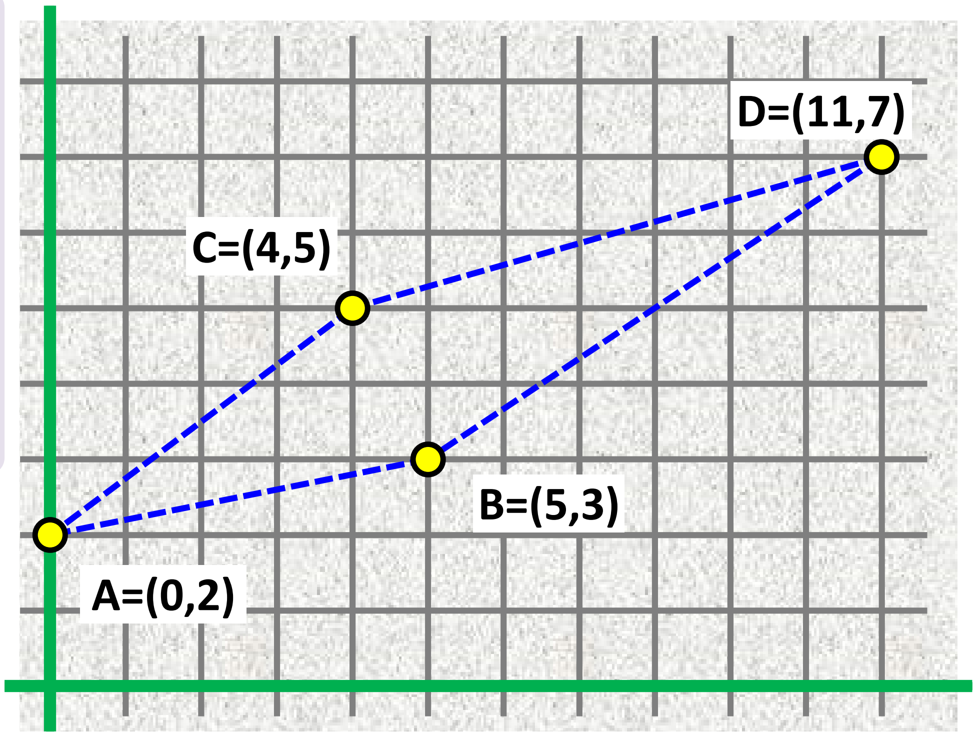
Distance (A, B) =

$= \mathrm{sqrt}((Ax–Bx)^2 + (Ay–By)^2)$

= distance (B, A)



Comparing distances

Compare squares of distances
(faster, integer coordinates → no floats!)

dist(A, B) < dist(B, C) ⇔ dist(A, C)^2 < dist(B, C)^2

$dist(A, B)^2 = (0–5)^2 + (2–3)^2 = 26$

$dist(A, C)^2 = (0–4)^2 + (2–5)^2 = 25$

$dist(C, D)^2 = (4–11)^2 + (5–7)^2 = 53$

$dist(B, D)^2 = (5–11)^2 + (3–7)^2 = 52$

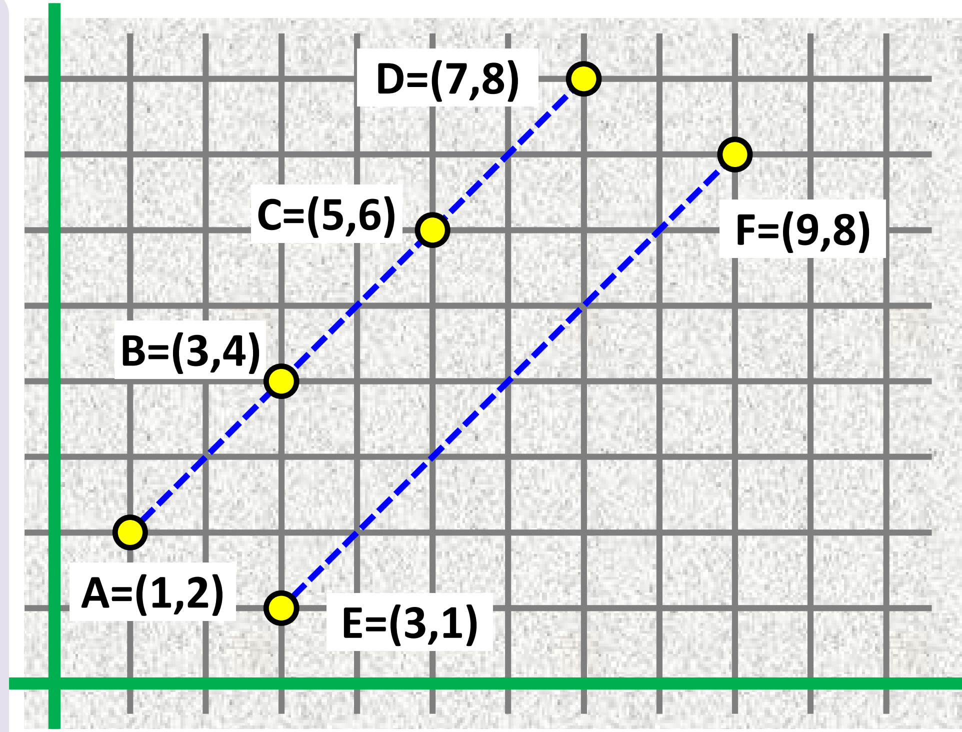
Comparing distances example

dist(A, D) = dist(A,B) + dist(B,C) + dist(D,E)

= sqrt(8) + sqrt(8) + sqrt(8)

dist(E, F) = sqrt(72)

theoretically: dist(E, F) = dist(A, D)



Implementation with double (IEEE 754 floating-point standard):

sqrt(8) + sqrt(8) + sqrt(8) = 8.48528137423857**1**
sqrt(72) = 8.48528137423857**0**

Bits in double representations:
01000000001000001111100001110110110011001101111101101100110110**10**
01000000001000001111100001110110110011001101111101101100110110**01**

AB = vector (A,B) = B – A

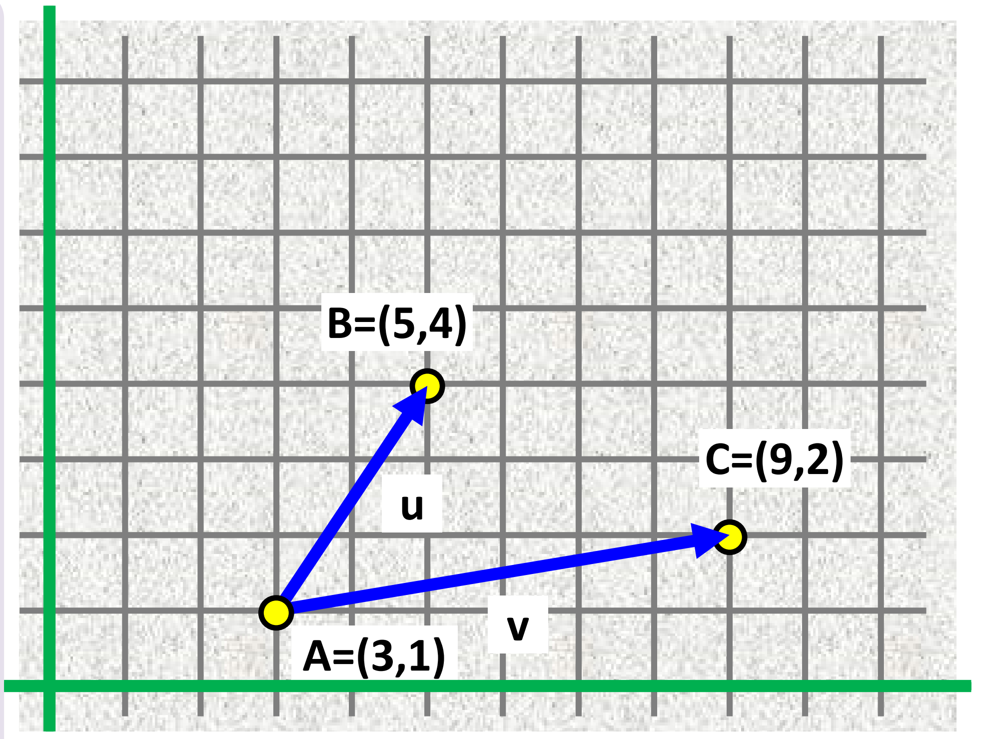
AB = $(Bx–Ax, By–Ay)^T$

Vector **norm** = vector **length**

‖**AB**‖ = ‖**BA**‖

‖**AB**‖ = sqrt($(Bx–Ax)^2 + (By–Ay)^2$)

