Geometry of Computer Vision

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Contents

1 Notation

$\varnothing$	...	empty set [1]
$\exp U$	...	the set of all subset of set U [1]
$\mathbb{Z}$	...	whole numbers [1]
$\mathbb{Q}$	...	rational numbers [2]
$\mathbb{R}$	...	real numbers [2]
i	...	imaginary unit
$(S, +, \ )$	...	space of geometric scalars
A	...	affine space (space of geometric vectors)
$(A_o, \oplus, \odot)$	...	space of geometric vectors bound to point o
$(V, \boxplus, \boxdot)$	...	space of free vectors
$\vec{x}$	...	vector
$\mathtt{A}$	...	matrix
$\mathtt{A}_{ij}$	...	ij element of $\mathtt{A}$
$\mathtt{A}^\top$	...	transpose of $\mathtt{A}$
$\lvert\mathtt{A}\rvert$	...	determinant of $\mathtt{A}$
$\mathtt{I}$	...	identity matrix
$\mathtt{R}$	...	rotation matrix
$\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$	...	basis (an ordered triple of independent generator vectors)
$\vec{x}_\beta$	...	column matrix of coordinates of $\vec{x}$ w.r.t. the basis β
$\vec{x} \cdot \vec{y}$	...	Euclidean scalar product of $\vec{x}$ and $\vec{y}$ ($\vec{x} \cdot \vec{y} = \vec{x}_\beta^\top \vec{y}_\beta$ in an orthonormal basis β)
$\vec{x} \times \vec{y}$	...	cross (vector) product of $\vec{x}$ and $\vec{y}$
$U \times V$	...	Cartesian product of sets U and V
$\lVert\vec{x}\rVert$	...	Euclidean norm of $\vec{x}$ ($\lVert\vec{x}\rVert = \sqrt{\vec{x} \cdot \vec{x}}$)
orthogonal vectors	...	mutually perpendicular and all of equal length
orthonormal vectors	...	unit orthogonal vectors

2 Change of coordinates induced by the change of basis

Let us discuss the relationship between the coordinates of a vector in a linear space, which is induced by passing from one basis to another. We shall derive the relationship between the coordinates in a three-dimensional linear space $\mathbb{R}^3$ over real numbers, which is the most important when modeling the geometry around us. The formulas for all other n-dimensional spaces are obtained by passing from 3 to n.

§ **1 Coordinates** Let us consider an ordered basis $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ of V^3. A vector $\vec{x} \in V^3$ is uniquely expressed as a linear combination of the basic vectors by its real *coordinates* x, y, z, i.e. $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$, and can be represented as an ordered triple of coordinates, as a coordinate vector $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$.

§ **2 Two bases** Having two ordered bases $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ and $\beta' = \begin{bmatrix} \vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3 \end{bmatrix}$ leads to expressing one vector $\vec{x}$ in two ways as $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and $\vec{x} = x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3$. The vectors of the basis β can also be expressed in the basis β' using their coordinates. Let us introduce

$$\begin{aligned} \vec{b}_1 &= a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 \\ \vec{b}_2 &= a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 \\ \vec{b}_3 &= a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3 \end{aligned} \tag{2.1}$$

§ **3 Change of coordinates** We will next use the above equations to relate the coordinates of $\vec{x}$ w.r.t. the basis β to the coordinates of $\vec{x}$ w.r.t. the basis β'

$$\begin{aligned} \vec{x} &= x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3 \\ &= x\,(a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3) + y\,(a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3) + z\,(a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3) \\ &= (a_{11}\,x + a_{12}\,y + a_{13}\,z)\,\vec{b}'_1 + (a_{21}\,x + a_{22}\,y + a_{23}\,z)\,\vec{b}'_2 + (a_{31}\,x + a_{32}\,y + a_{33}\,z)\,\vec{b}'_3 \\ &= x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3 \end{aligned}$$

Since coordinates are unique, we get

$$\begin{aligned} x' &= a_{11}\,x + a_{12}\,y + a_{13}\,z \\ y' &= a_{21}\,x + a_{22}\,y + a_{23}\,z \\ z' &= a_{31}\,x + a_{32}\,y + a_{33}\,z \end{aligned}$$

Coordinate vectors $\vec{x}_\beta$ and $\vec{x}_{\beta'}$ are thus related by the following matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which we concisely write as

$$\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_{\beta} \tag{2.2}$$

The columns of matrix A can be viewed as coordinate vectors representing basic vectors, $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of β in the basis β'

$$\mathtt{A} = \begin{bmatrix} | & | & | \\ \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \\ | & | & | \end{bmatrix}$$

and the matrix multiplication can be interpreted as the linear combination of columns of A by coordinates of $\vec{x}$ w.r.t. β

$$\vec{x}_{\beta'} = x\,\vec{b}_{1_{\beta'}} + y\,\vec{b}_{2_{\beta'}} + z\,\vec{b}_{3_{\beta'}}$$

Matrix A plays such an important role here that it deserves its own name. Matrix A is very often called the *change of basis matrix from basis β to β'* or the *transition matrix from basis β to basis β'* [3, 4] since it can be used to pass from coordinates w.r.t. β to coordinates w.r.t. β' by Equation 2.2.

However, literature [5, 6] calls A the *change of basis matrix from basis β' to β*, i.e. it (seemingly illogically) swaps the bases. This choice is motivated by the fact that A can also be used to obtain the vectors of β from vectors of β' using Equation 2.1 as

$$\begin{aligned}
\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} &= \Big[a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 \quad a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 \\
&\qquad a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3\Big] \qquad (2.3) \\
&= \begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= \begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} \mathtt{A} \qquad (2.4) \\
\begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} &= \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} \mathtt{A}^{-1} \qquad (2.5)
\end{aligned}$$

where the multiplication of a row of column vectors by a matrix from the right in Equation 2.5 has the meaning given by Equation 2.3 above. Yet another variation of the naming appeared in [7, 8] where $\mathtt{A}^{-1}$ was named the *change of basis matrix from basis β to β'*.

We have to conclude that the meaning associated with the *change of basis matrix* varies in the literature and hence we will avoid this confusing name and talk about A as about the *matrix transforming coordinates of a vector from basis β to basis β'*.

There is the following interesting variation of Equation 2.5

$$\begin{bmatrix}\vec{b}'_1 \\ \vec{b}'_2 \\ \vec{b}'_3\end{bmatrix} = \mathtt{A}^{-\top} \begin{bmatrix}\vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3\end{bmatrix} \tag{2.6}$$

where the basic vectors of β and β' are understood as elements of column vectors. For instance, vector $\vec{b}'_1$ is obtained as

$$\vec{b}'_1 = \bar{a}_{11}\,\vec{b}_1 + \bar{a}_{12}\,\vec{b}_2 + \bar{a}_{13}\,\vec{b}_3 \tag{2.7}$$

where $[\bar{a}_1,\, \bar{a}_2,\, \bar{a}_3]$ is the first row of $\mathtt{A}^{-\top}$.

§ **4 Example** We demonstrate the relationship between vectors and bases on a concrete example. Consider two bases α and β represented by coordinate vectors, which we write into matrices

$$\begin{aligned}\alpha &= \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\\ \beta &= \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\0 & 1 & 1\end{bmatrix},\end{aligned}$$

and a vector $\vec{x}$ with coordinates w.r.t. the basis α

$$\vec{x}_\alpha = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

We see that basic vectors of α can be obtained as the following linear combinations of basic vectors of β

$$\begin{aligned}\vec{a}_1 &= +1\,\vec{b}_1 + 0\,\vec{b}_2 + 0\,\vec{b}_3\\ \vec{a}_2 &= +1\,\vec{b}_1 - 1\,\vec{b}_2 + 1\,\vec{b}_3\\ \vec{a}_3 &= -1\,\vec{b}_1 + 0\,\vec{b}_2 + 1\,\vec{b}_3\end{aligned}$$

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\0 & -1 & 0\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}\mathtt{A}$$

Coordinates of $\vec{x}$ w.r.t. β are hence obtained as

$$\vec{x}_\beta = \mathtt{A}\,\vec{x}_\alpha, \qquad \mathtt{A} = \begin{bmatrix}1 & 1 & -1\\0 & -1 & 0\\0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1 & 1 & -1\\0 & -1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

We see that

$$\begin{aligned}\alpha &= \beta\,\mathtt{A}\\ \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\0 & -1 & 0\\0 & 1 & 1\end{bmatrix}\end{aligned}$$

3 Affine space

Let us study affine space, an important structure underlying geometry and its algebraic representation. Affine space is closely connected to linear space. The connection is so intimate that they are sometimes not even distinguished. Consider, for instance, function $f\colon \mathbb{R} \to \mathbb{R}$ with non-zero $a, b \in \mathbb{R}$

$$f(x) = a\,x + b \tag{3.1}$$

It is often called "linear" but it is not a *linear function* [5, 9, 3] since for every $\alpha \in \mathbb{R}$ there holds

$$f(\alpha\,x) = \alpha\,a\,x + b \neq \alpha\,(a\,x + b) = \alpha\,f(x) \tag{3.2}$$

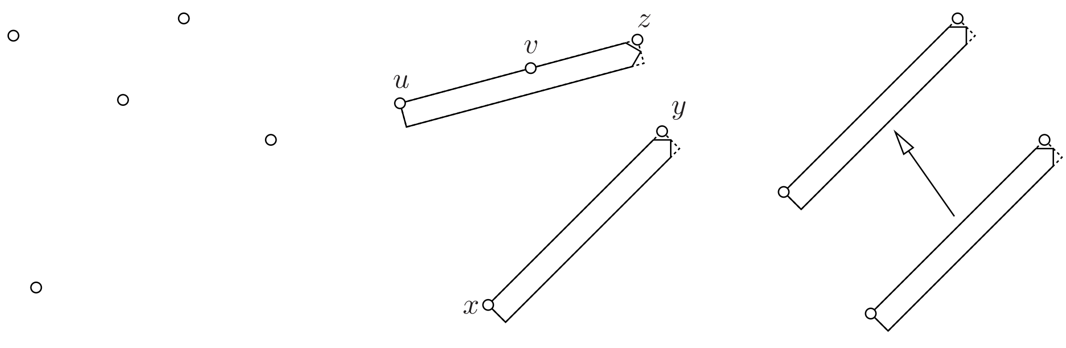
In fact, f is an *affine function*, which becomes linear function only for $b = 0$.

In geometry, we need to be very precise and we have to clearly distinguish affine from linear. Let us therefore first review the very basics of linear spaces, and in particular their relationship to geometry, and then move to the notion of affine space.

3.1 Vectors

Let us start with geometric vectors and study rules of their manipulation.

Figure 3.1(a) shows space of points P, which we live in and intuitively understand. We know what is an oriented line segment, which we also call a *marked ruler* (or just a ruler). A marked ruler is oriented from its origin towards its end, which is actually a mark (represented by an arrow) on a thought infinite ruler, Figure 3.1(b). We assume that we are able to align the ruler with any pair of points x, y, so that



(a) (b) (c)

Figure 3.1: (a) Space consists of points. Rulers (marked oriented line segments) can be aligned (b) and translated (c) and thus used to transfer, but not measure, distances.

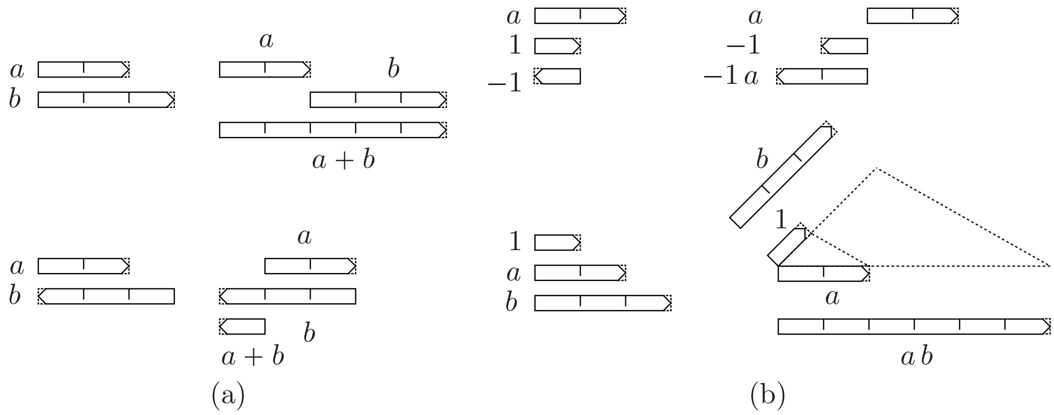


Figure 3.2: Scalars are represented by oriented rulers. They can be added (a) and multiplied (b) purely geometrically by translating and aligning rulers. Notice that we need to single out unit scalar "1" to perform geometric multiplication.

the ruler begins in x and a mark is made at the point y. We also know how to align a marked ruler with any pair of distinct points u, v such that the ruler begins in u and aligns with the line connecting u and v in the direction towards point v. The mark on so aligned ruler determines point z, which is collinear with points u, v. We know how to translate, Figure 3.1(c), a ruler in this space.

To define geometric vectors, we need to define geometric scalars.

3.1.1 Geometric scalars

Geometric scalars S are horizontal oriented rulers. The ruler, which has its origin identical with its end is called 0. Geometric scalars are equipped with two geometric operations, addition $a + b$ and multiplication $a\,b$, defined for every two elements $a, b \in S$.

Figure 3.2(a) shows addition $a + b$. We translate ruler b to align origin of b with the end of a and obtain ruler $a + b$.

