{\mathbf{n}'} u\big](\mathbf{r}') - \partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\big[u\big](\mathbf{r}')\,\mathrm{d}s(\mathbf{r}') = \begin{cases} u^{\text{int}}(\mathbf{r}), & \mathbf{r}\in\Omega \\ u^{\text{ext}}(\mathbf{r}), & \mathbf{r}\in\mathbb{R}^3\backslash\overline{\Omega} \\ \frac{u^{\text{int}}+u^{\text{ext}}}{2}(\mathbf{r}), & \mathbf{r}\in\partial\Omega \end{cases}}$$

where $\quad [u] = u^{\text{int}}(\mathbf{r}') - u^{\text{ext}}(\mathbf{r}'), \qquad [\partial_{\mathbf{n}'} u] = \partial^{-}_{\mathbf{n}'} u^{\text{int}}(\mathbf{r}') - \partial^{+}_{\mathbf{n}'} u^{\text{ext}}(\mathbf{r}')$

Extended representation theorem

- Regular case, $\mathbf{r} \in \mathbb{R}^3 \backslash \partial\Omega$.

$$\mathbf{p} = \sigma \nabla V, \quad \nabla \cdot \mathbf{p} = 0, \quad \text{in } \mathbb{R}^3 \backslash \partial\Omega$$

$$\boxed{\begin{aligned} -p(\mathbf{r}) &= \int_{\partial\Omega} \sigma \partial_{\mathbf{n},\mathbf{n}'} G(\mathbf{r},\mathbf{r}') \big[V\big](\mathbf{r}') - \partial_{\mathbf{n}} G(\mathbf{r},\mathbf{r}') \big[p\big](\mathbf{r}') \, \mathrm{d}s(\mathbf{r}') \\ V(\mathbf{r}) &= \int_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}') \big[V\big](\mathbf{r}') + \sigma^{-1} G(\mathbf{r},\mathbf{r}') \big[p\big](\mathbf{r}') \, \mathrm{d}s(\mathbf{r}') \end{aligned}}$$

where $p = \mathbf{p} \cdot \mathbf{n}, \quad \big[V\big] = V^- - V^+, \quad \big[p\big] = p^- - p^+$

Extended representation theorem

- Limit case, $\mathbf{r} \in \partial\Omega$.

$$\mathbf{p} = \sigma \nabla V, \quad \nabla \cdot \mathbf{p} = 0, \quad \text{in } \mathbb{R}^3 \backslash \partial\Omega$$

$$\boxed{\begin{aligned} -p^{\pm}(\mathbf{r}) &= \pm \frac{[p]}{2} + \int_{\partial\Omega} \sigma \partial_{\mathbf{n},\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\big[V\big] - \partial_{\mathbf{n}} G(\mathbf{r},\mathbf{r}')\big[p\big] \,\mathrm{d}s(\mathbf{r}') \\ V^{\pm}(\mathbf{r}) &= \mp \frac{[u]}{2} + \int_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\big[V\big] + \sigma^{-1} G(\mathbf{r},\mathbf{r}')\big[p\big] \,\mathrm{d}s(\mathbf{r}') \end{aligned}}$$

where $\quad p = \mathbf{p} \cdot \mathbf{n}, \quad \big[V\big] = V^{-} - V^{+}, \quad \big[p\big] = p^{-} - p^{+}$

Extended representation theorem

- Operator form, $\quad \mathbf{r} \in \partial\Omega.$

$$-p^{\pm}(\mathbf{r}) = \pm\frac{[p]}{2} + \int\limits_{\partial\Omega} \sigma\partial_{\mathbf{n},\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\big[V\big] - \partial_{\mathbf{n}} G(\mathbf{r},\mathbf{r}')\big[p\big]\,\mathrm{d}s(\mathbf{r}')$$

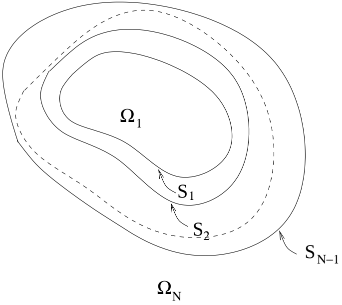
$$V^{\pm}(\mathbf{r}) = \mp\frac{[u]}{2} + \int\limits_{\partial\Omega} -\partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}')\big[V\big] + \sigma^{-1} G(\mathbf{r},\mathbf{r}')\big[p\big]\,\mathrm{d}s(\mathbf{r}')$$