‖**AB**‖ = distance(A, B)



u = **AB** = B – A = $(5 – 3, 4 – 1)^T = (2, 3)^T$

v = **AC** = C – A = $(9 – 3, 2 – 1)^T = (6, 1)^T$

‖**u**‖ = sqrt($2^2 + 3^2$) = sqrt(13)

‖**v**‖ = sqrt($6^2 + 1^2$) = sqrt(37) in Euclidean space

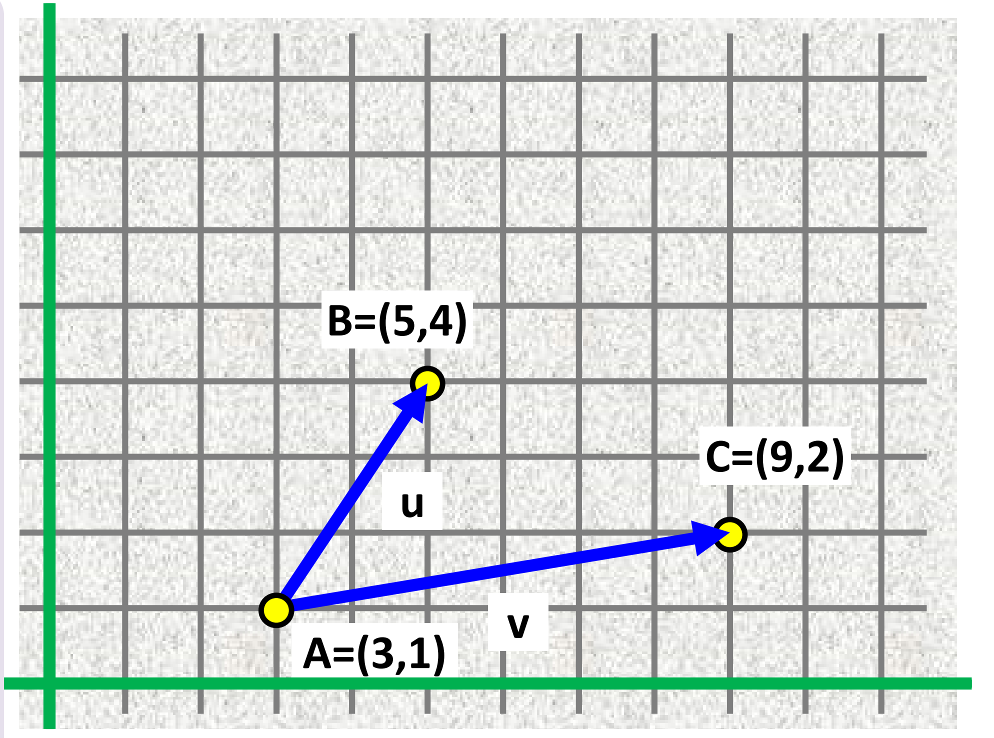
Dot product ≡ scalar product

$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

// commutative

sum(i = 1..dimension, **u**[i]***v**[i])

$= u_x v_x + u_y v_y$ **// in 2D**



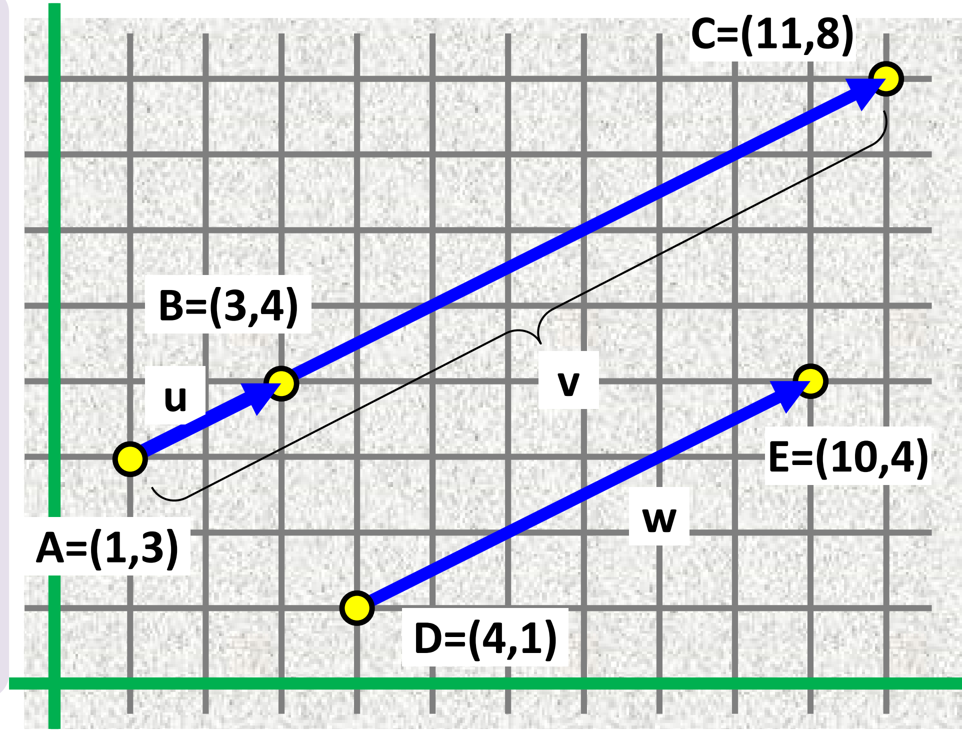
$\mathbf{u} = \mathbf{AB} = B - A = (5 - 3, 4 - 1)^T = (2, 3)^T$

$\mathbf{v} = \mathbf{AC} = C - A = (9 - 3, 2 - 1)^T = (6, 1)^T$

$\langle \mathbf{u}, \mathbf{v} \rangle = 2*6 + 3*1 = 15$

Vectors **u** and **v** are **collinear**
if and only if
u is a non-zero multiple of **v**
(and vice versa)
or equivalently:

determinant $\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = 0$



$\mathbf{u} = (2, 1)^T$
$\mathbf{v} = (10, 5)^T$
$\mathbf{w} = (6, 3)^T$

$$\det(\mathbf{u}, \mathbf{v}) = \det((2, 1)^T, (10, 5)^T) = \det\begin{pmatrix} 2, & 10 \\ 1, & 5 \end{pmatrix} = 2*5 - 1*10 = 0$$

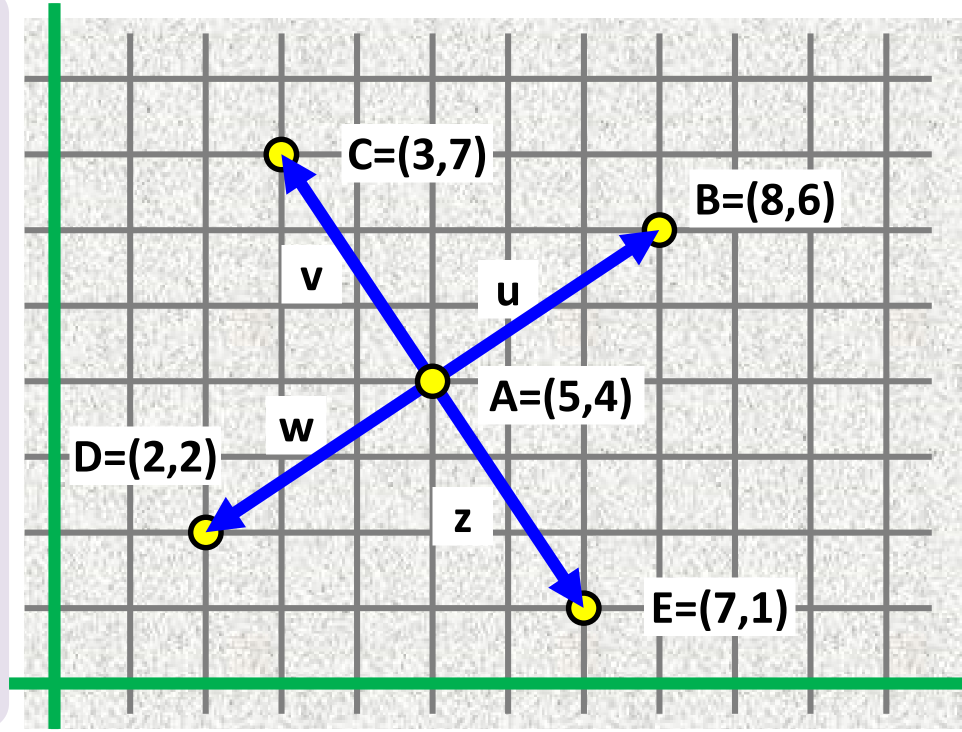
$$\det(\mathbf{u}, \mathbf{w}) = \det((2, 1)^T, (6, 3)^T) = \det\begin{pmatrix} 2, & 6 \\ 1, & 3 \end{pmatrix} = 2*3 - 1*6 = 0$$

$$\det(\mathbf{v}, \mathbf{w}) = \det((10, 5)^T, (6, 3)^T) = \det\begin{pmatrix} 10, & 6 \\ 5, & 3 \end{pmatrix} = 10*3 - 5*6 = 0$$