Figure 3.2(b) shows multiplication $a\,b$. To perform multiplication, we choose a unit ruler "1" and construct its additive inverse -1 using $1 + (-1) = 0$. This introduces orientation to scalars. Scalars aiming to the same side as 1 are *positive* and scalars aiming to the same side as -1 are *negative*. Scalar 0 is neither positive, nor negative. Next we define multiplication by -1 such that $-1\,a = -a$, i.e. -1 times a equals the additive inverse of a. Finally, we define multiplication of non-negative (i.e. positive and zero) rulers a, b as follows. We align a with 1 such that origins of 1 and a coincide and they contain an acute non-zero angle. We align b with 1 and construct ruler $a\,b$ by translation as shown in the figure.

All constructions were purely geometrical and were performed with real rulers. We can verify that so defined addition and multiplication of geometric scalars satisfy all rules of addition and multiplication of real numbers. Geometric scalars form field [6, 10] w.r.t. to $a + b$ and $a\,b$.

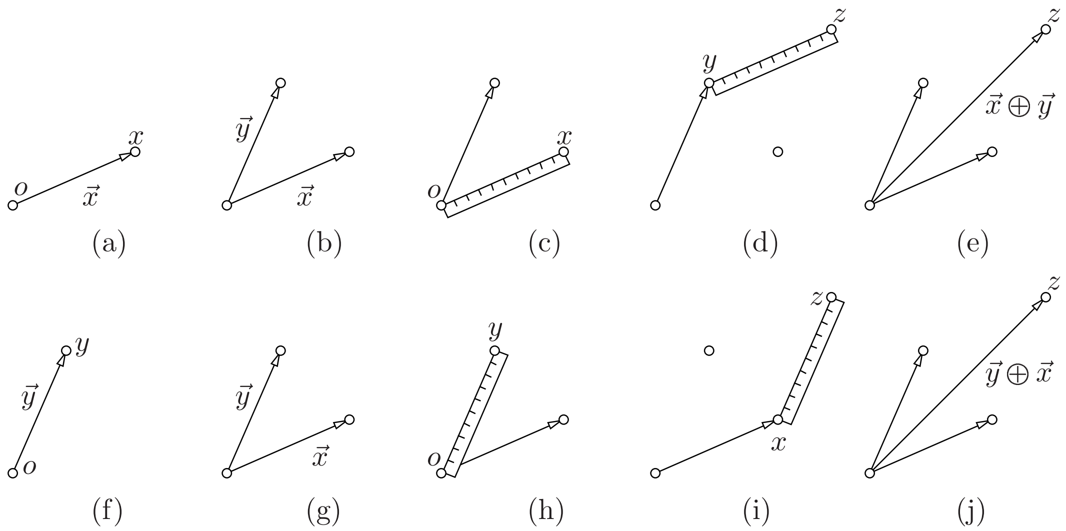


Figure 3.3: Bound vectors are (ordered) pairs of points (o, x), i.e. arrows $\vec{x} = (o, x)$. Addition of the bound vectors $\vec{x}$, $\vec{y}$ is realized by parallel transport (using a ruler). We see that the result is the same whether we add $\vec{x}$ to $\vec{y}$ or $\vec{y}$ to $\vec{x}$. The addition is commutative.

3.1.2 Geometric vectors

Ordered pairs of points, such as (x, y) in Figure 3.3(a), are called *geometric vectors* and denoted as $\overrightarrow{xy}$, i.e. $\overrightarrow{xy} = (x, y)$. Symbol $\overrightarrow{xy}$ is often replaced by a simpler one, e.g. by $\vec{a}$. The set of all geometric vectors is denoted by A.

3.1.3 Bound vectors

Let us now choose one point o and consider all pairs (o, x), where x can be any point, Figure 3.3(a). We obtain a subset A_o of A, which we call *geometric vectors bound to o*, or just *bound vectors* when it is clear to which point they are bound. We will write $\vec{x} = (o, x)$. Figure 3.3(f) shows another bound vector $\vec{y}$. The pair (o, o) is special. It will be called the *zero bound vector* and denoted by $\vec{0}$. We will introduce two operations $\oplus, \odot$ with bound vectors.

First we define *addition of bound vectors* $\oplus\colon A_o \times A_o \to A_o$. Let us add vector $\vec{x}$ to $\vec{y}$ as shown on Figure 3.3(b). We take a ruler and align it with $\vec{x}$, Figure 3.3(c). Then we translate the ruler to align its begin with point y, Figure 3.3(d). The end of the ruler determines point z. We define a new bound vector, which we denote $\vec{x} \oplus \vec{y}$ to correspond to the pair (o, z), Figure 3.3(e). Figures 3.3(f-j) demonstrate that addition gives the same result when we exchange (*commute*) vectors $\vec{x}$ and $\vec{y}$, i.e. $\vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x}$. We notice that for every point x, there is exactly one point x' such that $(o, x) \oplus (o, x') = (o, o)$, i.e. $\vec{x} \oplus \vec{x}' = \vec{0}$. Bound vector $\vec{x}'$ is the *inverse* to $\vec{x}$ and is denoted as $-\vec{x}$. Bound vectors are invertible w.r.t. operation $\oplus$. Finally, we see that $(o, x) \oplus (o, o) = (o, x)$, i.e. $\vec{x} \oplus \vec{0} = \vec{x}$. Vector $\vec{0}$ is the *identity element* of the operation $\oplus$. Clearly, operation $\oplus$ behaves exactly as addition of scalars – it is a commutative group [6, 10].

Secondly, we define the *multiplication of bound vector by a geometric scalar* $\odot\colon S \times A_o \to A_o$, where S are geometric scalars and A_o are bound vectors. Operation $\odot$ is a mapping which takes a geometric scalar (a ruler) and a bound vector and delivers another bound vector.

Figure 3.4 shows that to multiply a bound vector $\vec{x} = (o, x)$ by a geometric scalar a, we consider the ruler b whose origin can be aligned with o and end with x. We multiply scalars a and b to obtain scalar $a\,b$ and align $a\,b$ with $\vec{x}$ such that the origin of $a\,b$ coincides with o and $a\,b$ extends along the line passing through $\vec{x}$. We obtain end point y of so placed $a\,b$ and construct the resulting vector $\vec{y} = a \odot \vec{x} = (o, y)$.

We notice that addition $\oplus$ and multiplication $\odot$ of horizontal bound vectors coincides exactly with addition and multiplication of scalars.

3.2 Linear space

We can verify that for every two geometric scalars $a, b \in S$ and every three bound vectors $\vec{x}, \vec{y}, \vec{z} \in A_o$ with their respective operations, there holds the following eight rules

$$\begin{aligned}
\vec{x} \oplus (\vec{y} \oplus \vec{z}) &= (\vec{x} \oplus \vec{y}) \oplus \vec{z} \\
\vec{x} \oplus \vec{y} &= \vec{y} \oplus \vec{x} \\
\vec{x} \oplus \vec{0} &= \vec{x} \\
\vec{x} \oplus -\vec{x} &= \vec{0} \\
1 \odot \vec{x} &= \vec{x} \\
(a\,b) \odot \vec{x} &= a \odot (b \odot \vec{x}) \\
a \odot (\vec{x} \oplus \vec{y}) &= (a \odot \vec{x}) \oplus (a \odot \vec{y}) \\
(a + b) \odot \vec{x} &= (a \odot \vec{x}) \oplus (b \odot \vec{x})
\end{aligned}$$

These rules are known as axioms of *linear space* [5, 9, 11]. Bound vectors are one particular model of linear space. There are many other very useful models, e.g. n-tuples of real or rational numbers for any natural n, polynomials, series of real numbers and real functions. We will give some particularly simple examples useful in geometry later.

The next concept we will introduce are *coordinates of bound vectors*. To illustrate this concept, we will work in a plane. Figure 3.5 shows two non-collinear bound vectors $\vec{b}_1$, $\vec{b}_2$, which we call *basis*, and another bound vector $\vec{x}$. We see that there is only one way how to choose scalars x_1 and x_2 such that vectors $x_1 \odot \vec{b}_1$ and $x_2 \odot \vec{b}_2$

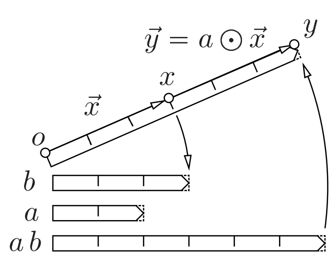


Figure 3.4: Multiplication of the bound vector $\vec{x}$ by a geometric scalar a is realized by aligning rulers to vectors and multiplication of geometric scalars.

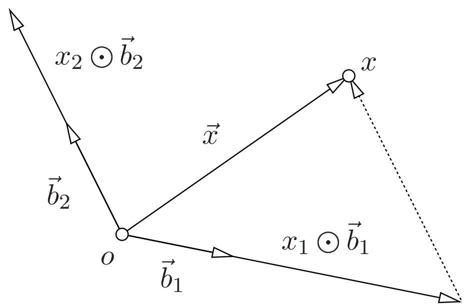


Figure 3.5: Coordinates are the unique scalars that combine independent basic vectors $\vec{b}_1$, $\vec{b}_2$ into $\vec{x}$.

add to $\vec{x}$, i.e.

$$\vec{x} = x_1 \odot \vec{b}_1 \oplus x_2 \odot \vec{b}_2$$

3.3 Free vectors

We can choose any point from P to construct bound vectors and all such choices will lead to the same manipulation of bound vector and to the same axioms of linear space. Figure 3.6 shows two such choices for points o and o'.

We take bound vectors $\vec{b}_1 = (o, b_1)$, $\vec{b}_2 = (o, b_2)$, $\vec{x} = (o, x)$ at o and construct bound vectors $\vec{b}'_1 = (o', b'_1)$, $\vec{b}'_2 = (o', b'_2)$, $\vec{x}' = (o', x')$ at o' by translating x to x', b_1 to b'_1 and b_2 to b'_2 by the same translation. Coordinates of $\vec{x}$ w.r.t. $[\vec{b}_1, \vec{b}_2]$ equal coordinates of $\vec{x}'$ w.r.t. $[\vec{b}'_1, \vec{b}'_2]$. This interesting property allows us to construct another model of linear space, whic plays an important role in geometry.

Let us now consider the set of all geometric vectors A. Figure 3.7(a) shows an example of a few points and a few geometric vectors. Let us *partition* the set A of geometric vectors into disjoint subsets $A_{(o,x)}$ such that we choose one bound vector (o, x) and put to $A_{(o,x)}$ all geometric vectors that can be obtained by translation of (o, x). Figure 3.7(b) shows two such partitions $A_{(o,x)}$, $A_{(o,y)}$. It is clear that

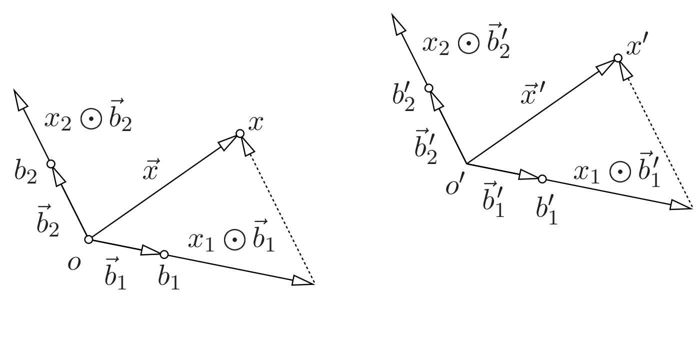


Figure 3.6: Two sets of bound vectors A_o and $A_{o'}$. Coordinates of $\vec{x}$ w.r.t. $[\vec{b}_1, \vec{b}_2]$ equal coordinates of $\vec{x}'$ w.r.t. $[\vec{b}'_1, \vec{b}'_2]$.

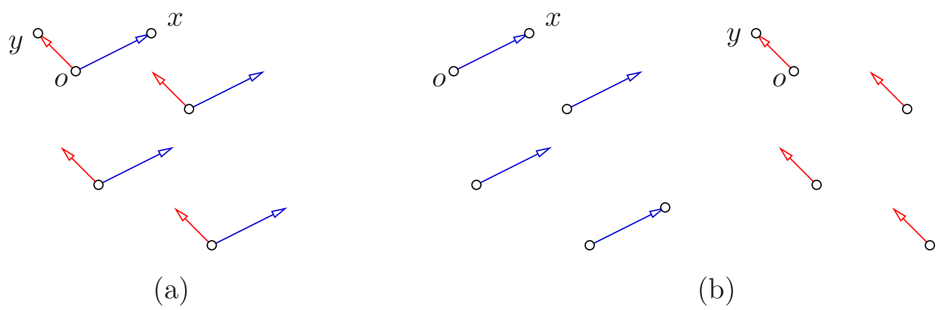


Figure 3.7: The set A of all geometric vectors (a) can be partitioned into subsets which are called free vectors. Two free vectors $A_{(o,x)}$ and $A_{(o,y)}$, i.e. subsets of A, are shown in (b).

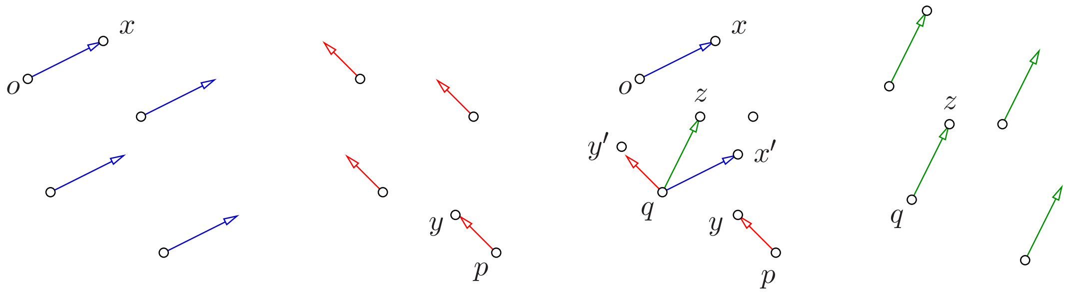


Figure 3.8: Free vector $A_{(o,x)}$ is added to free vector $A_{(p,y)}$ by translating (o,x) to (q,x') and (p,y) to (q,y'), adding bound vectors $(q,z) = (q,x') \oplus (q,y')$ and setting $A_{(o,x)} \boxplus A_{(p,y)} = A_{(q,z)}$

$A_{(o,x)} \cap A_{(o,x')} = \varnothing$ for $x \neq x'$ and that every geometric vector is in some (and in exactly one) subset $A_{(o,x)}$.