$$\boxed{\begin{aligned} -p^{\pm}(\mathbf{r}) &= \sigma\mathcal{N}\big[V\big] + \left(\pm\tfrac{\mathcal{I}}{2} - \mathcal{D}^*\right)\big[p\big] \\ V^{\pm}(\mathbf{r}) &= \left(\mp\tfrac{\mathcal{I}}{2} - \mathcal{D}\right)\big[V\big] + \sigma^{-1}\mathcal{S}\big[p\big] \end{aligned}}$$

where $\quad p = \sigma\partial_{\mathbf{n}} V$

BEM problem

Solve: $$\Delta V = f \quad \text{in } \cup\,\Omega_i$$
$$\left[\sigma\partial_{\mathbf{n}} V\right] = \left[V\right] = 0 \quad \text{on } S_i$$



$$f = \sum_{i=1}^{N} f_{\Omega_i}$$
$$v_{\Omega_i}(\mathbf{r}) = -f_{\Omega_i} * G\,(\mathbf{r})$$

BEM — Single layer formulation

$$v_s = \sum_{i=1}^{N} v_{\Omega_i}/\sigma_i \quad \text{verifies} \quad \sigma\Delta v_s = f$$

Consider: $u_s = V - v_s \quad \rightarrow \quad [u_s]_j = 0$

$$\rightarrow \quad u_s = \sum_{i=1}^{N} \mathcal{S}_{ji}\xi_{S_i} \qquad \xi_{S_i} = [p]_i$$

From: $[\sigma\partial_{\mathbf{n}} V] = 0 \quad \rightarrow \quad [\sigma\partial_{\mathbf{n}} u_s] = -[\sigma\partial_{\mathbf{n}} v_s]$

From the representation theorem:

$$\boxed{\partial_{\mathbf{n}} v_s = \frac{\sigma_j + \sigma_{j+1}}{2(\sigma_{j+1} - \sigma_j)}\xi_{S_j} - \sum_{i=1}^{N} \mathcal{D}^*_{ji}\xi_{S_i}}$$

BEM — Double layer formulation

$$v_d = \sum_{i=1}^N v_{\Omega_i} \quad \text{verifies} \quad \Delta v_d = f$$

Consider: $u_d = \sigma V - v_d \quad \rightarrow \quad [\partial_{\mathbf{n}} u_d]_j = 0$

$$\rightarrow \quad u_d = \sum_{i=1}^N \mathcal{D}_{ji} \mu_{S_i}$$

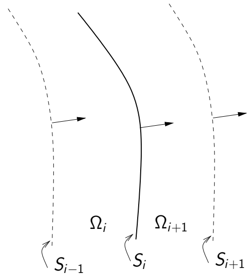
$$\mu_{S_i} = -[u_d]_i = (\sigma_{i+1} - \sigma_i) V_{S_i}$$

From: $[V] = 0 \quad \rightarrow \quad \sigma_{j+1}(u_d + v_d)^- = \sigma_j (u_d + v_d)^+$

From the representation theorem:

$$\boxed{v_d = \frac{\sigma_j + \sigma_{j+1}}{2} V_{S_j} - \sum_{i=1}^N (\sigma_{i+1} - \sigma_i) \mathcal{D}_{ji} V_{S_i}}$$

BEM – Symmetric formulation

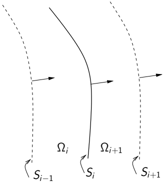


Consider: $$u_{\Omega_i} = \begin{cases} V - v_{\Omega_i}/\sigma_i & \text{in } \Omega_i \\ -v_{\Omega_i}/\sigma_i & \text{in } \mathbb{R}^3 \backslash \overline{\Omega}_i \end{cases}$$

$$[u_{\Omega_i}]_i = V_{S_i}, \quad [u_{\Omega_i}]_{i-1} = -V_{S_{i-1}}$$

Define: $$p_{S_i} = \sigma_i [\partial_{\mathbf{n}} u_{\Omega_i}]_i = \sigma_i (\partial_{\mathbf{n}} V)^-_{S_i} = \sigma_{i+1} (\partial_{\mathbf{n}} V)^+_{S_i}$$