Nonzero vectors **u**, **v**
are **perpendicular** to each other

u ⊥ v

iff scalar product ⟨**u**, **v**⟩ = 0



u = (3, 2)^T **u** □ **v**: ⟨ (3, 2)^T, (−2, 3)^T⟩ = 3*(−2) + 2*3 = 0

v = (−2, 3)^T **v** □ **w**: ⟨ (−2, 3)^T, (−3, −2)^T⟩ = (−2)*(−3) + 3*(−2) = 0

w = (−3, −2)^T **w** □ **z**: ⟨ (−3, −2)^T, (2, −3)^T⟩ = (−3)*2 + (−2)*(−3) = 0

z = (2, −3)^T **z** □ **u**: ⟨ (2, −3)^T, (3, 2)^T⟩ = 2*3 + (−3)*2 = 0

Area of a triangle given by vectors **u, v = AB**, **AC**

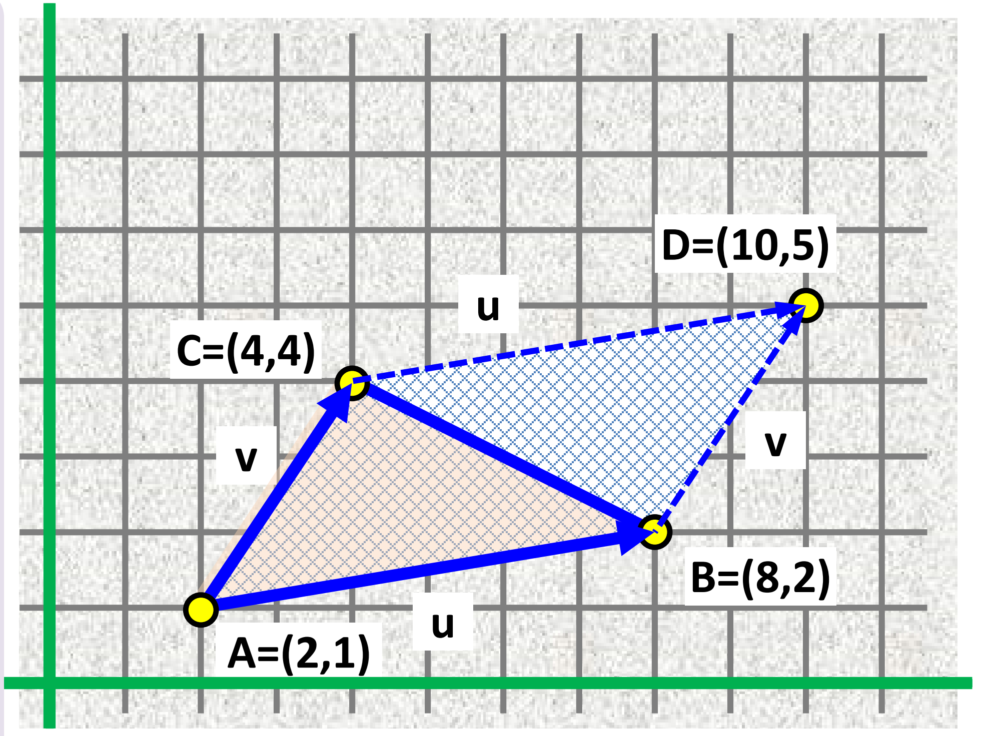
$\frac{1}{2} \cdot |\det(\mathbf{u}, \mathbf{v})|$

Area of parallelogram ABCD
(D = B + **v** = C + **u**)

$|\det(\mathbf{u}, \mathbf{v})|$

Vector mutual position matters:

$\det(\mathbf{v}, \mathbf{u}) = -\det(\mathbf{u}, \mathbf{v})$



Triangle ABC **area** =

= abs (det ($(6, 1)^T$, $(2, 3)^T$)) / 2 = abs(6*3 – 1*2) / 2 = 8 // vectors **AB**, **AC**

= abs (det ($(–6, –1)^T$, $(–4, 2)^T$)) / 2 = abs((–6)*2 – (–4)*(–1)) / 2 = 8 // vectors **BA**, **BC**

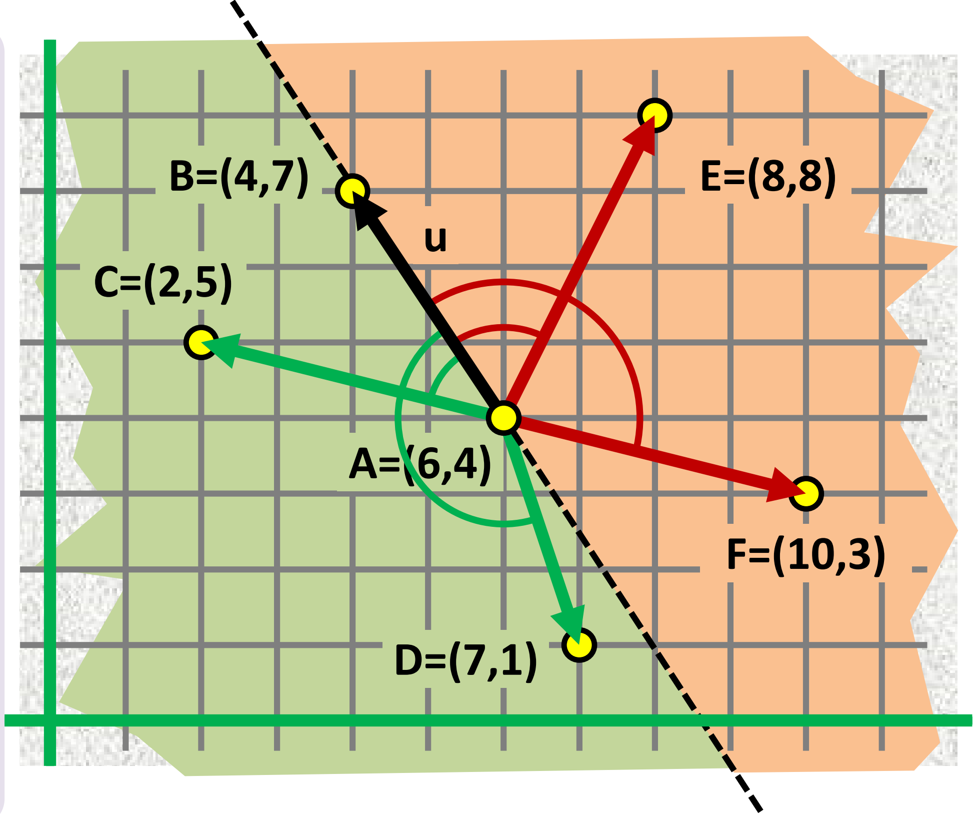
= abs (det ($(–2, –3)^T$, $(4, –2)^T$)) / 2 = abs((–2)*(–2) – (–3)*4)) / 2 = 8 // vectors **CA**, **CB**

Relative orientation of vectors

angle (**u**, **v**) ... how much to turn **u** *to the* ***left*** to obtain a vector parallel to **v**