Two geometric vectors (o,x) and (o',x') form two subsets $A_{(o,x)}$, $A_{(o',x')}$ which are equal if and only if (o,x) and (o',x') are related by a translation.

"To be related by a translation" is an equivalence relation [1]. All geometric vectors in $A_{(o,x)}$ are equivalent to (o,x).

There are as many sets in the partition as there are bound vectors at a point. We can define the partition by geometric vectors bound to any point o because if we choose another point o', then for every point x, there is exactly one point x' such that (o,x) can be translated to (o',x').

We denote the set of subsets $A_{(o,x)}$ by V. Let us see that we can equip set V with meaningful addition $\boxplus\colon V \times V \to V$ and multiplication $\boxdot\colon S \times V \to V$ by geometric scalars S such that it will become a model of linear space. Elements of V will be called *free vectors*.

We define sum of $\vec{x} = A_{(o,x)}$ and $\vec{y} = A_{(o,y)}$, i.e. $\vec{z} = \vec{x} \boxplus \vec{y}$ as the set $A_{(o,x)\oplus(o,y)}$. Multiplication of $\vec{x} = A_{(o,x)}$ by geometrical scalar a is defined analogically, i.e. $a \boxdot \vec{x}$ equals the set $A_{a \odot (o,x)}$. We see that the result of $\boxplus$ and $\boxdot$ does not depend on the choice of o. We have constructed linear space V of free vectors.

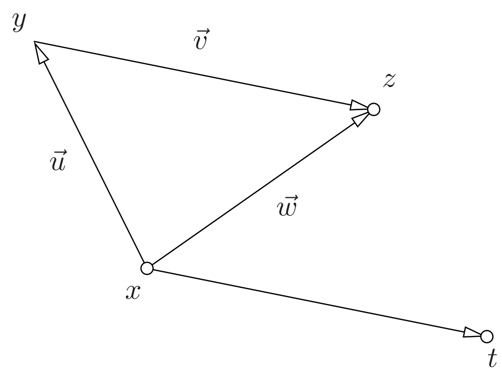


Figure 3.9: Free vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ defined by three points x, y and z satisfy triangle identity $\vec{u} \boxplus \vec{v} = \vec{w}$.

3.4 Affine space

We saw that bound vectors and free vectors were (models of) linear space. On the other hand, we see that the set of geometric vectors A is not (a model of) linear space because we do not know how to meaningfully add (by translation) geometric vectors which are not bound to the same point. The set of geometric vectors is *affine space*.

Affine space connects points, geometric scalars, bound geometric vectors and free vectors in a natural way.

Two points x and y, in this order, give one geometric vector (x, y), which determines exactly one free vector $\vec{v} = A_{(x,y)}$. We define function $\varphi\colon P \times P \to V$, which assigns to two points $x, y \in \mathcal{P}$ their corresponding free vector $\varphi(x, y) = A_{(x,y)}$.

Consider a point $o \in P$ and a free vector $\vec{x} \in V$. There is exactly one geometric vector (o, x), with o at the first position, in the free vector $\vec{x}$. Therefore, point o and free vector $\vec{x}$ uniquely define point x. We define function $\#\colon P \times V \to P$, which takes a point and a free vector and delivers another point. We write $o \# \vec{x} = x$ and see that $\vec{x} = \varphi(o, x)$.

Consider three points $x, y, z \in P$, Figure 3.9. We can produce three free vectors $\vec{u} = \varphi(x, y) = A_{(x,y)}$, $\vec{v} = \varphi(y, z) = A_{(x,y)}$, $\vec{w} = \varphi(x, z) = A_{(x,z)}$. Let us investigate the sum $\vec{x} \boxplus \vec{y}$. Chose representatives of the free vectors, such that they are all bound to x, i.e. bound vectors $(x, y) \in A_{x,y}$, $(x, t) \in A_{(y,z)}$ and $(x, z) \in A_{(x,z)}$. Notice that we could choose pairs of the original points to represent the first and the third free vector but we had to introduce a new pair of points (x, t) to represent the second free vector. Clearly, there holds $(x, y) \oplus (x, t) = (x, z)$. We now see, Figure 3.9, that (y, z) is related to (x, t) by a translation and therefore

$$\vec{u} \boxplus \vec{v} = A_{(x,y)} \boxplus A_{(y,z)} = A_{(x,y)} \boxplus A_{(x,t)} = A_{(x,z)} = \vec{w}$$

The above rules are known as *axioms of affine space* and can be used to define more general affine spaces.

Triple (P, L, φ) with a set of points P, linear space $(L, \boxplus, \boxdot)$ (over some field of scalars) and surjective (onto) [1] function $\varphi\colon P \times P \to L$, is affine space when there

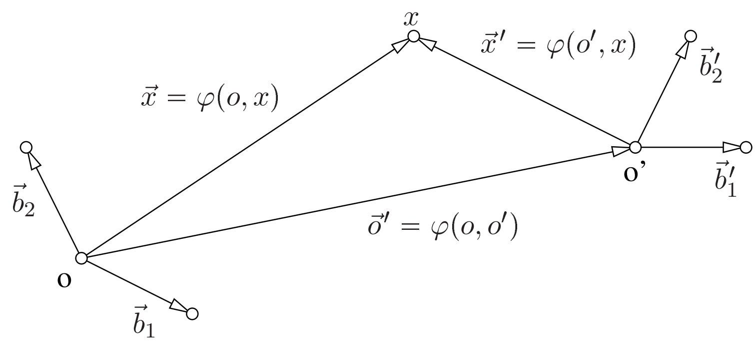


Figure 3.10: Point x is represented in two affine coordinate systems.

holds for every three points $x, y, z \in P$

$$\varphi(x,y) = \varphi(x,z) \quad \Rightarrow \quad y = z \tag{3.3}$$

$$\varphi(x,z) \quad = \quad \varphi(x,y) \boxplus \varphi(y,z) \tag{3.4}$$

Function $\#$ is defined by $\varphi(a, a\#\vec{x}) = \vec{x}$ for all $a \in A$ and $\vec{x} \in L$.

In our geometrical model of A discussed above, function φ assigned to pairs of points x, y their corresponding free vector $A_{(x,y)}$.

3.5 Coordinate system in affine space

We see that function φ assigns the same vector from L to many different pairs of points from P. To represent uniquely points by vectors, we select a point o, called *origin of affine coordinate system* and represent point $x \in P$ by $\vec{x} = \varphi(o, x)$. In our geometric model of A discussed above, we thus represent point x by bound vector (o, x) or by point o and free vector $A_{(o,x)}$.

To be able to compute with points, we now pass to the representation of points in A by coordinate vectors. We choose a basis $\beta = (\vec{b}_1, \vec{b}_2, \ldots)$ in L. That allow us to represent point $x \in P$ by a coordinate vector

$$\vec{x}_\beta = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}, \quad \text{such that} \quad \vec{x} = x_1\,\vec{b}_1 + x_2\,\vec{b}_2 + \cdots$$

The pair (o, β), where $o \in P$ and β a basis of L is called *affine coordinate system* (often shortly called just coordinate system) of affine space (P, L, φ).

Let us now study what happens when we choose another point o' and another basis $\beta' = (\vec{b}'_1, \vec{b}'_2, \ldots)$ to represent $x \in P$ by coordinate vectors, Figure 3.10. Point x is represented twice: by coordinate vector $\vec{x}_\beta = \varphi(o, x)_\beta = A_{(o,x)\beta}$ and by coordinate vector $\vec{x}'_{\beta'} = \varphi(o', x)_{\beta'} = A_{(o',x)\beta'}$.

To get the relationship between the coordinate vectors $\vec{x}_\beta$ and $\vec{x}'_{\beta'}$, we employ the triangle equality

$$\begin{aligned} \varphi(o,x) &= \varphi(o,o') \boxplus \varphi(o',x) \\ \vec{x} &= \vec{o}' \boxplus \vec{x}' \end{aligned}$$

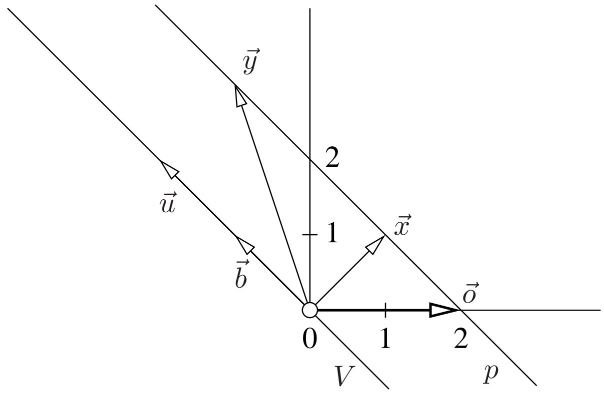


Figure 3.11: Affine space (P, V, φ) of solutions to a linear system is the set of vectors representing points on line p. In coordinate system $(\vec{o}, \vec{u})$, vector $\vec{x}$ has coordinate 1. The subspace V of solutions to the accompanied homogeneous system is the associate linear space. Function φ assigns to two points $\vec{o}$, $\vec{x}$ the vector $\vec{u} = \vec{y} - \vec{x}$.

which we can write in basis β as (notice that we replace $\boxplus$ by $+$ to emphasize that we are adding coordinate vectors)

$$\vec{x}_\beta \quad = \quad \vec{x}'_\beta + \vec{o}'_\beta$$

and use the matrix $\mathtt{A}$ transforming coordinates of vectors from basis β' to β to get the desired relationship

$$\vec{x}_\beta \quad = \quad \mathtt{A}\,\vec{x}'_{\beta'} + \vec{o}'_\beta \tag{3.5}$$

Columns of $\mathtt{A}$ correspond to coordinate vectors $\vec{b}'_{1\beta}, \vec{b}'_{2\beta}, \ldots$. When presented with a situation in real affine space, we can measure those coordinates by a ruler on a particular representation of L by geometrical vectors bound to, e.g. point o.

§ **5 Remark on notation** We were carefully distinguishing operations with $(+,\ )$ with scalars, $(\oplus, \odot)$ bound vectors, $(\boxplus, \boxdot)$ free vectors, and function $\#$ combining points and free vectors. This is very correct but rarely used. Often, only the symbols introduced for geometric scalars are used of all operations, i.e.

$$\begin{aligned} + &\equiv +, \oplus, \boxplus, \# \\ &\equiv \ \ , \odot, \boxdot \end{aligned}$$

3.6 Examples of affine spaces

Let us present a few important examples of affine spaces.

3.6.1 Affine space of solutions of a system of linear equations

We look at at the following system of linear equations in $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \tag{3.6}$$

we immediately see that there is an infinite number of solutions. They can be written as

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \tau \in \mathbb{R}$$

or as a sum of a particular solution $[2,0]^\top$ and the set of solutions $\vec{v} = \tau\,[-1,1]^\top$of accompanied homogeneous system

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.7}$$

Figure 3.11 shows that the affine space (P, V, φ) of solutions to the linear system (3.6) is the set of vectors representing points on line p. The subspace V of solutions to the accompanied homogeneous system (3.7) is the linear space associated to A by function φ, which assigns to two points $\vec{x}, \vec{y} \in A$ the vector $\vec{u} = \vec{y} - \vec{x} \in V$. If we choose $\vec{o} = [2,0]^\top$ as the origin in A and vector $\vec{b} = \varphi(\vec{o}, \vec{x}) = \vec{x} - \vec{o}$ as the basis of V, vector $\vec{x}$ has coordinate 1.

We see that in this example, points of A are actually vectors of $\mathbb{R}^2$, which are the solution to the system (3.6). The vectors of V are the vectors of $\mathbb{R}^2$, which are solutions to the associated homogeneous linear system (3.7).

3.6.2 Small affine spaces

Let us now look at small affine spaces. These are structures that satisfy axioms of affine space and has very small number of points. They do not represent geometry of normal space around us but satisfy axioms, which we have distilled out from the properties of the geometric affine space studied above. They illuminate the structure of space we live in.

§ **6 One-point affine space** The *one-point affine space* with $P = \{x\}$ has associated trivial linear space $L = \{\vec{0}\}$, which has only zero vector and function $\varphi(x, x) = \vec{0}$. This is *trivial affine space*.