BEM – Symmetric formulation



From the extended representation theorem:

$$
\begin{aligned}
\left(u_{\Omega_i}\right)^-_{S_i} &= \left(V - v_{\Omega_i}/\sigma_i\right)^-_{S_i} \\
&= \frac{V_{S_i}}{2} + \mathcal{D}_{i,i-1} V_{S_{i-1}} - \mathcal{D}_{ii} V_{S_i} - \sigma_i^{-1}\mathcal{S}_{i,i-1} p_{S_{i-1}} + \sigma_i^{-1}\mathcal{S}_{ii} p_{S_i} \\
\left(u_{\Omega_{i+1}}\right)^+_{S_i} &= \left(V - v_{\Omega_{i+1}}/\sigma_{i+1}\right)^+_{S_i} \\
&= \frac{V_{S_i}}{2} + \mathcal{D}_{ii} V_{S_i} - \mathcal{D}_{i,i+1} V_{S_{i+1}} - \sigma_{i+1}^{-1}\mathcal{S}_{ii} p_{S_i} + \sigma_{i+1}^{-1}\mathcal{S}_{i,i+1} p_{S_{i+1}}
\end{aligned}
$$

BEM – Symmetric formulation

Subtract:

$$\begin{aligned}\sigma_{i+1}^{-1}(v_{\Omega_{i+1}})_{S_i} - \sigma_i^{-1}(v_{\Omega_i})_{S_i} = \\ \mathcal{D}_{i,i-1} V_{S_{i-1}} - 2\mathcal{D}_{ii} V_{S_i} + \mathcal{D}_{i,i+1} V_{S_{i+1}} \\ - {\sigma_i}^{-1}\mathcal{S}_{i,i-1} p_{S_{i-1}} + (\sigma_i^{-1} + \sigma_{i+1}^{-1})\mathcal{S}_{ii} p_{S_i} - \sigma_{i+1}^{-1}\mathcal{S}_{i,i+1} p_{S_{i+1}}\end{aligned}$$

and also:

$$\begin{aligned}(\partial_{\mathbf{n}} v_{\Omega_{i+1}})_{S_i} - (\partial_{\mathbf{n}} v_{\Omega_i})_{S_i} = \\ \sigma_i \mathcal{N}_{i,i-1} V_{S_{i-1}} - (\sigma_i + \sigma_{i+1})\mathcal{N}_{ii} V_{S_i} + \sigma_{i+1}\mathcal{N}_{i,i+1} V_{S_{i+1}} \\ - \mathcal{D}^*_{i,i-1} p_{S_{i-1}} + 2\mathcal{D}^*_{ii} p_{S_i} - \mathcal{D}^*_{i,i+1} p_{S_{i+1}}\end{aligned}$$

BEM – Symmetric formulation

$$
\begin{bmatrix}
(\sigma_1+\sigma_2)\mathcal{N}_{11} & -2\mathcal{D}^*_{11} & -\sigma_2\mathcal{N}_{12} & \mathcal{D}^*_{12} & & & & & & \\
-2\mathcal{D}_{11} & (\sigma_1^{-1}+\sigma_2^{-1})\mathcal{S}_{11} & \mathcal{D}_{12} & -\sigma_2^{-1}\mathcal{S}_{12} & & & & & & \\
-\sigma_2\mathcal{N}_{21} & \mathcal{D}^*_{21} & (\sigma_2+\sigma_3)\mathcal{N}_{22} & -2\mathcal{D}^*_{22} & -\sigma_3\mathcal{N}_{23} & \mathcal{D}^*_{23} & & & & \\
\mathcal{D}_{21} & -\sigma_2^{-1}\mathcal{S}_{21} & -2\mathcal{D}_{22} & (\sigma_2^{-1}+\sigma_3^{-1})\mathcal{S}_{22} & \mathcal{D}_{23} & -\sigma_3^{-1}\mathcal{S}_{23} & & & & \\
 & & -\sigma_3\mathcal{N}_{32} & \mathcal{D}^*_{32} & (\sigma_3+\sigma_4)\mathcal{N}_{33} & -2\mathcal{D}^*_{33} & \dots & & & \\
 & & \mathcal{D}_{32} & -\sigma_3^{-1}\mathcal{S}_{32} & -2\mathcal{D}_{33} & (\sigma_3^{-1}+\sigma_4^{-1})\mathcal{S}_{33} & \dots & & & \\
 & & & & \vdots & \vdots & \ddots & & & \\
 & & & & & & & (\sigma_{N-1}+\sigma_N)\mathcal{N}_{N-1,N-1} & -2\mathcal{D}^*_{N-1,N-1} & -\sigma_N\mathcal{N}_{N-1,N} \\
 & & & & & & & -2\mathcal{D}_{N-1,N-1} & (\sigma_{N-1}^{-1}+\sigma_N^{-1})\mathcal{S}_{N-1,N-1} & \mathcal{D}_{N-1,N} \\
 & & & & & & & -\sigma_N\mathcal{N}_{N,N-1} & \mathcal{D}^*_{N,N-1} & \sigma_N\mathcal{N}_{N,N}
\end{bmatrix}
$$