det(**u**, **v**) > 0 ⇔ 0 < angle (**u**, **v**) < 180

det(**u**, **v**) < 0 ⇔ 180 < angle (**u**, **v**) < 360



$\mathbf{u} = (\mathbf{B}–\mathbf{A})^T = (–2, 3)^T$

det (**u**, **AC**) = det($(–2, 3)^T, (–4, 1)^T$) = –2*1 – 3*(–4) = 10 > 0

det (**u**, **AD**) = det($(–2, 3)^T, (1, –3)^T$) = –2*(–3) – 3*1 = 3 > 0

det (**u**, **AE**) = det($(–2, 3)^T, (2, 4)^T$) = –2*4 – 3*2 = –14 < 0

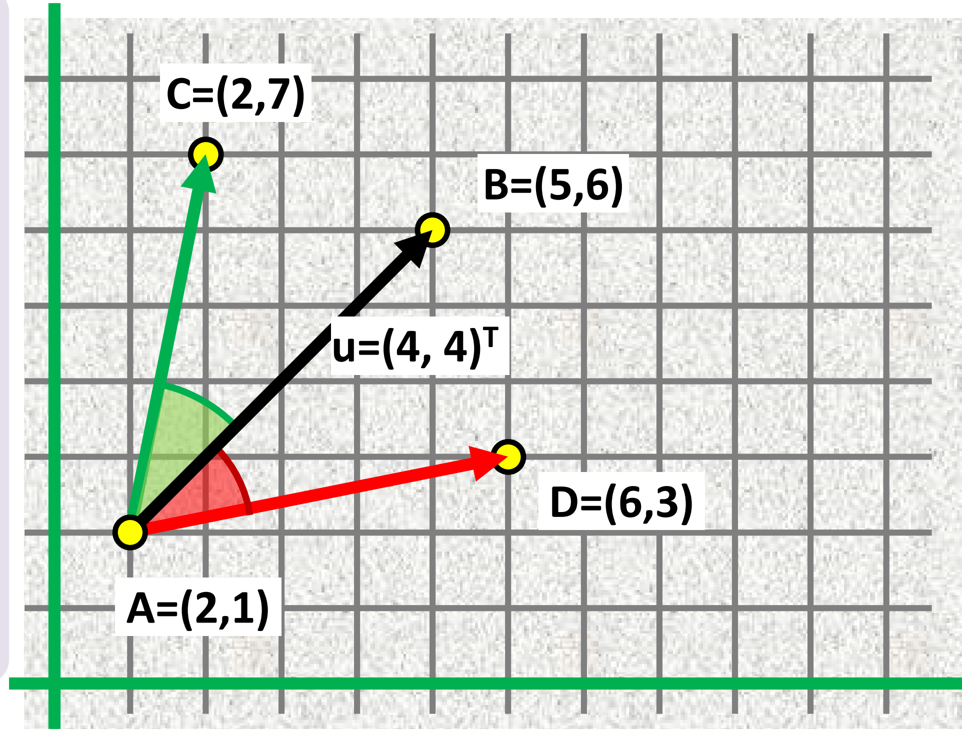
det (**u**, **AF**) = det($(–2, 3)^T, (4, –1)^T$) = –2*(–1) – 3*4 = –10 < 0

Angle of vectors

cos angle = ⟨**u**, **v**⟩ / (‖**u**‖ ‖**v**‖)

angle = arc cos (⟨**u**, **v**⟩ / (‖**u**‖ ‖**v**‖))

Relative orientation of **u** and **v** is **not** calculated



cos ∠ BAC = ⟨**u**, **AC**⟩ / (‖**u**‖ ‖**AC**‖) = cos ∠ CAB = ⟨**AC**,**u**⟩ / (‖**AC**‖ ‖**u**‖)
= ⟨(4, 4) ^{T}, (1, 5) ^{T}⟩ / (‖(4, 4) ^{T}‖ ‖ (1, 5) ^{T}‖)
= (4*1 + 4*5) / (sqrt(32) * sqrt(26)) = 24 / (8*sqrt(13)) = 3/sqrt(13)

cos ∠ BAD = ⟨**u**, **AD**⟩ / (‖**u**‖ ‖**AD**‖) = cos ∠ DAB = ⟨**AD**,**u**⟩ / (‖**AD**‖ ‖**u**‖)
= ⟨(4, 4) ^{T}, (5, 1) ^{T}⟩ / (‖(4, 4) ^{T}‖ ‖ (5, 1) ^{T}‖)
= (4*5 + 4*1) / (sqrt(32) * sqrt(26)) = 24 / (8*sqrt(13)) = 3/sqrt(13)

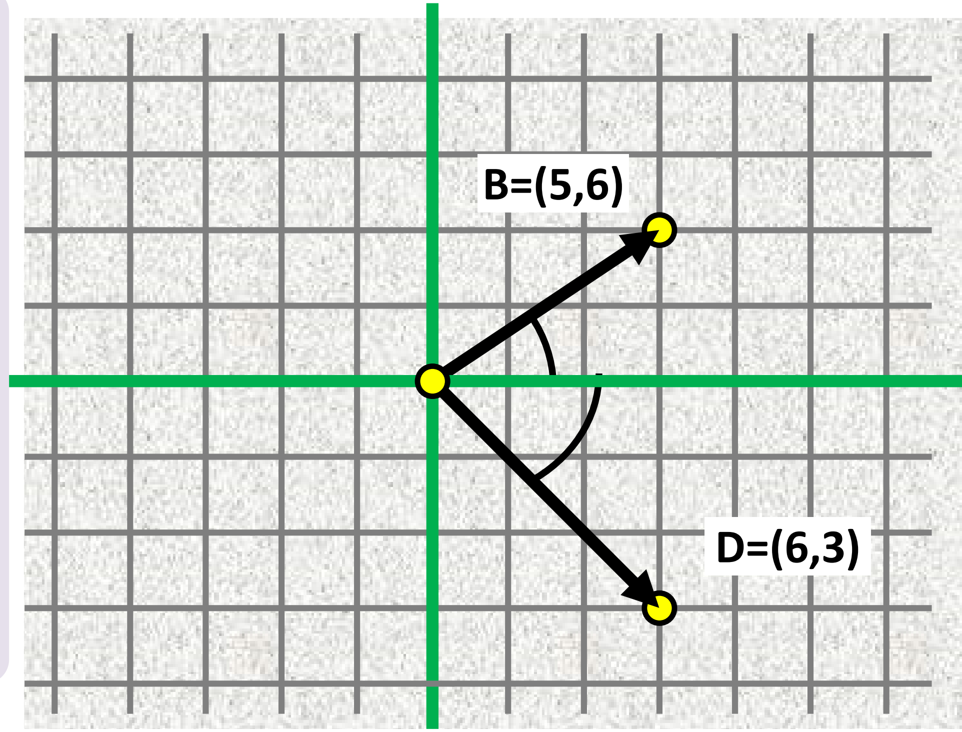
∠ BAC = ∠ BAD = arc cos (3/ sqrt(13)) = 0.588 rad = 33.69°

π rad = 180 deg
1 rad = 180/π deg
1 deg = π/180 rad

Angle of vector (x, y)
in quadrant I and IV

angle = arc tan(y/ x)

Implementations handle it completely:



Implementation

```
Caution! The parameters are (y,x), and not (x,y)!

math.atan2( 1, 1) *180/math.pi ==   45.0
math.atan2(-1,-1) *180/math.pi == -135.0
math.atan2( 1,-1) *180/math.pi ==  135.0    # Note x<-->y reversal!
math.atan2(-1, 1) *180/math.pi ==  -45.0    # Note x<-->y reversal!

math.atan2( 0, 0) *180/math.pi  ==   0.0    # despite being undefined ☺
```

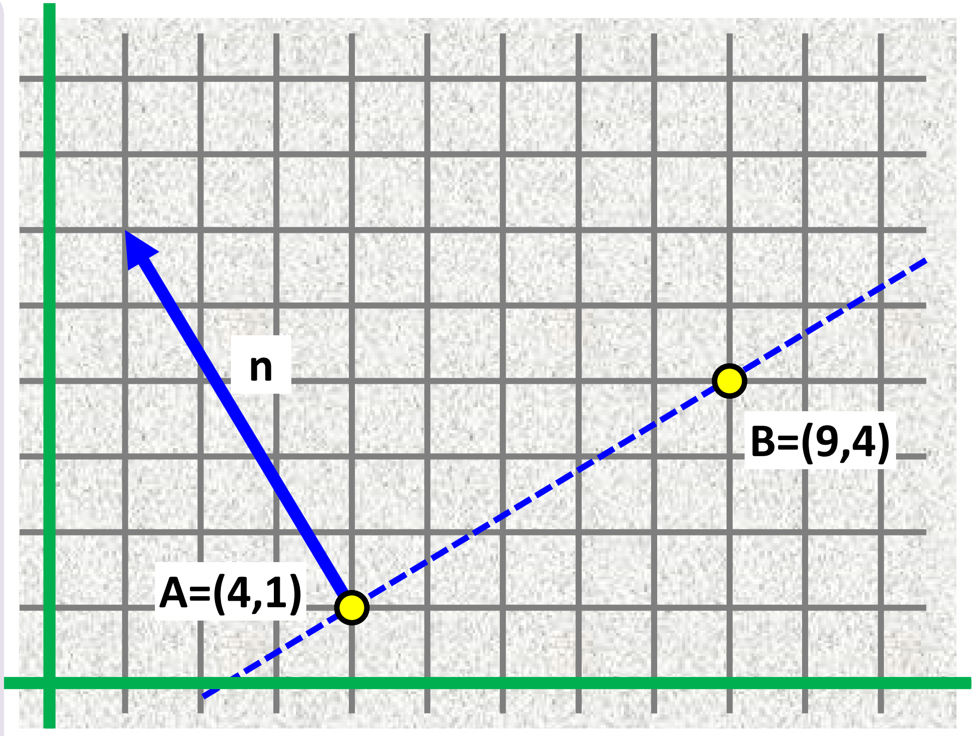
Two points **A** = (A_x, A_y), **B** = (B_x, B_y)