§ **7 Two-point affine space** Let us construct *two-point affine space*. If there was an affine space with $P = \{x, y\}$, then there would be a linear space L associated with P such that φ would exhaust all vectors in L and the axioms 3.3, 3.4 would hold. There are four ordered pairs (x, x), (y, y), (x, y), (y, x). Due to triangle equality $\varphi(x, x) = \varphi(y, y) = \vec{0}$ (consider, e.g., $\varphi(x, x) + \varphi(x, x) = \varphi(x, x)$) and $\varphi(x, y) = -\varphi(y, x)$ (consider $\varphi(x, y) + \varphi(y, x) = \varphi(x, x)$). Now, φ must exhaust all vectors in L. We have already seen that the $\varphi(x, x) = \varphi(y, y) = \vec{0}$. Assigning $\varphi(x, y) = \vec{0}$ is not possible since $\varphi(x, y) = \varphi(x, x)$ but $x \neq y$. The same holds true for $\varphi(y, x)$. Therefore, $\varphi(x, y) = \varphi(y, x) = \vec{u}$ for some $\vec{0} \neq \vec{u} \in L$. This calls for a linear space with exactly two vectors. Let us consider linear space $L = \{0, 1\}$ with addition defined by $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$ and $1 + 1 = 0$ and multiplication defined by $0 \cdot 0 = 0$, $0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$ (Check that all axioms of linear space hols true). We know this linear space also as binary numbers with modulo-two arithmetic. Define φ as $\varphi(x, x) = \varphi(y, y) = 0$, $\varphi(x, y) = \varphi(y, x) = 1$. We see that φ is onto. There are no two pairs that would agree on the first component and would be mapped by φ to the same vector. Hence axiom 3.3 holds true. Finally,

$\varphi(x,x) + \varphi(x,x) = 0+0 = 0 = \varphi(x,x)$, $\varphi(x,x) + \varphi(x,y) = 0+1 = 1 = \varphi(x,y)$, $\varphi(x,y) + \varphi(y,x) = 1+1 = 0 = \varphi(x,x)$, $\varphi(x,y) + \varphi(y,y) = 1+0 = 1 = \varphi(x,y)$, $\varphi(y,y) + \varphi(y,y) = 0+0 = 0 = \varphi(y,y)$ verify that axiom 3.4 holds true as well.

§ **8 Three-point affine space** If there was affine space with $P = \{x,y,z\}$, then there would be a linear L space associated with P such that $\varphi(x,y)$ would exhaust all vectors in L. There are nine ordered pairs (x,x), (y,y), (z,z), (x,y), (y,x), (y,z), (z,y), (x,z), (z,x). Due to triangle equality $\varphi(x,x) = \varphi(y,y) = \varphi(z,z) = \vec{0}$ and $\varphi(x,y) = -\varphi(y,x)$, $\varphi(y,z) = -\varphi(z,y)$, $\varphi(x,z) = -\varphi(z,x)$. Clearly function φ can't assign a pair of different points to $\vec{0}$. Hence $\vec{0} \neq \varphi(x,y) = \vec{u} \in L$ and $\vec{0} \neq \varphi(x,z) = \vec{v} \in L$ and $\vec{v} \neq \vec{u}$ since $x \neq z$. Let us try $\varphi(y,z) = \vec{u}$. Then $\vec{u} + \vec{u} = \varphi(x,y) + \varphi(y,z) = \varphi(y,z) = \vec{u}$ implies $\vec{u} = \vec{0}$, which is not possible. Let us try $\varphi(y,z) = \vec{v}$. Then $\vec{0} = \vec{v} - \vec{v} = \varphi(y,z) + \varphi(z,x) = \varphi(y,x) = -\vec{u}$ implies $\vec{u} = \vec{0}$, which is also not possible. Hence, $\varphi(y,z) = \vec{w}$ with $\vec{w} \neq \vec{0}$, $\vec{w} \neq \vec{u}$ and $\vec{w} \neq \vec{v}$. Consider the linear space $L = \{0,1\}^2$, i.e. ordered pairs of binary numbers with coordinate-wise addition and multiplication in modulo-two arithmetic (Verify that it indeed is a linear space). Now, make $x = [1,0]^\top$, $y = [0,1]^\top$ and $z = [1,1]^\top$ and $\varphi(u,v) = v - u$ for $u,v \in A$. We see that $\varphi(u,u) = u - u = [0,0]^\top$ for every $u \in A$, $\varphi(u,v) = v - u = -(u-v) = -\varphi(v,u)$. We get $\varphi(x,x) = \varphi(y,y) = \varphi(z,z) = [0,0]^\top$, $\varphi(x,y) = \varphi(y,x) = [1,1]^\top$, $\varphi(y,z) = \varphi(z,y) = [1,0]^\top$, $\varphi(x,z) = \varphi(z,x) = [0,1]^\top$. We see that φ is onto. There are no two pairs that would agree on the first component and would be mapped by φ to the same vector. Hence axiom 3.3 holds true. Finally, $\varphi(u,v) + \varphi(v,w) = v - u + w - v = w - u = \varphi(u,w)$ verifies that axiom 3.4 holds true as well.

§ **9 Four-point affine space** To construct the four-point affine space, we extend the previous example of the three-point affine space. Let us set $P = \{x,y,z,w\}$, with $x = [1,0]^\top$, $y = [0,1]^\top$, $z = [1,1]^\top$, $w = [0,0]^\top$ and $\varphi(u,v) = v - u$ for $u,v \in A$. We see that $\varphi(u,u) = u - u = [0,0]^\top$ for every $u \in A$, $\varphi(u,v) = v - u = -(u-v) = -\varphi(v,u)$ and $\varphi(u,v) + \varphi(v,w) = v - u + w - v = w - u = \varphi(u,w)$. Clearly φ is onto and the triangle equality, axiom 3.4, holds true. We only need to verify axiom 3.3. Let us consider sets of point pairs that map to the same vector. On pairs from x,y,z, the φ coincides with three-point affine plane and hence for them axiom 3.3 holds true. We need to check assignments $\varphi(x,w) = [1,0]^\top$, $\varphi(y,w) = [0,1]^\top$, $\varphi(z,w) = [1,1]^\top$ and $\varphi(w,w) = [0,0]^\top$. We see that pre-image of $[1,0]^\top$, $\varphi^{-1}([1,0]^\top) = \{(y,z),(z,y),(x,w),(w,x)\}$ has no repetition of the first component of its elements. The same can be verified for preimages of $[0,1]^\top$ and $[1,1]^\top$. Axiom 3.3 holds true.

The above linear space is not the only linear space with four vectors. The set $L' = \{0,1,t,t+1\}$ of polynomials in t with addition modulo polynomial t^2+t+1 forms a linear space over itself. It is a finite field with four elements (Verify the axioms). Set $P = L'$ and $\varphi(u,v) = v - u$. Clearly φ is onto and axiom 3.4 holds true. Sets $\varphi^{-1}(0) = \{(x,x),(y,y),(z,z),(w,w)\}$, $\varphi^{-1}(1) = \{(y,z),(z,y),(x,w),(w,x)\}$, etc., demonstrate that axiom 3.3 holds true.

The above two examples are interesting because they represent two quite different affine spaces. Consider that the first example clearly represents a two-dimensional linear space since vectors $[0,1]^\top$ and $[1,0]^\top$ are linearly independent. The second example, on the other hand, represents only one-dimensional space since all vectors

	l_1	l_2	l_3	l_4	l_5	l_6
P_1	$\circ$	$\circ$	$\circ$			
P_2	$\circ$			$\circ$	$\circ$	
P_3		$\circ$		$\circ$		$\circ$
P_4			$\circ$		$\circ$	$\circ$

(a)

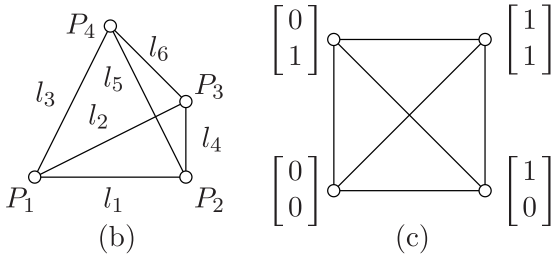


Figure 3.12: Four-point affine plane models. (a) Incidence table. (b) Geometrical incidence model. (c) Linear algebraic model.

in the space can be generates as multiples of 1 as $0 = 0\,1$, $1 = 1\,1$, $t = t\,1$, $t + 1 = (t + 1)\,1$.

We also see that the first example of the four-point affine space is actually the smallest affine plane, i.e. the smallest affine space that is based on a two dimensional linear space. It is clear that two dimensional linear space has to have at least four vectors. Since it has to have the zero vector, two basic vectors and the sum of the two basic vectors must give yet another non-zero vector.

3.7 Incidence axioms of affine spaces

We saw that affine space with the same number of points can be represented by linear spaces of different dimensions. This suggests that models of affine spaces can be quite different. Indeed there are some very interesting affine spaces which can't even be represented by a linear space. To get such spaces, however, we have to pass to a different set of axioms that will not be incorporating linear space into the foundations of affine space.

We can verify that all affine spaces according to the definition in Section 3.4 satisfy the following rules that we shall adopt as axioms to arrive at more cases of affine spaces. Let us first define a few new concepts. Then we will formulate axioms of synthetic affine spaces and show interesting examples.

Let P be a set. We define triple $(\mathcal{P}, \mathcal{L}, \mathcal{E})$ of points $\mathcal{P} \subset \exp P$, lines $L \subset \exp P$ and planes $\mathcal{E} \subset \exp P$. Points, lines and planes are subsets of $\mathcal{P}$. In particular, points are singleton sets, i.e. the sets containing exactly one element. We say that point $p \in \mathcal{P}$ is *incident* [12] with (lies in, is on, belongs to, is contained in) line $l \in \mathcal{L}$, and write $p \circ l$ when $p \cap l \neq \emptyset$, Similarly, p is incident with $\pi \in \mathcal{E}$ when $p \cap \pi \neq \emptyset$. Line $l \in \mathcal{L}$ *intersects* (meets) line $m \in \mathcal{L}$, when $l \cap m \neq \emptyset$, line $l \in \mathcal{L}$ intersects plane $\pi \in \mathcal{E}$ when $l \cap \pi \neq \emptyset$ and plane $\sigma \in \mathcal{E}$ intersects $\pi \in \mathcal{E}$ when $\sigma \cap \pi \neq \emptyset$. We say that line l is *parallel* to line m if they do not intersect and are contained in a plane. We say that points are collinear (coplanar) if they are contained in a line (plane).

We say that $(\mathcal{P}, \mathcal{L}, \mathcal{E})$ is *synthetic affine space* when the following *incidence axioms of affine space* [13] hold for the triple.

AS1 Every two distinct points are contained in exactly one line.

AS2 Every three distinct non-collinear points are contained in exactly one plane.

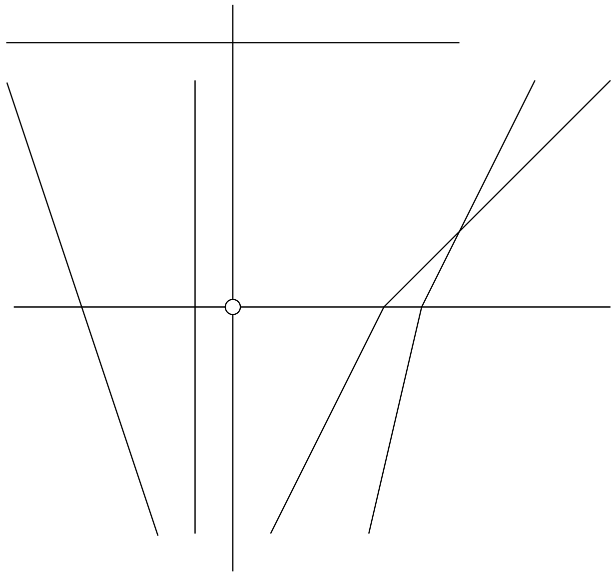


Figure 3.13: Moulton plane

AS3 If p is a point not contained in a line l, then there is a unique line m such that p is contained in m and m is parallel to l.

AS4 If k, l, m are distinct lines with k parallel to l and l parallel to m, then k is parallel to m.

This new definition encompasses all case defined above but also some other affine spaces that can't be represented by linear space with φ given by vector subtraction.

Figure 3.7 shows three models of the four-point affine plane. In Figure 3.7(a), the plane is defined by an incidence table. Another model is purely set-theoretical. We set $\mathcal{P} = \{\{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}\}$ and line segments connecting points are lines $\mathcal{L} = \{\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}\}$. There is only one plane $\mathcal{E} = P = \{P_1, P_2, P_3, P_4\}$. We can verify that axioms AS1–AS4 are satisfied but it is clear that axioms AS2 and AS4 wre met trivially in this case.

Figure 3.7(b) shows a geometrical model in which the four-point affine plane is represented by a tetrahedron. Vertices of the tetrahedron represent points. The edges of the tetrahedron represent lines. This model is actually just another visualization of the corresponding incidence table.

Figure 3.7(c) shows four-point affine space represented by two dimensional linear space over binary numbers. Notice that every point is represented by a vector and φ can be defined as difference of vectors representing points. This is a *linear model of affine space*.

Interestingly, it is possible to drop AS2 and AS4, simplify AS3, and to add enaothe axiom to arrive at a set of axioms of *synthetic affine planes* which are satisfied by all affine spaces represented by two dimensional linear spaces with φ equal to the difference of vectors representing points

AP1 Every two distinct points are contained in exactly one line.

AP2 There are three non-collinear points.

AP3 If p is a point not contained in a line l, then there is a unique line m such that m does not intersect l.

Synthetic affine planes represent an important class of affine spaces, some of which do not have a linear model. Consider, for instance, the Moulton plane [13]. Set

$$
\begin{aligned}
\mathcal{P} &= \mathbb{R}^2 \\
\mathcal{L} &= \{[x, y]^\top \in \mathbb{R}^2 \mid a, b, c \in \mathbb{R},\ a^2 + b^2 \neq 0, \\
&\quad\ a\,x + t(y)\,b\,y = c,\ t(y) = 1 \text{ for } y \geqslant 0 \text{ and } t(y) = 2 \text{ for } y < 0\,\} \\
[x, y]^\top \circ l \in \mathcal{L} &\text{ iff } [x, y]^\top \subset l
\end{aligned}
$$

See Figure 3.13 for examples of a few lines of the Moulton plane. We can check that Moulton plane satisfies axioms of synthetic affine space as well as axioms of synthetic affine plane. Its points are represented by vectors of $\mathbb{R}^2$ but lines are not one-dimensional subspaces of $\mathbb{R}^2$ and φ is not obtained by subtracting vectors representing the points.