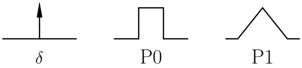
$$
\cdot \begin{bmatrix} V_{S_1} \\ p_{S_1} \\ V_{S_2} \\ p_{S_2} \\ V_{S_3} \\ p_{S_3} \\ \vdots \\ V_{S_N} \end{bmatrix}
=
\begin{bmatrix}
(\partial_{\mathbf{n}} v_{\Omega_1})_{S_1} - (\partial_{\mathbf{n}} v_{\Omega_2})_{S_1} \\
\sigma_2^{-1}(v_{\Omega_2})_{S_1} - \sigma_1^{-1}(v_{\Omega_1})_{S_1} \\
(\partial_{\mathbf{n}} v_{\Omega_2})_{S_2} - (\partial_{\mathbf{n}} v_{\Omega_3})_{S_2} \\
\sigma_3^{-1}(v_{\Omega_3})_{S_2} - \sigma_2^{-1}(v_{\Omega_2})_{S_2} \\
(\partial_{\mathbf{n}} v_{\Omega_3})_{S_3} - (\partial_{\mathbf{n}} v_{\Omega_4})_{S_3} \\
\sigma_4^{-1}(v_{\Omega_4})_{S_3} - \sigma_3^{-1}(v_{\Omega_3})_{S_3} \\
\vdots \\
(\partial_{\mathbf{n}} v_{\Omega_N})_{S_N}
\end{bmatrix}
$$

Discretization in BEM

- Discretize unknowns ($\varphi_i =$ P0,P1)

$$\xi(\mathbf{r}) = \sum_i \xi_i \varphi_i(\mathbf{r})$$

- Test functions ($\psi_j = \delta$,P0,P1)



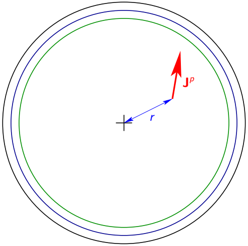
$$\left\langle \frac{\xi}{2}(\mathbf{r}) - \int\limits_{\partial\Omega} \xi(\mathbf{r}')\partial_{\mathbf{n}} G(\mathbf{r},\mathbf{r}')\,\mathrm{d}s(\mathbf{r}'), \psi_j \right\rangle = \langle \partial_{\mathbf{n}} V_0, \psi_j \rangle$$

$$\sum_i \xi_i \left(\tfrac{1}{2}\langle \phi_i, \psi_j \rangle - \int\limits_{\partial\Omega\times\partial\Omega} \varphi_i(\mathbf{r}')\partial_{\mathbf{n}} G(\mathbf{r},\mathbf{r}')\psi_j(\mathbf{r})\,\mathrm{d}s^2(\mathbf{r}',\mathbf{r}') \right) = \langle \partial_{\mathbf{n}} V_0, \psi_j \rangle$$

- Linear system of equations: $\mathsf{A}\boldsymbol{\xi} = \mathbf{b}$

BEM accuracy

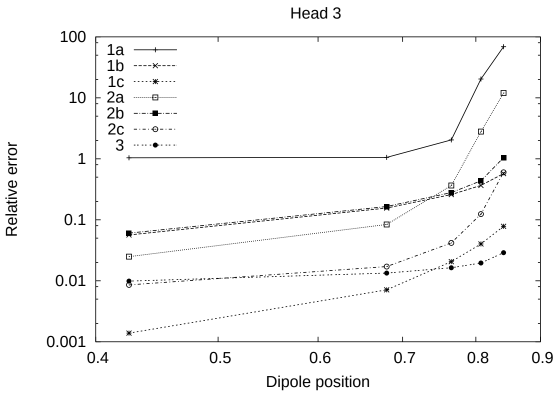
- Analytical solution



- Relative error $\quad \left\| V - V_{\mathsf{anal}} \right\|_{\ell_2} / \left\| V_{\mathsf{anal}} \right\|_{\ell_2}$
- Dipoles at $\quad 0.50R,\ 0.80R,\ 0.90R,\ 0.95R,\ 0.98R$
- Spherical head phantoms with

$$N_{\mathsf{V}} = 3 \times 42, 3 \times 162, 3 \times 642$$

BEM accuracy



Fast Multipole Method – Motivation

$$\boxed{\mathbf{\Gamma x} = \mathbf{y}}$$

- Solution methods
  - **Direct**, e.g. LU decomposition.
    Complexity $O(P^3)$, memory $O(P^2)$
  - **Iterative**, e.g. GMRES, uses products $\mathbf{\Gamma v}$.
    Complexity $O(MP^2)$, memory $O(P)$