→ **line equation**

$$ax + by + c = 0$$

1. Normal vector **n** = $(a, b)^T$
2. A lies on the line AB ⇒
   $a \cdot A_x + b \cdot A_y + c = 0 \Rightarrow$
   $c = -a \cdot A_x - b \cdot A_y \Rightarrow$

$$a \cdot x + b \cdot y - a \cdot A_x - b \cdot A_y = 0$$



AB = $(5,3)^T$
n = (–3, 5), equation: $-3x + 5y + c = 0$

$c = -(-3)*4 - 5*1 = 7$ equation: $-3x + 5y + 7 = 0$

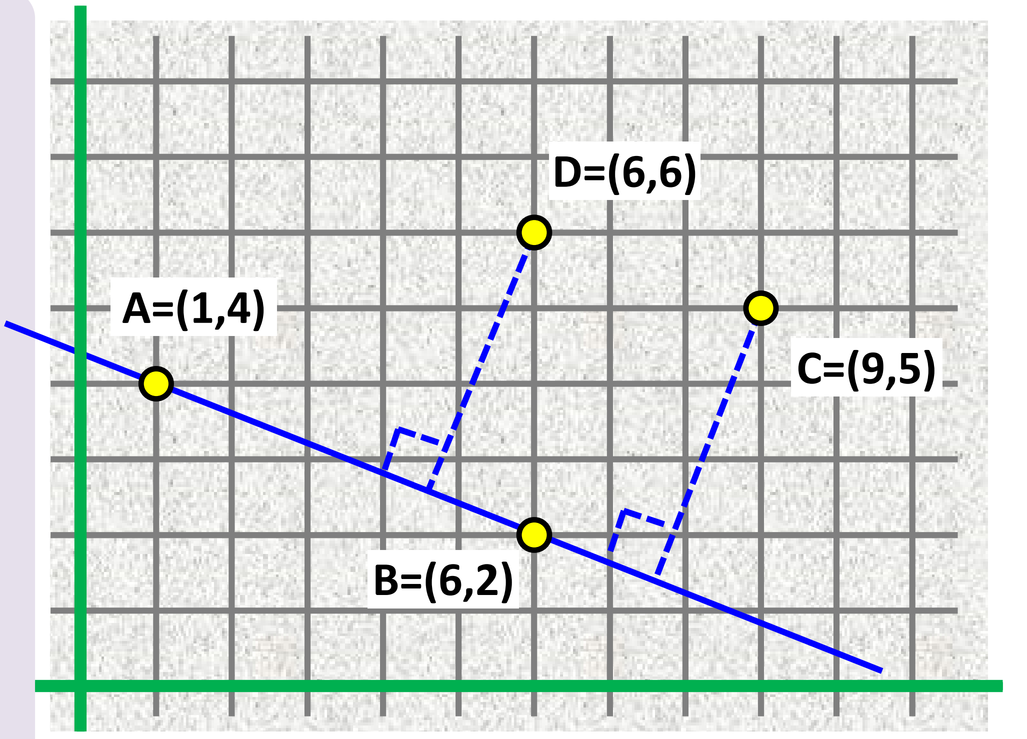
(Check: plug coords of B = (9, 4) into the equation: $-3*9 + 5*4 + 7 = -27+20+7 = 0$)

Distance point to line

Point P: (Px, Py)
Line: ax + by + c = 0

Distance(P, line) =

$| a \cdot Px + b \cdot Py + c | \; / \; \| (a, b)^T \|$



line AB: 2x + 5y – 22 = 0
C = (9, 5)
D = (6, 6)

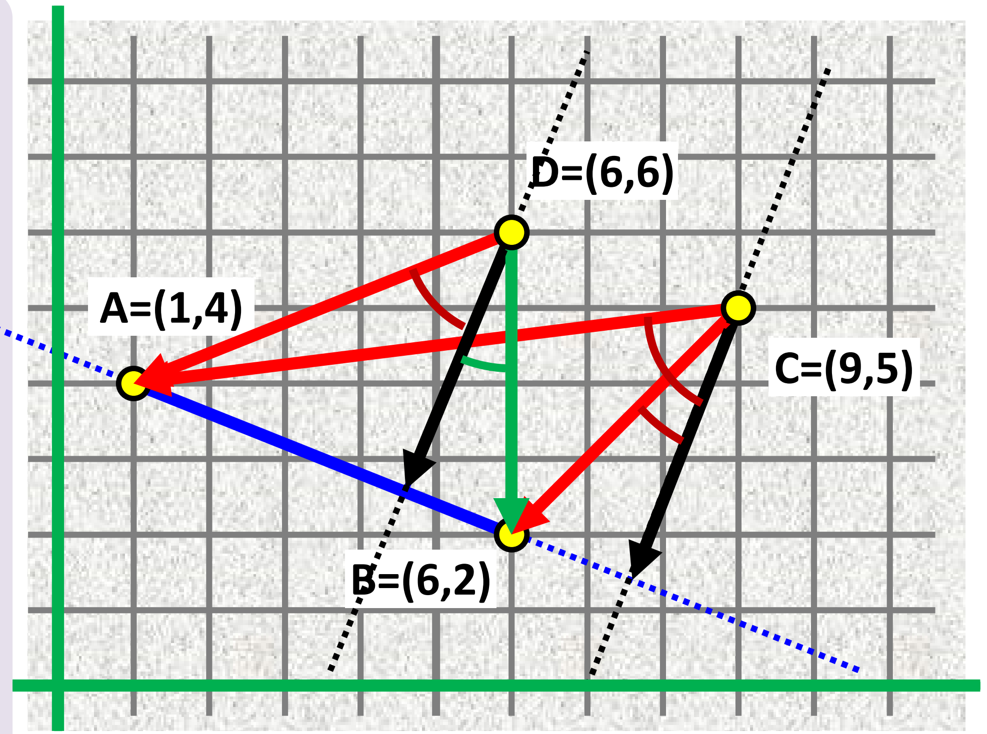
dist(C, AB) = abs(2*9 + 5*5 – 22) / sqrt(2*2 + 5*5) = 21/sqrt(29) ≈ 3.8996

dist(D, AB) = abs(2*6 + 5*6 – 22) / sqrt(2*2 + 5*5) = 20/sqrt(29) ≈ 3.7139

Distance point to segment

Point P: (Px, Py)
Segment: AB

Normal vector **n** to AB
If vectors **PA** and **PB**
are "on the same side" wrt **n**
then
dist(P, AB) = min (dist(P, A), dist(P, B))
else
dist(P, AB) = dist(P, line AB)



Being "on the same side" wrt **n** means
that the angle between **n** and **PA** and the angle between **n** and **PB**
are either both between 0° and 180° or both between 180° and 360°.
In other words, the sign of det (**n**, **PA**) and det (**n**, **PB**) is the same, or simply
det (**n**, **PA**) * det (**n**, **PB**) > 0.

Line-line intersection P

$a1 \cdot x + b1 \cdot y + c1 = 0$,

$a2 \cdot x + b2 \cdot y + c2 = 0$

Solution of syst. of two lin. eq. in x and y, using Cramer rule:

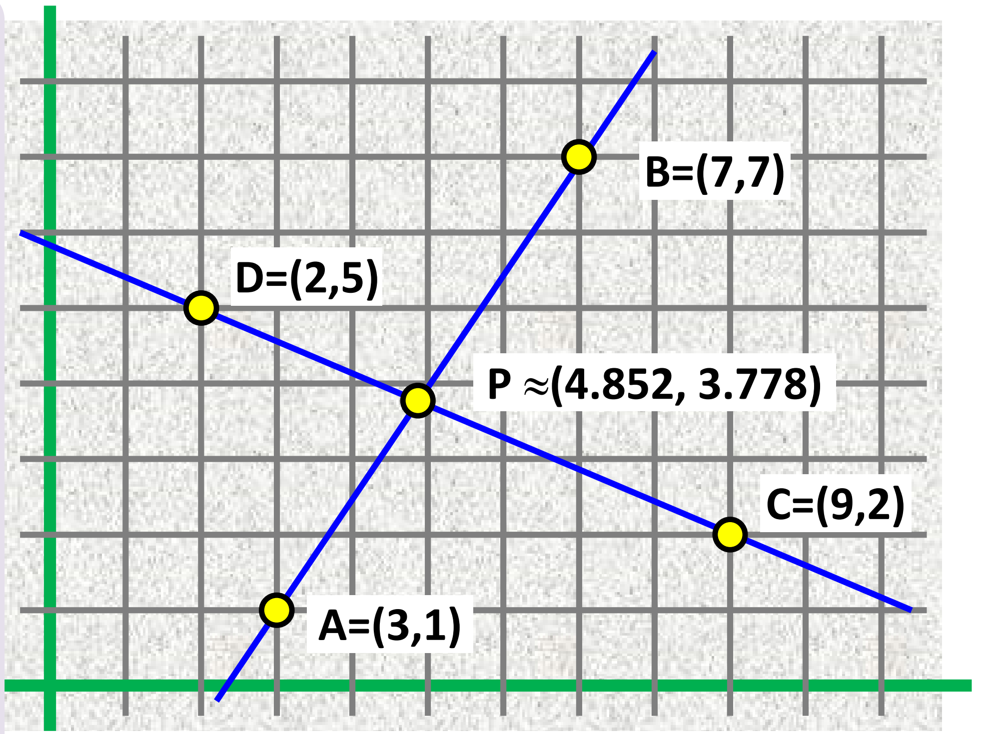
$\det(\mathbf{n1}^T, \mathbf{n2}^T) = a1*b2 - a2*b1$