4 Image coordinate system

Digital image `Im` is a matrix of pixels. We assume that `Im` is obtained by measuring intensity of light by sensors (pixels) arranged in a grid, Figure 4.1.

We will work with images in two ways. First, we will work with intensity values, which are stored in the memory as a three-dimensional array of bytes indexed by the row index `i`, the column index `j` and the color index `k`, Figure 4(a). Color index attains three values $1, 2, 3$, with 1 corresponding to red, 2 corresponding to green and 3 corresponding to blue.

In `Matlab`, image `Im` is accessed using the row index `i`, the column index `j` and color index `k` as `>>Im(i,j,k)`. The most top left pixel has row as well as column index equal to 1. The red channel of the pixel with row index `2` and column index `3` is accessed as `>>Im(2,3,1)`.

§ 10 Image coordinate system For geometrical computation, we introduce an *image coordinate system* as in Figure 4(b). The origin of the coordinate system is chosen to assign coordinates $1, 1$ to the center of the most top left pixel. Horizontal axis $\vec{b}_1$ goes from left to right. The vertical axis $\vec{b}_2$ goes from top down. Pixel, which is accessed as `>>Im(i,j,k)` is in the image coordinate system represented by the vector $\vec{u} = [\mathtt{j}, \mathtt{i}]^\top$. Digital image with H rows and W columns will in `Matlab` be indexed as `>>Im(1:H,1:W,1:3)` and `>>size(Im)` will return `[H W 3]`. In the image coordinate system the center of the most bottom right pixel will have coordinates $[W, H]^\top$.

The image coordinate system coincides with the `Matlab` coordinate system `image`, i.e. commands

```
>> axis ij
>> axis equal
>> plot(j,i,'.b')
```

plot a blue dot on the pixel accessed as `Im(i,j,k)`;

The image coordinate system is non-standard in two dimensions since it is a left-handed system. The reason for such a unnatural choice is that this system will be next augmented into a three-dimensional right-handed coordinate system in such a way that the $\vec{b}_3$ axis will be pointing towards the scene.

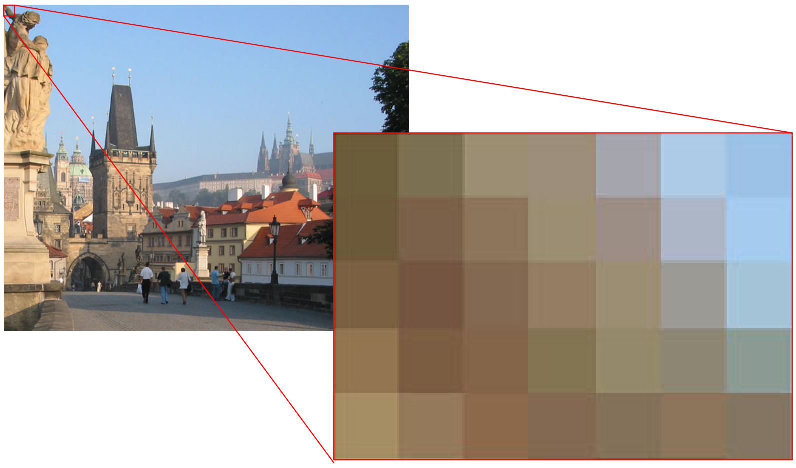
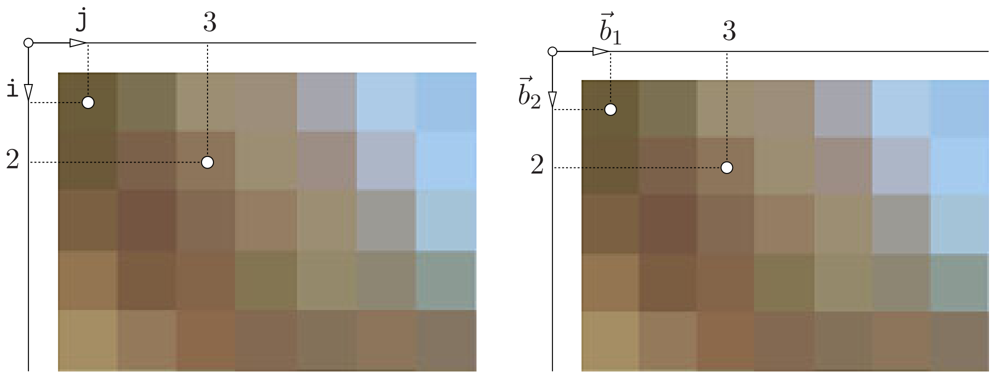


Figure 4.1: Image is digitized by a rectangular array of pixels



(a) Image `Im` is a matrix of pixels. In `Matlab`, it is accessed using the row index `i`, the column index `j` and color index `k` as `>>Im(i,j,k)`. The most top left pixel has row as well as column index equal to 1. The red channel of the pixel with row index `2` and column index `3` is accessed as `>>Im(2,3,1)`.

(b) The image coordinate system is defined with horizontal axis $\vec{b}_1$ and vertical axis $\vec{b}_2$. The origin of the coordinate system is chosen to to assign coordinates $1, 1$ to the most top left pixel. Notice that pixel, which is accessed as `>>Im(2,3,1)`, is represented in the image coordinate system by the vector $\vec{u} = [3, 2]^\top$.

Figure 4.2: Image coordinate system.

5 Perspective camera

Modern photographic camera, Figure 5.1, is an interesting and advanced device. We shall abstract from all physical and technical details of image formation and will concentrate solely on its geometry. From the point of view of geometry, a perspective camera projects point X from space into an image point x by intersecting the line connecting X with the *projection* center (red) and a planar image plane (green), Figure 5.1(b).

5.1 Perspective camera model

Let us now develop a mathematical model of the perspective camera. The model will allow us to project space point X into image point x and to find the ray p in space along the which point x has been projected.

§ **11 Camera coordinate system** Figure 5.2 shows the geometry of the perspective camera. Point X is projected along ray p from three-dimensional space to point x in two-dimensional image. Point x is obtained as the intersection of ray p with planar *projection plane* π. Ray p is constructed by joining point X with the *projection center* C.

The image plane is equipped with an image coordinate system (§ 10), (o, α), where o is the origin and $\alpha = [\vec{b}_1, \vec{b}_2]$ is the basis of the image coordinate system. Notice that the basis α is shown as non-orthogonal. We want to develop a general camera model, which will be applicable even in the situation when image coordinate system is not rectangular. Point x is represented by vector $\vec{u}$ in (o, α)

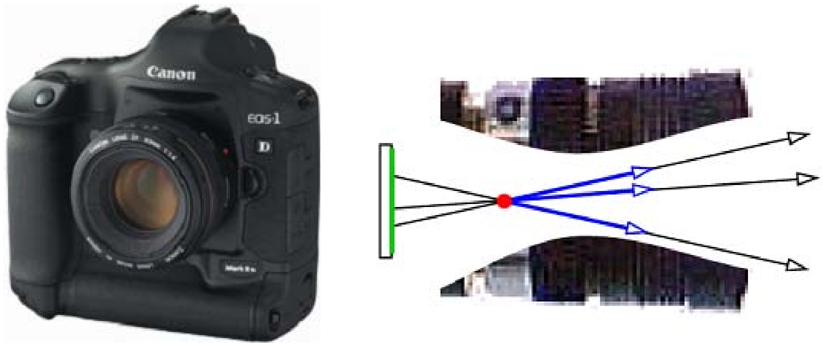
$$\vec{u} = u\,\vec{b}_1 + v\,\vec{b}_2 \qquad \text{i.e.} \qquad \vec{u}_\alpha = \begin{bmatrix} u \\ v \end{bmatrix} \tag{5.1}$$

Three-dimensional space is equipped with a *world coordinate system* (O, δ), where O is the origin and $\delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$ is a three-dimensional orthonormal basis. Point X is represented by vector $\vec{X}$ in (O, δ). The camera projection center is represented by vector $\vec{C}$ in (O, δ).

Let us next define the *camera coordinate system*. The system will be derived from the image coordinate system to make the construction of coordinates of the direction vector $\vec{x}$ of p extremely simple.

Camera coordinate system (C, β) has the origin in the projection center C and its basis $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ is constructed by re-using the two basis vectors of α and adding the third basic vector $\vec{b}_3$, which corresponds to vector $\overrightarrow{Co}$. We see that vectors in β form a basis when point C is not in π, which is satisfied for every meaningful perspective camera. Notice also that the camera coordinate system, is three-dimensional.

Image points o and x are in plane π, which is in three-dimensional space, and hence, we can consider them points of the space. Point x is in (C, β) represented by



(a) (b)

Figure 5.1: Perspective camera (a) is geometrically a point (red) and a projection pane (green) (b).

vector $\vec{x}$, which is the direction vector of the projection ray p along which point X has been projected into x. We see that vectors $\vec{u}$, $\vec{x}$, $\vec{b}_3$ form a triangle such that

$$\begin{aligned} \vec{x} &= \vec{u} + \vec{b}_3 && (5.2) \\ &= u\,\vec{b}_1 + v\,\vec{b}_2 + 1\,\vec{b}_3 && (5.3) \end{aligned}$$

and therefore

$$\vec{x}_\beta = \vec{x}_{[\vec{b}_1,\vec{b}_2,\vec{b}_3]} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}. \tag{5.4}$$

Notice that basis β has been constructed in a very special way to facilitate construction of $\vec{x}_\beta$. We can use u, v directly since β re-uses vectors of α and the third coordinate is always 1 by the construction of $\vec{b}_3$. Although we do not know exact position of C w.r.t. the image plane, we know that it is not in the plane π and hence a meaningful camera coordinate system constructed this way exists.

Notice next that the camera coordinate system is right-handed. This is because when looking at a scene from a point C through the projection plane, the image is constructed by intersecting projection rays with the projection plane, which is in front and hence the vector $\vec{b}_3$ points towards the scene. We see that vectors of β form a right-handed system.

Let us mention that we have used deeper properties of linear and affine spaces. In particular, we were making use of the concept of *free vector* in the following way. We look at vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{u}$ as on a free vectors. Therefore, coordinates of the representative of $\vec{u}$ beginning in o with respect to representatives of $\vec{b}_1$, $\vec{b}_2$ beginning in o equal the coordinates of the representative of $\vec{u}$ beginning in C with respect to representatives of $\vec{b}_1$, $\vec{b}_2$ beginning in C. Hence u, v reappear as the first two coordinates of $\vec{x}$.

For usual consumer cameras, vector $\vec{b}_3$ is often much longer than vectors $\vec{b}_1, \vec{b}_2$ and often not orthogonal to them. Therefore, basis β is in general neither orthonormal nor orthogonal! This has severe consequences since we can't measure angles and distances in the space using β, unless we find out what are the lengths of its vectors and what are the angles between them.

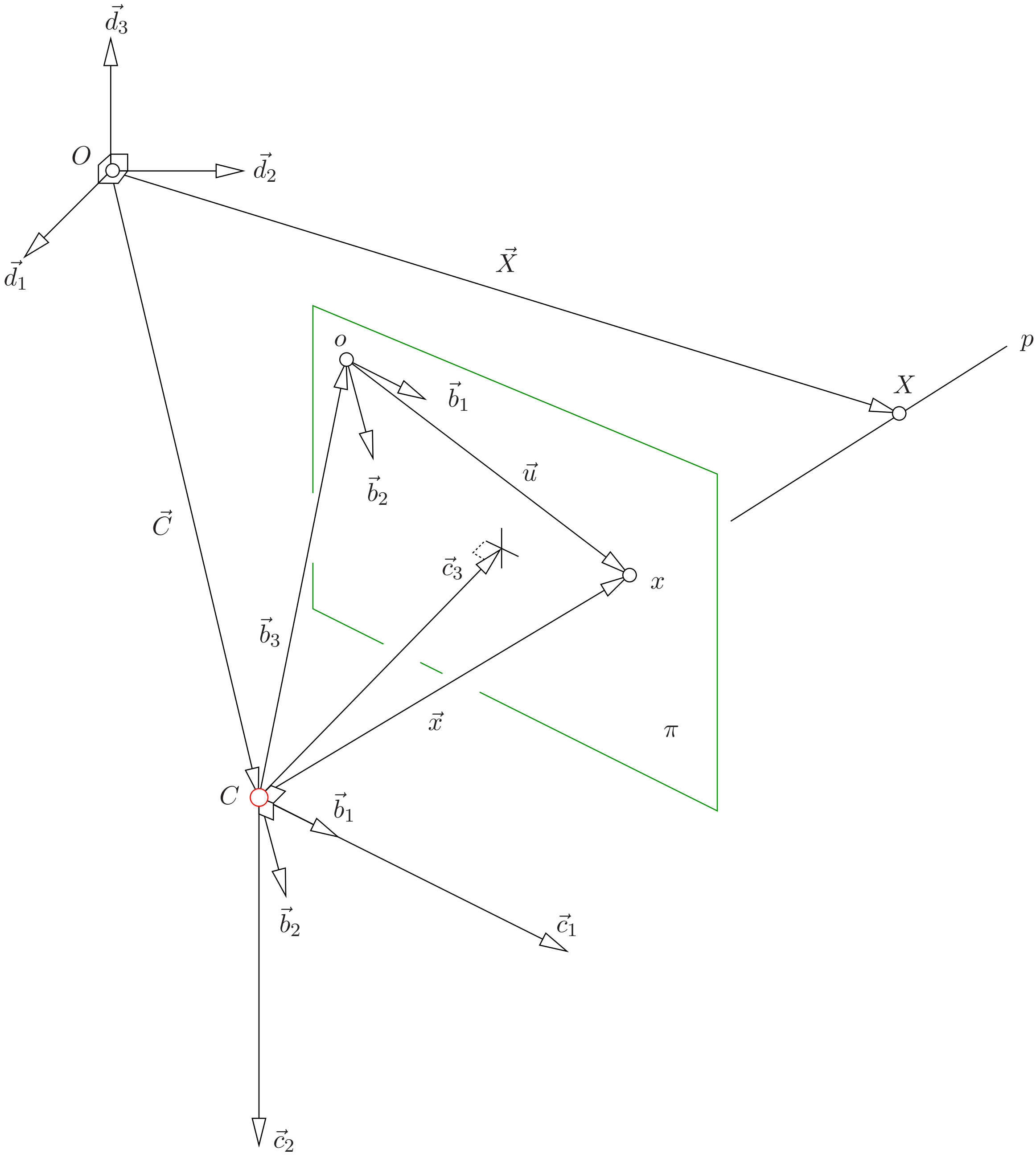


Figure 5.2: Coordinate systems of perspective camera.