  Number of elements P, number of iterations M.
- Fast Multipole Method
  - Calculate $\mathbf{y} = \mathbf{\Gamma v}$ in $O(P \log P)$ time.
  - Approximative, hierarchical

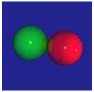
Multipole expansion

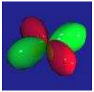
Typical element:

$$\boldsymbol{\Gamma}_{i,j} = \iint\limits_{\substack{\mathbf{r} \in \operatorname{supp} \psi_i \\ \mathbf{r}' \in \operatorname{supp} \varphi_j}} \nabla' \frac{1}{\|\mathbf{r}-\mathbf{r}'\|} \cdot \mathbf{n}_j \, \varphi_j(\mathbf{r}') \psi_i(\mathbf{r}) \mathrm{ds}^2(\mathbf{r}', \mathbf{r})$$

$$\boxed{\nabla' \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = -\sum_{\substack{n=0\ldots\infty \\ m=-n\ldots n}} \nabla' I_n^{-m}(\mathbf{M}_p - \mathbf{r}')\, O_n^m(\mathbf{r} - \mathbf{M}_p)}$$

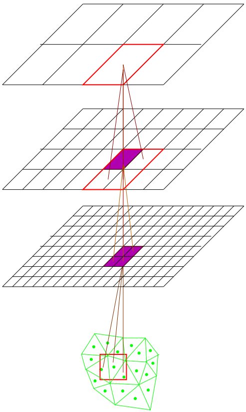
Spherical harmonics I_n^{-m}, O_n^m:





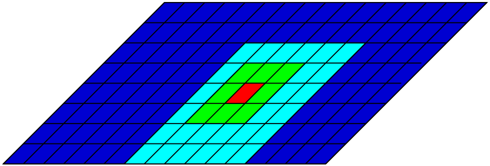
FMM – Algorithm

Build an oct-tree:



FMM – Algorithm

Levels involved:



Level 2

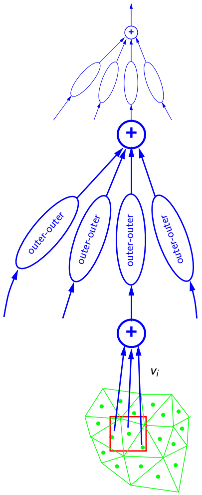
Level 1 – suburb

Treated locally

Cell considered

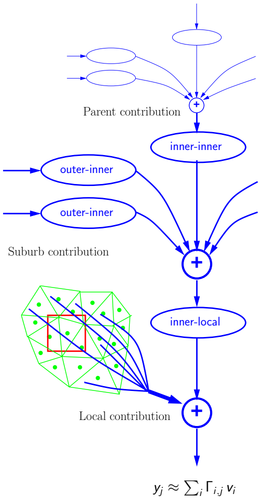
FMM – Algorithm

Sweep-up $\rightarrow$ outer fields

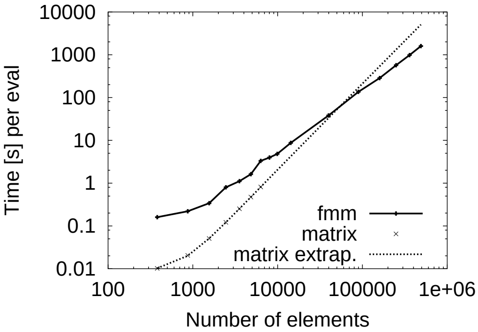


FMM – Algorithm

Sweep-down $\longrightarrow$ result



FMM – Speed-Up



Single-level FMM, $O(P^{4/3})$, faster for $P \gtrapprox 70000$ triangles.

MEG/EEG, Conclusions

- MEG, EEG, brain function analysis
- Excellent time-resolution, bad spatial resolution
- Standard diagnostic use
- Combination with other methods (fMRI, PET) desirable
- Localization is a hard inverse problem
- Solution methods — FDM, FEM, BEM methods
- BEM formulations, single/double/symmetric
- Implementation, discretization
- Fast Multipole Method for acceleration