if $\det(\mathbf{n1}^T, \mathbf{n2}^T) == 0$ then collinear

if $\det(\mathbf{n1}^T, \mathbf{n2}^T)\ != 0$ then

$Px = (b1*c2 - b2*c1) / \det(\mathbf{n1}^T, \mathbf{n2}^T)$

$Py = (c1*a2 - c2*a1) / \det(\mathbf{n1}^T, \mathbf{n2}^T)$



line eq: $\mathbf{n}^T = (a, b)$, $ax + by - a \cdot Ax - b \cdot Ay = 0$

line AB: $\mathbf{n1}^T = (-6, 4)$; $-6x + 4y + 14 = 0$

line CD: $\mathbf{n2}^T = (3, 7)$; $3x + 7y - 41 = 0$

$\det(\mathbf{n1}^T, \mathbf{n2}^T) = (-6)*7 - 4*3 = -54$

$Px = (4*(-41) - 7*14) / (-54) = -262 / -54 \approx 4.852$

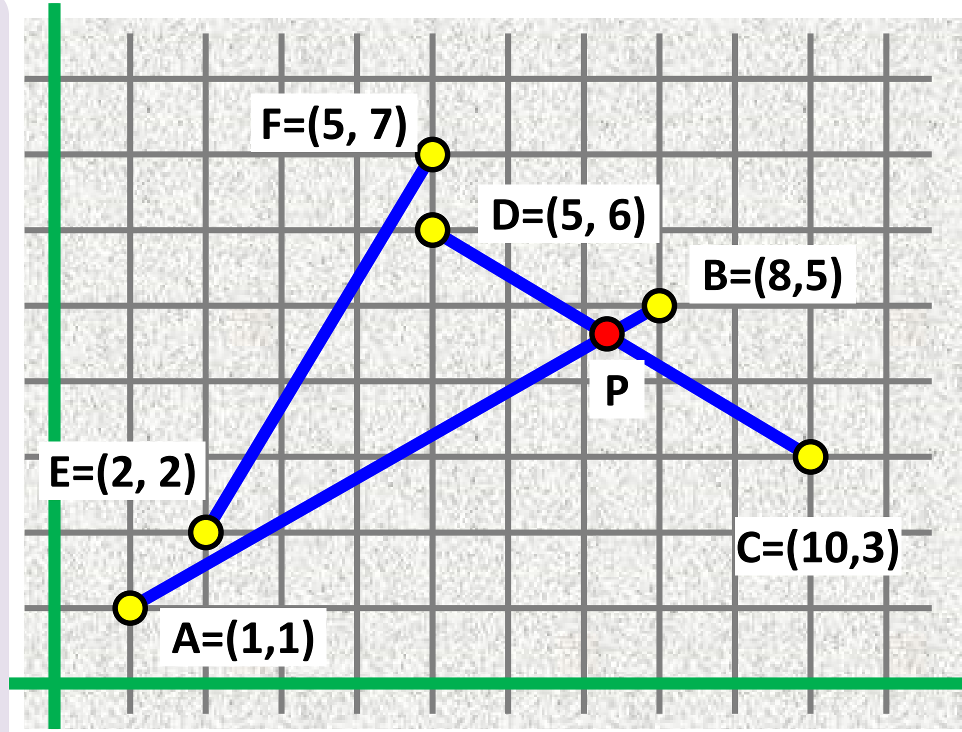
$Py = (14*3 - (-41)(-6)) / (-54) = -204 / -54 \approx 3.778$

Segment - segment intersection

Does (A,B) intersect (C, D) ?

-- C and D should not lie on the same side of line (A, B)
-- A and B should not lie on the same side of line (C, D)

Apply Relative orientation of vectors (using determinant, slide 8.)
Also check collinearity of (A,B) and (C, D).




```
# filter out non-intersection
if det((B-A)ᵀ, (C-A)ᵀ) * det((B-A)ᵀ, (D-A)ᵀ) > 0   or   det((B-A)ᵀ, (C-A)ᵀ) * det((B-A)ᵀ, (D-A)ᵀ) > 0:
  return false
# manage (possible?) collinearity (=line AB is also line CD)
if det((B-A)ᵀ, (D-C)ᵀ) == 0:
  if (A==C and  B!= D) or  (A==D and  B!=C):  P = A; return true
  if (B==C and  A!= D) or  (B==D and  A!=C):  P = B; return true
  return false  # no intersection or infinitely many
# no collinearity, calculate intersection P coordinates
P = intersection( lineAB, lineCD ); return true     # even here, P may be equal to one of A,B,C,D,
```

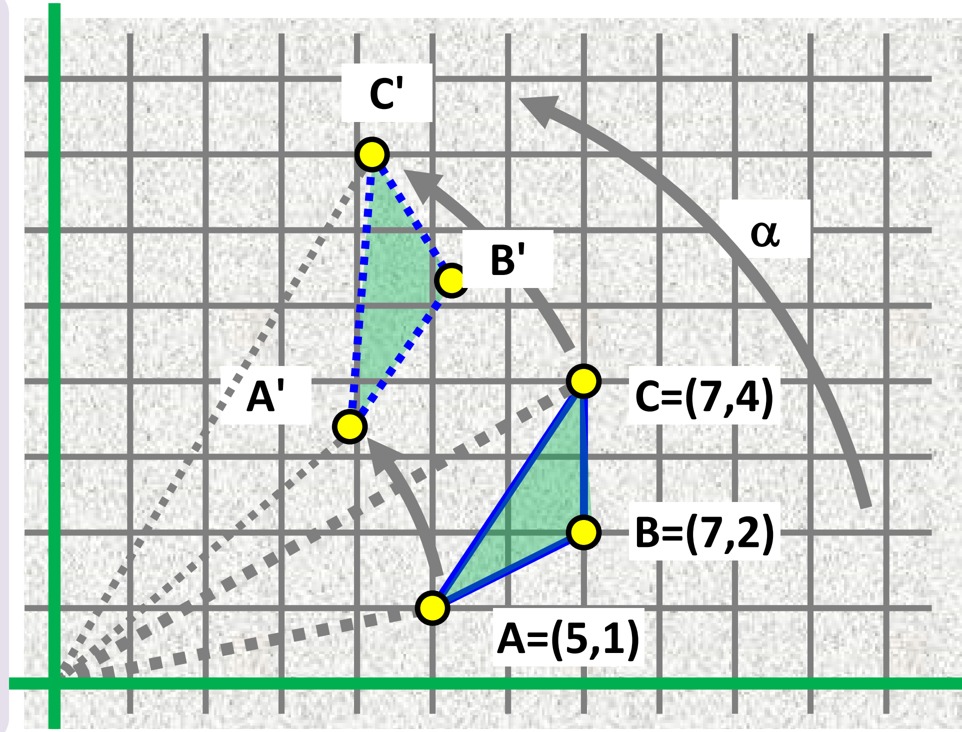
Rotate point A = (Ax, Ay) counterclockwise (!) around origin **by a given angle α**:

$$\begin{pmatrix} \cos\alpha, & -\sin\alpha \\ \sin\alpha, & \cos\alpha \end{pmatrix}\begin{pmatrix} Ax \\ Ay \end{pmatrix} =$$

(cosα*Ax – sinα*Ay, sinα*Ax + cosα*Ay)

rotate left by 90 deg, multiply by matrix

$$\begin{pmatrix} \cos 90^\circ, & -\sin 90^\circ \\ \sin 90^\circ, & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$