§ **12 Perspective projection** Point X has been projected along p into x. Since $\vec{x}$ is a direction vector of p, point X can be represented in (C, β) by

$$\eta\,\vec{x} \tag{5.5}$$

for some real η. Value of η corresponds to scaled depth of X, i.e. the distance of X from the plane passing through C and generated by $\vec{b}_1$, $\vec{b}_2$ in units equal to the distance of C from π. Value η is not known since it "has been lost" in the process of projection[1] but will serve us to parametrize the projection ray in order to get coordinates of all possible points in space that could project into x.

Let us now relate the coordinates $\vec{u}_\alpha$, which are measured in the image, to the coordinates $\vec{X}_\delta$, which are measured in the world coordinate system. First consider vectors $\vec{X}$, $\vec{C}$ and $\vec{x}$. They are all coplanar and hence

$$\eta\,\vec{x} \;=\; \vec{X} - \vec{C} \tag{5.6}$$

To pass to coordinates, we will use the camera coordinate system, in which we can write

$$\eta\,\vec{x}_\beta \;=\; \vec{X}_\beta - \vec{C}_\beta \tag{5.7}$$

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} \;=\; \vec{X}_\beta - \vec{C}_\beta \tag{5.8}$$

Next we shall pass to the coordinates w.r.t. basis δ on the right hand side of Equation 5.8 by introducing a matrix $\mathtt{A}$, which transforms coordinates of a general vector $\vec{y}$ from basis δ to basis β, i.e.

$$\vec{y}_\beta = \mathtt{A}\,\vec{y}_\delta \tag{5.9}$$

We know from linear algebra (§ 3) that such matrix exists

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} \;=\; \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta)$$

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} \;=\; \mathtt{A} \begin{bmatrix} \mathtt{I} \,|\, -\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} \;=\; \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.10}$$

$$\eta\,\vec{x}_\beta \;=\; \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.11}$$

with 3×4 *camera (projection) matrix*

$$\mathtt{P} = \begin{bmatrix} \mathtt{A} & | - \mathtt{A}\,\vec{C}_\delta \end{bmatrix} \tag{5.12}$$

§ **13 Projection equation** Equation 5.10 describes the relationship between measurement $\vec{u}_\alpha$ in the image and measurement $\vec{X}_\delta$ in space. It says that $\vec{X}_\delta$ is projected into $\vec{u}_\alpha$ since there exists η such that Equation 5.10 holds. Notice that η multiple of the vector on the left of Equation 5.10 is obtained by a linear mapping represented by matrix $\mathtt{P}$ from vector $\vec{X}_\delta$ on the right.

[1] It can be recovered when a point X is observed by two cameras with different projection centers.

When computing $\vec{u}_\alpha$ from $\vec{X}_\delta$, we actually eliminate η using the last row of the (matricidal) equation (5.10)

$$\vec{u}_\alpha = \begin{bmatrix} \dfrac{\mathtt{p}_1^\top \mathtt{X}}{\mathtt{p}_3^\top \mathtt{X}} \\ \dfrac{\mathtt{p}_2^\top \mathtt{X}}{\mathtt{p}_3^\top \mathtt{X}} \end{bmatrix} \tag{5.13}$$

where we introduced rows of $\mathtt{p}_1$, $\mathtt{p}_2$, $\mathtt{p}_3$ of $\mathtt{P}$ and a 4×1 vector $\mathtt{X}$ as follows

$$\mathtt{P} = \begin{bmatrix} \mathtt{p}_1^\top \\ \mathtt{p}_2^\top \\ \mathtt{p}_3^\top \end{bmatrix} \quad \text{and} \quad \mathtt{X} = \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.14}$$

Notice that the projection equation is not linear. It is a rational function of the first order polynomials on elements of $\mathtt{X}$.

§ **14 Projection ray** Having an image point $\vec{u}_\alpha$, we can construct its projection ray p in space. The ray consists of all points $\vec{Y}$ that can project to $\vec{u}_\alpha$. In (C, β), the ray is emanating from the zero vector. We parametrize it by real η and express it in (O, δ) by vector $\vec{X}_\delta$

$$\begin{aligned} \vec{Y}_\beta &= \eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \eta\, \vec{x}_\beta \\ \vec{X}_\delta &= \eta\, \mathtt{A}^{-1} \vec{x}_\beta + \vec{C}_\delta \end{aligned} \tag{5.15}$$

Notice that $\vec{X}_\delta$ (5.15) can also be obtained for a given η by solving the system of linear equations

$$\eta\, \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.16}$$

for $\vec{X}_\delta$.

5.2 Computing camera projection matrix from image of six points

Let us now consider the task of finding the $\mathtt{P}$ from measurements. We shall consider the situation when we can measure points in space as well as their projection in the image. Consider a pair of such measurements $[x, y, z]^\top \overset{corr}{\leftrightarrow} [u, v]^\top$. There holds

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{Q} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{Q}\,\mathtt{X} \tag{5.17}$$

for some real λ, 3×4 matrix $\mathtt{Q}$ and 4×1 coordinate vector $\mathtt{X}$. Notice that we introduced new symbols λ and $\mathtt{Q}$ to emphasize that they are determined by Equation 5.17 up to a non-zero scale

$$\mathtt{Q} = \xi\, \mathtt{P} \tag{5.18}$$

We will see that this will have further consequences.

Introduce symbols for rows of $\mathtt{Q}$

$$\mathtt{Q} = \begin{bmatrix} \mathtt{q}_1^\top \\ \mathtt{q}_2^\top \\ \mathtt{q}_3^\top \end{bmatrix} \tag{5.19}$$

and rewrite the above matrix equation as

$$\begin{aligned} \lambda\, u &= \mathtt{q}_1^\top \mathtt{X} \\ \lambda\, v &= \mathtt{q}_2^\top \mathtt{X} \\ \lambda &= \mathtt{q}_3^\top \mathtt{X} \end{aligned}$$

Eliminate λ from the first two equations using the third one

$$\begin{aligned} (\mathtt{q}_3^\top \mathtt{X})\, u &= \mathtt{q}_1^\top \mathtt{X} \\ (\mathtt{q}_3^\top \mathtt{X})\, v &= \mathtt{q}_2^\top \mathtt{X} \end{aligned}$$

move all to the left hand side and reshape it using $\mathtt{x}^\top \mathtt{y} = \mathtt{y}^\top \mathtt{x}$

$$\begin{aligned} \mathtt{X}^\top \mathtt{q}_1 - (u\, \mathtt{X}^\top)\, \mathtt{q}_3 &= 0 \\ \mathtt{X}^\top \mathtt{q}_2 - (v\, \mathtt{X}^\top)\, \mathtt{q}_3 &= 0 \end{aligned}$$

Introduce vector of parameters (which are elements of $\mathtt{Q}$)

$$\mathtt{q} = \begin{bmatrix} \mathtt{q}_1^\top & \mathtt{q}_2^\top & \mathtt{q}_3^\top \end{bmatrix}^\top$$

and express the above two equations in matrix form

$$\begin{aligned} \begin{bmatrix} x & y & z & 1 & 0 & 0 & 0 & 0 & -u\,x & -u\,y & -u\,z & -u \\ 0 & 0 & 0 & 0 & x & y & z & 1 & -v\,x & -v\,y & -v\,z & -v \end{bmatrix} \mathtt{q} &= 0 \\ \mathtt{M}\ \ \mathtt{q} &= 0 \end{aligned} \tag{5.20}$$

Every correspondence $[x, y, z]^\top \overset{corr}{\leftrightarrow} [u, v]^\top$ brings two rows into the matrix $\mathtt{M}$ (5.20). We need therefore at least 6 correspondences in general position to obtain 11 linearly independent rows in Equation 5.20 to obtain a one-dimensional space of solutions.

If $\mathtt{Q}$ is a solution to Equation 5.20, then $\tau\, \mathtt{Q}$ is also a solution and both determine the same projection since

$$(\tau\, \mathtt{Q})\ \mathtt{X} = \tau\, (\mathtt{Q}\, \mathtt{X}) = \tau\, (\lambda\, \vec{x}_\beta) = (\tau\lambda)\, \vec{x}_\beta$$

Assuming $\mathtt{P} = \tau\, \mathtt{Q}$ leads to $\lambda = \eta/\tau$. We see that we can't recover $\mathtt{P}$ but only its non-zero multiple. Therefore, when solving Equation 5.20, we are looking for one-dimensional subspace of 3×4 matrices of *rank* 3. Such a subspace determines one projection. Also note that the zero matrix does not represent any interesting projection.

Notice that when considering more correspondences, $\mathtt{M}$ becomes

$$\mathtt{M} = \begin{bmatrix} x_1 & y_1 & z_1 & 1 & 0 & 0 & 0 & 0 & -u_1 x_1 & -u_1 y_1 & -u_1 z_1 & -u_1 \\ x_2 & y_2 & z_2 & 1 & 0 & 0 & 0 & 0 & -u_2 x_2 & -u_2 y_2 & -u_2 z_2 & -u_2 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & x_1 & y_1 & z_1 & 1 & -v_1 x_1 & -v_1 y_1 & -v_1 z_1 & -v_1 \\ 0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 1 & -v_2 x_2 & -v_2 y_2 & -v_2 z_2 & -v_2 \\ & & & & & & \vdots & & & & & \end{bmatrix} \mathtt{q} = 0$$

which can be more concisely rewritten as

$$
\mathtt{M} = \begin{bmatrix} \mathtt{X}_1^\top & \mathtt{0}^\top & -u_1\,\mathtt{X}_1^\top \\ \mathtt{X}_2^\top & \mathtt{0}^\top & -u_2\,\mathtt{X}_2^\top \\ & \vdots & \\ \mathtt{0}^\top & \mathtt{X}_1^\top & -v_1\,\mathtt{X}_1^\top \\ \mathtt{0}^\top & \mathtt{X}_2^\top & -v_2\,\mathtt{X}_2^\top \\ & \vdots & \end{bmatrix} \tag{5.21}
$$

with $\mathtt{0}^\top = [0,0,0,0]$.

5.3 Camera pose

The projection formula 5.10 reveals that the perspective projection depends on matrix $\mathtt{A}$ and vector $\vec{C}_\delta$. The vector $\vec{C}_\delta$ represents the position of the camera projection center w.r.t. the world coordinate system. Columns of matrix $\mathtt{A}$ are coordinates of the basic vectors of δ in the basis β

$$
\mathtt{A} \;=\; \begin{bmatrix} \vec{d}_{1_\beta} & \vec{d}_{2_\beta} & \vec{d}_{3_\beta} \end{bmatrix}
$$

To recover the orientation of the camera, we will introduce the focal length f (in world units) and replace the product $f\,\mathtt{A}$ by the product of two 3×3 matrices $\mathtt{K}$ and $\mathtt{R}$

$$
f\,\mathtt{A} = \mathtt{K}\,\mathtt{R} \tag{5.22}
$$

We will see that this seemingly artificial construction is indeed justified.

Rotation matrix $\mathtt{R}$ determines the orientation of the camera in space and altogether with $\vec{C}_\delta$ defines the *camera pose*. The *camera calibration matrix* matrix $\mathtt{K}$ does not change when moving its camera in the space.

To obtain $\mathtt{K}$ and $\mathtt{R}$, we define the *camera cartesian coordinate system* (C,γ) with center (again) in the camera projection center C and with basis $\gamma = [\vec{c}_1,\vec{c}_2,\vec{c}_3]$ such that

$$
\begin{aligned}
\vec{c}_1 &= k_{11}\,\vec{b}_1 \\
\vec{c}_2 &= k_{12}\,\vec{b}_1 + k_{22}\,\vec{b}_2 \\
\vec{c}_3 &= k_{13}\,\vec{b}_1 + k_{23}\,\vec{b}_2 + 1\,\vec{b}_3
\end{aligned} \tag{5.23}
$$

Parameters k_{ij} are determined to make the basis γ orthogonal. Notice that vector $\vec{c}_3$ is orthogonal to π since it is orthogonal to $\vec{c}_1,\vec{c}_2$, which span π, by construction. Also notice that γ is (in general) not an orthonormal basis since the length of its vectors equals the distance of C from π, i.e. the *focal length* f in the world units.

Equations 5.23 define matrix $\mathtt{K}$ as

$$
\mathtt{K} \;=\; \begin{bmatrix} \vec{c}_{1_\beta} & \vec{c}_{2_\beta} & \vec{c}_{3_\beta} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{5.24}
$$

The world cartesian coordinate system has basic vectors of unit length. The camera cartesian coordinate system (C, γ) has basic vectors of length equal to f. Therefore,

$$\begin{bmatrix} \vec{d}_{1_\gamma} & \vec{d}_{2_\gamma} & \vec{d}_{3_\gamma} \end{bmatrix} = \frac{1}{f}\,\mathtt{R} = \begin{bmatrix} \mathtt{r}_1^\top/f \\ \mathtt{r}_2^\top/f \\ \mathtt{r}_3^\top/f \end{bmatrix} \tag{5.25}$$

for some 3×3 orthonormal matrix $\mathtt{R}$ with rows $\mathtt{r}_1^\top$, $\mathtt{r}_2^\top$, $\mathtt{r}_3^\top$.