Rotate ABC by 30°

$$\begin{pmatrix} \cos 30^\circ, & -\sin 30^\circ \\ \sin 30^\circ, & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \text{sqrt}(3)/2, & -1/2 \\ 1/2, & \text{sqrt}(3)/2 \end{pmatrix} \approx \begin{pmatrix} 0.886, & -0.5 \\ 0.5, & 0.886 \end{pmatrix}$$

A' = (0.886*5 – 0.5*1, 0.5*5 + 0.886*1) = (3.93, 3.386)

B' = (0.886*7 – 0.5*2, 0.5*7 + 0.886*2) = (5.202, 5.272)

C' = (0.886*7 – 0.5*4, 0.5*7 + 0.886*4) = (4.202, 7.044)

Simple polygon
(No two of its non-adjacent boundary segments touch or intersect each other)

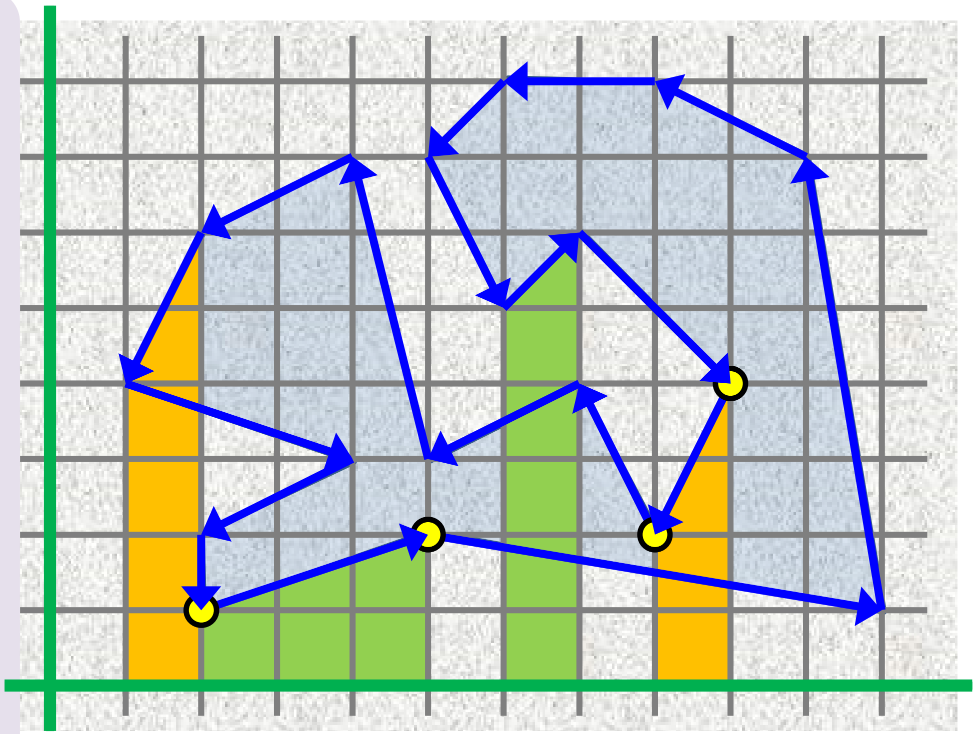
Area

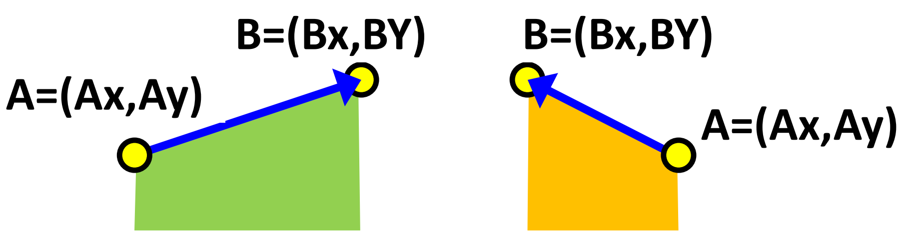
Choose boundary direction
consider forward and backward trapezoids

add all forward trapezoids area
subtract all backward trapezoids area

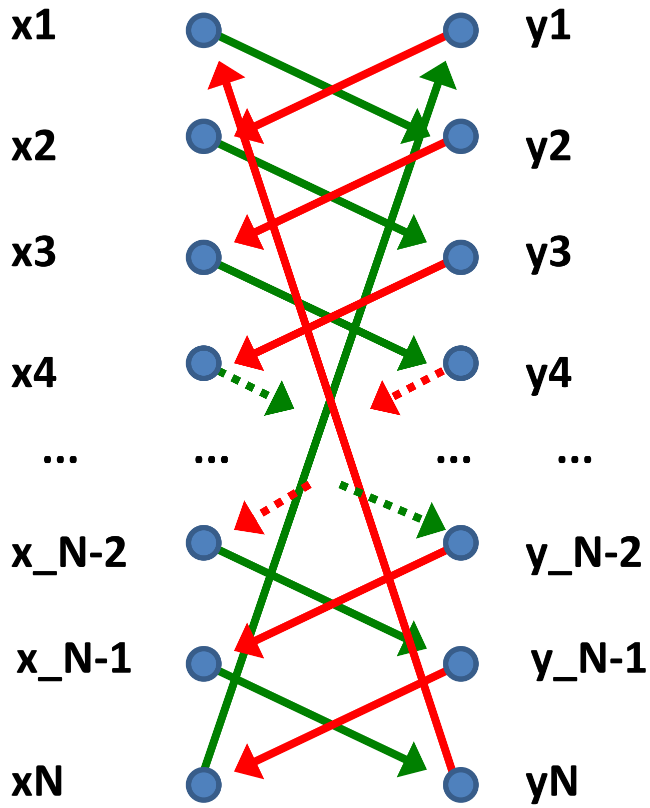
or equivalently

add all trapezoid areas
under assumption that the area of backward trapezoids is negative

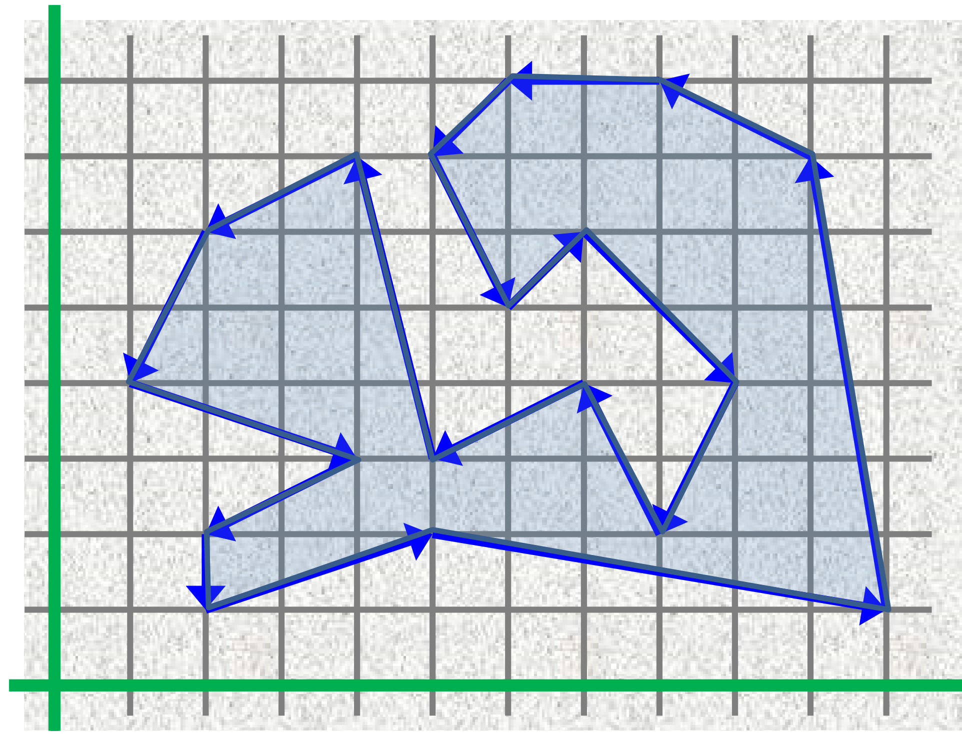




Trapezoid area |(Ay+By)*(Bx–Ax)| / 2



Simple polygon
(No two of its non-adjacent boundary segments touch or intersect each other)

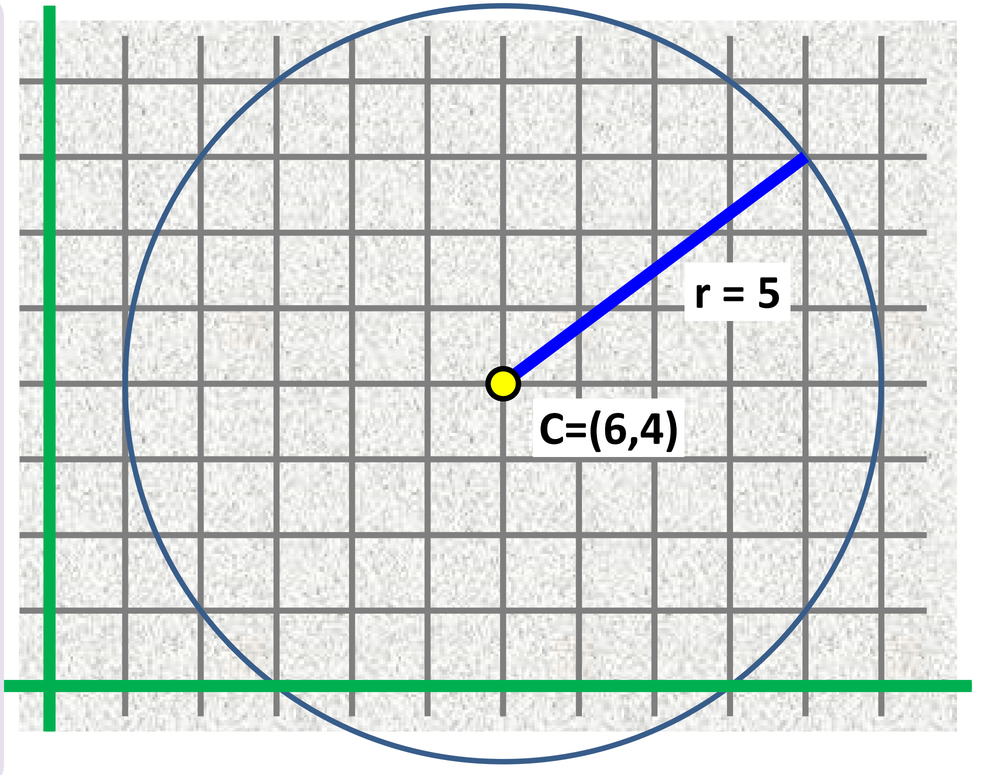


Area: Shoelace formula

$$\frac{1}{2} * \left(x1*y2 + x2*y3 + \ldots\ x\_N\text{-}1*yN + xN*y1 \quad - x2*y1 - x3*y2 - \ldots - xN*y\_N\text{-}1 - x1*yN\right)$$

Circle equation

$(Cx - x)^2 + (Cy - y)^2 = r^2$



Circle tangent in point T

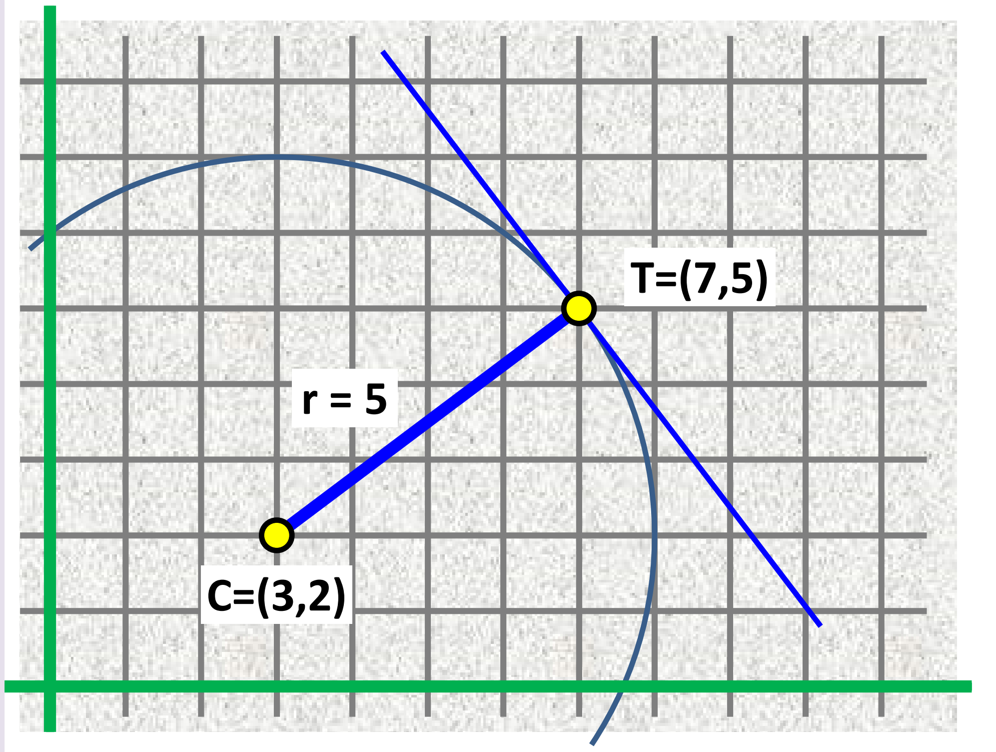
Circle eq: $(Cx - x)^2 + (Cy - y)^2 = r^2$

tangent line equation

$$(Tx - Cx)(x - Cx) + (Ty - Cy)(y - Cy) = r^2$$

tangent line equation
$ax + by + c = 0$

$a = Tx - Cx$
$b = Ty - Cy$
$c = Cx^2 + Cy^2 - r^2 - Cx \cdot Tx - Cy \cdot Ty$



tangent line equation,

$$(7-3)(x-3) + (5-2)(y-2) = 25$$
$$4x - 12 + 3y - 6 = 25$$
$$4x + 3y - 43 = 0$$

Polar of a point T wrt a circle

Polar line connects points T' and T''.
Lines TT' and TT'' are tangent lines to the given circle

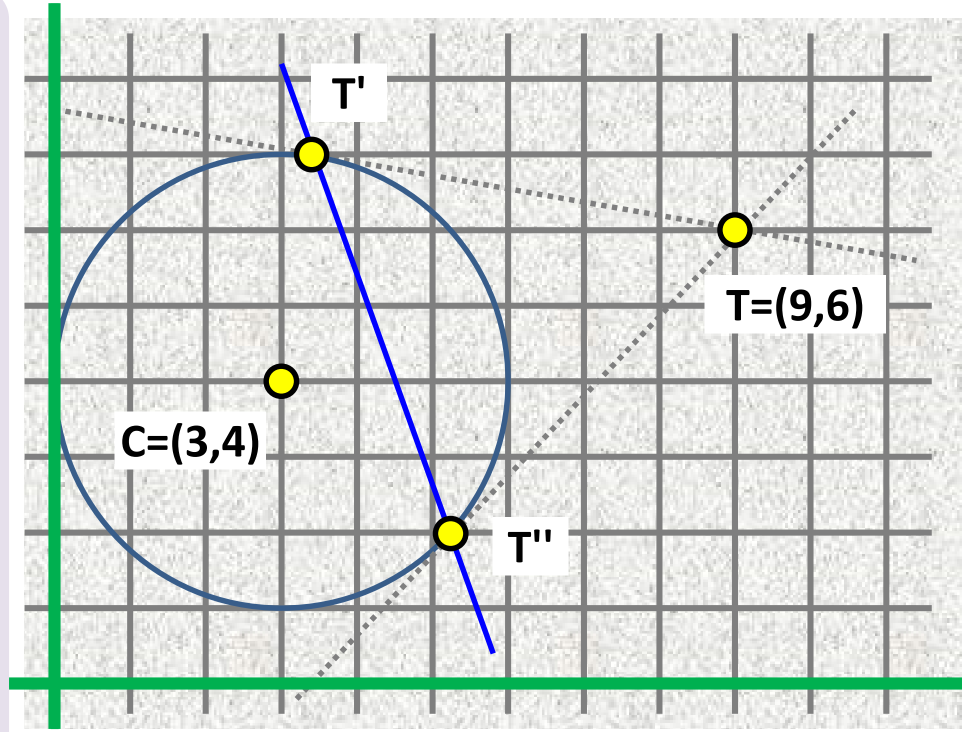
Circle eq: $(Cx - x)^2 + (Cy - y)^2 = r^2$

polar line equation

$$(Tx - Cx)(x - Cx) + (Ty - Cy)(y - Cy) = r^2$$

polar line equation
$ax + by + c = 0$

$a = Tx - Cx$
$b = Ty - Cy$
$c = Cx^2 + Cy^2 - r^2 - Cx \cdot Tx - Cy \cdot Ty$



polar line equation
wrt circle $(3 - x)^2 + (4 - y)^2 = 3^2$
$T = (9, 6)$

$$6x + 2y + 9 + 16 - 9 - 27 - 24 = 0$$
$$6x + 2y - 35 = 0$$