Consider that

$$\mathtt{A} = \begin{bmatrix} \vec{d}_{1_\beta} & \vec{d}_{2_\beta} & \vec{d}_{3_\beta} \end{bmatrix} = \mathtt{K} \begin{bmatrix} \vec{d}_{1_\gamma} & \vec{d}_{2_\gamma} & \vec{d}_{3_\gamma} \end{bmatrix} = \frac{1}{f}\,\mathtt{K}\,\mathtt{R} \tag{5.26}$$

We can view the matrices $\frac{1}{f}\mathtt{R}$ and $\mathtt{K}$ as coordinate transformation matrices, which transform a general vector $\vec{y}$ from the coordinates w.r.t. δ to γ and then to β, i.e.

$$\vec{y}_\beta = \mathtt{K}\,\vec{y}_\gamma = \frac{1}{f}\,\mathtt{K}\,\mathtt{R}\,\vec{y}_\delta \tag{5.27}$$

The basis γ is orthogonal and all basic vectors have the same length, which is equal to f. It follows from the orthogonality of the basis γ that $\vec{c}_1 \cdot \vec{c}_1 = f^2$, $\vec{c}_1 \cdot \vec{c}_2 = 0$ and $\vec{c}_2 \cdot \vec{c}_2 = f^2$ and hence using Equation 5.23 leads to

$$\begin{aligned} k_{11}\,\|\vec{b}_1\| - f &= 0 \\ k_{11}^2\,k_{22}\,(\vec{b}_1 \cdot \vec{b}_2) + k_{12}\,f^2 &= 0 \\ k_{11}^2\,k_{22}^2\,\|\vec{b}_2\|^2 - (k_{12}^2 + k_{11}^2)\,f^2 &= 0 \end{aligned} \tag{5.28}$$

Let us solve Equations 5.28 for k_{11}, k_{12} and k_{22}. The first Equation in (5.28) provides k_{11}. Substitution the square of the first equation into the second one an dividing it by k_{11}^2 provides the second equation of (5.29). To get k_{22} from the third equation of (5.29), we express k_{11} from the first equation of (5.28) and k_{22} from the second equation of (5.28) and substitute them into the third equation of (5.28). Altogether, we get

$$\begin{aligned} k_{11}\,\|\vec{b}_1\| - f &= 0 \\ k_{12}\,\|\vec{b}_1\|^2 + k_{22}\,(\vec{b}_1 \cdot \vec{b}_2) &= 0 \\ k_{22}^2\,(\|\vec{b}_1\|^2\,\|\vec{b}_2\|^2 - (\vec{b}_1 \cdot \vec{b}_2)^2) - f^2\,\|\vec{b}_1\|^2 &= 0 \end{aligned} \tag{5.29}$$

Looking at the third equation of (5.28) we see that

$$k_{22}^2 = \frac{f^2\|\vec{b}_1\|^2}{\|\vec{b}_1\|^2\|\vec{b}_2\|^2 - (\vec{b}_1 \cdot \vec{b}_2)^2} = \frac{f^2}{\|\vec{b}_2\|^2 - \|\vec{b}_2\|^2 \cos^2\angle(\vec{b}_1, \vec{b}_2)} \tag{5.30}$$

$$k_{22} = \frac{f}{\|\vec{b}_2\| \sin\angle(\vec{b}_1, \vec{b}_2)} \tag{5.31}$$

since γ was constructed to make k_{22} positive. The second equation of (5.28) now gives

$$k_{12} = -k_{22}\frac{\vec{b}_1 \cdot \vec{b}_2}{\|\vec{b}_1\|^2} = -k_{22}\frac{\|\vec{b}_2\| \cos\angle(\vec{b}_1, \vec{b}_2)}{\|\vec{b}_1\|} \tag{5.32}$$

$$= -\frac{f\,\cos\angle(\vec{b}_1, \vec{b}_2)}{\|\vec{b}_1\| \sin\angle(\vec{b}_1, \vec{b}_2)} \tag{5.33}$$

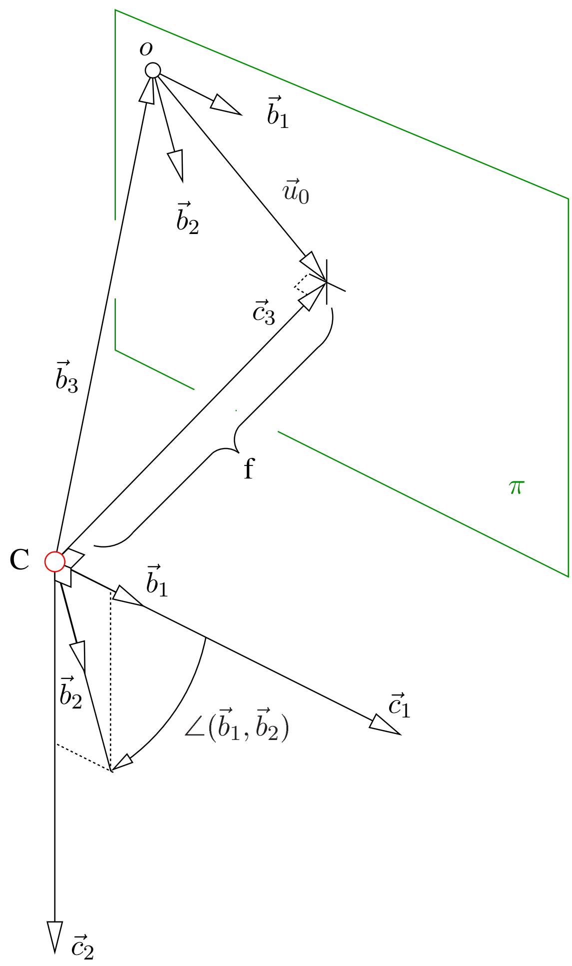


Figure 5.3: Camera internal parameters are related to the geometry of basis β.

Finally k_{11} follows from (5.29)

$$k_{11} = \frac{f}{\|\vec{b}_1\|}$$

Considering Figure 5.3 and Equation 5.23, we see that the coordinates of the vector $\vec{u}_0$, corresponding to the *principal point*, which is the perpendicular projection of C onto π, are in β

$$\vec{u}_{0\beta} = \begin{bmatrix} k_{13} \\ k_{23} \\ 0 \end{bmatrix}, \text{ i.e. } \vec{u}_{0\alpha} = \begin{bmatrix} k_{13} \\ k_{23} \end{bmatrix} \tag{5.34}$$

The horizontal pixel size corresponds to $\|\vec{b}_1\|$. Quantity k_{11} thus transforms the world units into the horizontal image units. It can be understood as f expressed in the horizontal image units. The angle between the image axes $\vec{b}_1, \vec{b}_2$ is obtained from $k_{11}/k_{12} = -\tan\angle(\vec{b}_1, \vec{b}_2)$. The ratio of the lengths of the image axes is determined by $\|\vec{b}_2\|/\|\vec{b}_1\| = \sqrt{k_{11}\,(k_{11}+k_{12})/k_{22}}$.

Let us now return to Equation 5.10 and substitute there the above results to arrive at the final projection equation

$$\eta\,\vec{x}_\beta = \mathtt{P}\begin{bmatrix}\vec{X}_\delta \\ 1\end{bmatrix} \tag{5.35}$$

$$\eta\begin{bmatrix}\vec{u}_\alpha \\ 1\end{bmatrix} = \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{5.36}$$

$$f\,\eta\begin{bmatrix}\vec{u}_\alpha \\ 1\end{bmatrix} = f\ \mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{5.37}$$

$$f\,\eta\begin{bmatrix}\vec{u}_\alpha \\ 1\end{bmatrix} = \mathtt{K}\,\mathtt{R}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{5.38}$$

$$\zeta\begin{bmatrix}\vec{u}_\alpha \\ 1\end{bmatrix} = \mathtt{K}\,\mathtt{R}\,(\vec{X}_\delta - \vec{C}_\delta) \tag{5.39}$$

$$\zeta\begin{bmatrix}\vec{u}_\alpha \\ 1\end{bmatrix} = \mathtt{K}\,\mathtt{R}\begin{bmatrix}\mathtt{I}\,|-\vec{C}_\delta\end{bmatrix}\begin{bmatrix}\vec{X}_\delta \\ 1\end{bmatrix} \tag{5.40}$$

We have introduced a new parameter $\zeta = f\,\eta$, which is the depth of X in world units. We conclude that

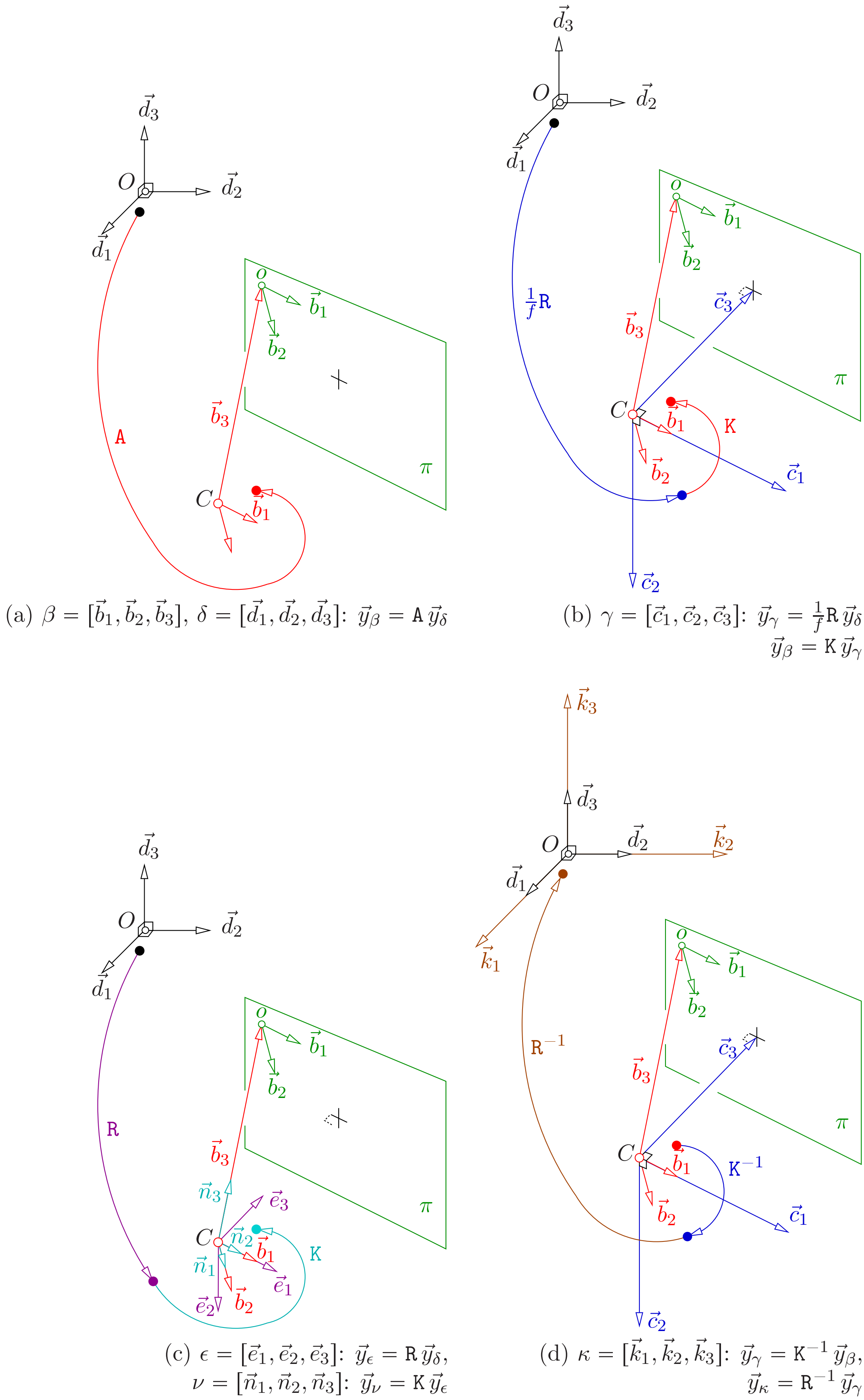
$$\mathtt{P} = \begin{bmatrix}\frac{1}{f}\mathtt{K}\,\mathtt{R} & |-\frac{1}{f}\mathtt{K}\,\mathtt{R}\,\vec{C}_\delta\end{bmatrix} \tag{5.41}$$

Notice that the last row $\mathtt{a}_3^\top$ of $\mathtt{A}$ provides f since

$$\mathtt{A} = \begin{bmatrix}\mathtt{a}_1^\top \\ \mathtt{a}_2^\top \\ \mathtt{a}_3^\top\end{bmatrix} = \frac{1}{f}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathtt{r}_1^\top \\ \mathtt{r}_2^\top \\ \mathtt{r}_3^\top\end{bmatrix} = \frac{1}{f}\begin{bmatrix} k_{11}\mathtt{r}_1 + k_{12}\mathtt{r}_2 + k_{13}\mathtt{r}_3 \\ k_{22}\mathtt{r}_2 + k_{23}\mathtt{r}_3 \\ \mathtt{r}_3\end{bmatrix}$$

and hence $\|\mathtt{a}_3^\top\| = \frac{1}{f}$. Therefore $\|\mathtt{P}(3, 1:3)\| = \frac{1}{f}$.

§ 15 Coordinate systems generated by applying $\mathtt{K}\,\mathtt{R}$ to $\vec{y}_\delta$ and $\mathtt{R}^{-1}\mathtt{K}^{-1}$ to $\vec{y}_\beta$ We have seen that the decomposition of $\mathtt{A}$ to $\mathtt{K}$ and $\mathtt{R}$ introduced the camera cartesian



(a) $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$, $\delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$: $\vec{y}_\beta = \mathtt{A}\,\vec{y}_\delta$

(b) $\gamma = [\vec{c}_1, \vec{c}_2, \vec{c}_3]$: $\vec{y}_\gamma = \frac{1}{f}\mathtt{R}\,\vec{y}_\delta$
$\vec{y}_\beta = \mathtt{K}\,\vec{y}_\gamma$

(c) $\epsilon = [\vec{e}_1, \vec{e}_2, \vec{e}_3]$: $\vec{y}_\epsilon = \mathtt{R}\,\vec{y}_\delta$,
$\nu = [\vec{n}_1, \vec{n}_2, \vec{n}_3]$: $\vec{y}_\nu = \mathtt{K}\,\vec{y}_\epsilon$

(d) $\kappa = [\vec{k}_1, \vec{k}_2, \vec{k}_3]$: $\vec{y}_\gamma = \mathtt{K}^{-1}\,\vec{y}_\beta$,
$\vec{y}_\kappa = \mathtt{R}^{-1}\,\vec{y}_\gamma$

Figure 5.4: Coordinate systems generated by applying $\frac{1}{f}\mathtt{R}$, $\mathtt{K}$, $\mathtt{R}$, $\mathtt{R}^{-1}$ and $\mathtt{K}^{-1}$.

coordinate system (C, γ), Figure 5.4(b)

$$\vec{y}_\gamma = \frac{1}{f} \mathtt{R}\, \vec{y}_\delta \tag{5.42}$$

$$\vec{y}_\beta = \mathtt{K}\, \vec{y}_\gamma \tag{5.43}$$

There are three more coordinate system to consider when looking at how matrices $\mathtt{R}$, $\mathtt{K}$, and their inverses $\mathtt{R}^{-1}$, $\mathtt{K}^{-1}$, apply to vectors $\vec{y}_\delta$ and $\vec{y}_\beta$, Figure 5.4.

Let us first consider coordinates of a vector $\vec{y}$ w.r.t. basis δ and apply successively $\mathtt{R}$ and $\mathtt{K}$. Coordinate vector $\mathtt{R}\,\vec{y}_\delta$ can be interpreted as coordinates of $\vec{y}$ w.r.t. a new basis $\epsilon = [\vec{e}_1, \vec{e}_2, \vec{e}_3]$, Figure 5.4(c). Applying further $\mathtt{K}$ to $\vec{y}_\epsilon$ gives the coordinate vector $\mathtt{K}\,\vec{y}_\epsilon$, which can be interpreted as $\vec{y}$ w.r.t. yet another new basis $\nu = [\vec{n}_1, \vec{n}_2, \vec{n}_3]$. We get from ν to β by using $\frac{1}{f}\,\mathtt{I}$

$$\vec{y}_\epsilon = \mathtt{R}\, \vec{y}_\delta \tag{5.44}$$

$$\vec{y}_\nu = \mathtt{K}\, \vec{y}_\epsilon \tag{5.45}$$

$$\vec{y}_\beta = \frac{1}{f} \mathtt{I}\, \vec{y}_\nu \tag{5.46}$$

We have introduced two new coordinate systems (O, ν), $\nu = [\vec{n}_1, \vec{n}_2, \vec{n}_3]$ and (O, ϵ), $\epsilon = [\vec{e}_1, \vec{e}_2, \vec{e}_3]$.

Next we consider coordinates of a vector $\vec{y}$ w.r.t. basis β and apply successively $\mathtt{K}^{-1}$ and $\mathtt{R}^{-1}$. Coordinate vector $\mathtt{K}^{-1}\,\vec{y}_\beta$ gives $\vec{y}_\gamma$. Coordinate vector $\mathtt{R}^{-1}\,\vec{y}_\gamma$ can be interpreted as coordinates of $\vec{y}$ w.r.t. a new basis $\kappa = [\vec{k}_1, \vec{k}_2, \vec{k}_3]$, Figure 5.4(d). To get from $\vec{y}_\kappa$ to $\vec{y}_\delta$ we need to employ $f\,\mathtt{I}$

$$\vec{y}_\gamma = \mathtt{K}^{-1}\, \vec{y}_\beta \tag{5.47}$$

$$\vec{y}_\kappa = \mathtt{R}^{-1}\, \vec{y}_\gamma \tag{5.48}$$

$$\vec{y}_\delta = f\,\mathtt{I}\, \vec{y}_\kappa \tag{5.49}$$

We have thus introduced a new coordinate system (O, κ), $\epsilon = [\vec{k}_1, \vec{k}_2, \vec{k}_3]$.

Figure 5.5 summarizes the relationship between coordinates of a vector and between bases associated with a perspective cameras.

§ 16 Recovering camera pose from its projection matrix Let us next consider that we have already computed the camera projection matrix

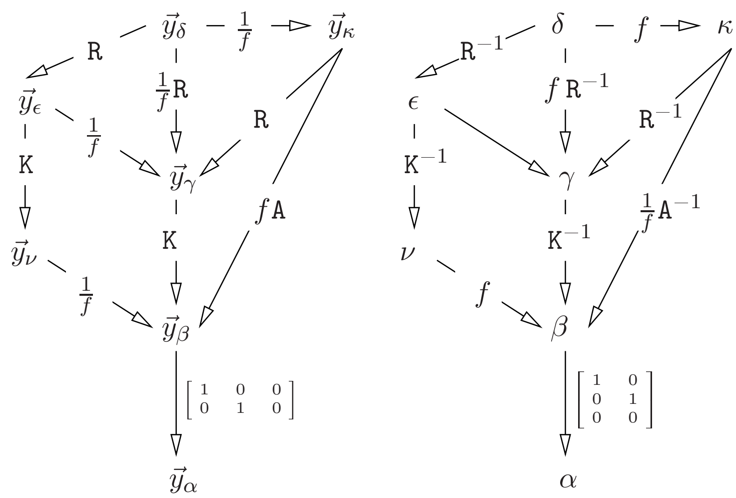
$$\mathtt{Q} = \xi\, \mathtt{P} = \xi\, \frac{1}{f}\, \mathtt{K}\,\mathtt{R}\, [\mathtt{I} \,|\, \vec{C}_\delta]$$

consisting of a 3×3 matrix $\mathtt{M}$ and 3×1 vector $\mathtt{m}$

$$\mathtt{Q} = [\mathtt{M} \,|\, \mathtt{m}] \tag{5.50}$$

To recover camera pose from $\mathtt{Q}$, we need to get $\vec{C}_\delta$ from $\mathtt{m}$ and to decompose $\mathtt{Q}$ into the product of $\mathtt{K}$ in the form of (5.24) and $\mathtt{R}$ such that $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$ and $|\mathtt{R}| = 1$. Consider $\mathtt{M}$ in the form

$$\mathtt{M} = \begin{bmatrix} \mathtt{m}_1^\top \\ \mathtt{m}_2^\top \\ \mathtt{m}_3^\top \end{bmatrix} \tag{5.51}$$



(a) (b)

Figure 5.5: Relationships between (a) coordinates in different bases. e.g. $\vec{y}_\beta = \mathtt{K}\,\vec{y}_\gamma$ and (b) bases themselves, e.g. $\beta = \gamma\,\mathtt{K}^{-1}$, associated with a perspective camera.

Next we notice that the last row of $\mathtt{K}\,\mathtt{R}$ has unit norm since it is equal to the last row of rotation $\mathtt{R}$. Therefore, we need to divide $\mathtt{M}$ by the norm of its last row to get a matrix decomposable into a product of $\mathtt{K}\,\mathtt{R}$. Moreover, it follows from the construction of β that $k_{11} > 0$ and $k_{22} > 0$. Thus, determinant $|\mathtt{K}\,\mathtt{R}| = |\mathtt{K}|\,|\mathtt{R}| = k_{11}\,k_{22} > 0$. Therefore, we also need to multiply $\mathtt{M}$ by the sign of its determinant to get a matrix decomposable into a product of $\mathtt{K}\,\mathtt{R}$.

$$\frac{\operatorname{sign}|\mathtt{M}|}{\|\mathtt{m}_3\|}\,\mathtt{M} \;=\; \frac{\operatorname{sign}|\mathtt{M}|}{\|\mathtt{m}_3\|}\begin{bmatrix} \mathtt{m}_1^\top \\ \mathtt{m}_2^\top \\ \mathtt{m}_3^\top \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathtt{r}_1^\top \\ \mathtt{r}_2^\top \\ \mathtt{r}_3^\top \end{bmatrix} \tag{5.52}$$

which provides the following set of equations

$$\frac{\mathtt{m}_2^\top \mathtt{m}_3}{\|\mathtt{m}_3\|^2} = k_{22}\,\mathtt{r}_2^\top \mathtt{r}_3 + k_{23}\,\mathtt{r}_3^\top \mathtt{r}_3 = k_{23} \tag{5.53}$$

$$\frac{\mathtt{m}_1^\top \mathtt{m}_3}{\|\mathtt{m}_3\|^2} = k_{13} \tag{5.54}$$

$$\frac{\mathtt{m}_2^\top \mathtt{m}_2}{\|\mathtt{m}_3\|^2} = k_{22}^2 + k_{23}^2 \tag{5.55}$$

$$\frac{\mathtt{m}_1^\top \mathtt{m}_2}{\|\mathtt{m}_3\|^2} = k_{12}\,k_{22} + k_{13}\,k_{23} \tag{5.56}$$

$$\frac{\mathtt{m}_1^\top \mathtt{m}_1}{\|\mathtt{m}_3\|^2} = k_{11}^2 + k_{12}^2 + k_{13}^2 \tag{5.57}$$

from which k_{11}, k_{12}, k_{13}, k_{22}, k_{23} can be easily computed considering that the most of consumer digital cameras have $k_{11} > 0$, $k_{22} > 0$, $k_{13} > 0$, $k_{23} > 0$.

Having k_{ij} computed, we recover $\mathtt{R}$ from $\mathtt{M}$ as

$$\mathtt{R} = \mathtt{K}^{-1}\frac{\operatorname{sign}|\mathtt{M}|}{\|\mathtt{m}_3\|}\mathtt{M} \tag{5.58}$$

Camera projection center can be computed in two ways. Either we get

$$\vec{C}_\delta = -\,\mathtt{M}^{-1}\mathtt{m} \tag{5.59}$$

or we obtain it by finding a basis $\mathtt{c}$ of the one-dimensional right null space of matrix $\mathtt{Q}$, i.e. solving

$$\mathtt{Q}\,\mathtt{c} = 0 \tag{5.60}$$

and then computing

$$\begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \frac{1}{\mathtt{c}_4}\mathtt{c} \tag{5.61}$$

where $\mathtt{c}_4$ is the fourth coordinate of vector $\mathtt{c}$.

5.4 Camera calibration and angle between projection rays

We have introduced matrices $\mathtt{P}$, $\mathtt{R}$ and $\mathtt{K}$, which determine the projection from space to images. However, since $\mathtt{K}$ is introduced with $\mathtt{K}_{33} = 1$, the pair $(\mathtt{K}, \mathtt{R})$ does not contain all information about the camera, which can be obtained by direct measurement of its physical components in a world coordinate system equipped with a known *world unit length* $\mathbf{1}_W$. The missing element is the scale of $\mathtt{K}$, which is equivalent to knowing the value of focal length or the size of pixel, i.e. f, $\|\vec{b}_1\|$ or $\|\vec{b}_2\|$, in $\mathbf{1}_W$.

Knowing $\mathtt{K}$ and f allows to recover $\|\vec{b}_1\|$ from Equations 5.23 as $\|\vec{b}_1\| = f/k_{11}$. Knowing $\mathtt{K}$ and $\|\vec{b}_1\|$, on the other hand, gives $f = \|\vec{b}_1\|/k_{11}$.

Therefore, full calibration of the camera is encoded, e.g., in one of the following triplets: $(\mathtt{K}, \mathtt{R}, f)$, $(\mathtt{K}, \mathtt{R}, k_{11})$ or $(\mathtt{K}, \mathtt{R}, k_{22})$.

We defined the camera calibration matrix $\mathtt{K}$ with $\mathtt{K}_{33} = 1$ because we often do not have access to world units when working with images without knowing anything about the camera which was used to make them. Moreover, a number of important tasks can be done without knowing the world unit.

§ **17 Angle between projection rays** Consider two image points $\vec{u}_{1\beta}$ and $\vec{u}_{2\beta}$. The direction vectors of the rays are in β given by

$$\vec{x}_{1\beta} = \begin{bmatrix} \vec{u}_{1\beta} \\ 1 \end{bmatrix}, \quad \vec{x}_{2\beta} = \begin{bmatrix} \vec{u}_{2\beta} \\ 1 \end{bmatrix} \tag{5.62}$$

To obtain the angle between the direction vectors by evaluating the scalar product of the vectors, we need to pass to an orthogonal basis. The "closest" orthogonal basis is γ. Hence

$$\cos\angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\gamma}^\top \vec{x}_{2\gamma}}{\|\vec{x}_{1\gamma}\|\|\vec{x}_{2\gamma}\|} = \frac{\vec{x}_{1\beta}^\top\,\mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{2\beta}}{\|\mathtt{K}^{-1}\vec{x}_{1\beta}\|\|\mathtt{K}^{-1}\vec{x}_{2\beta}\|} \tag{5.63}$$

5.5 Calibrated camera pose computation

We have seen how to find (uncalibrated) perspective camera pose from projections of known six points. In fact, we have recovered the calibration of the camera. Next we shall show that when the calibration is known, we are able to find the pose of

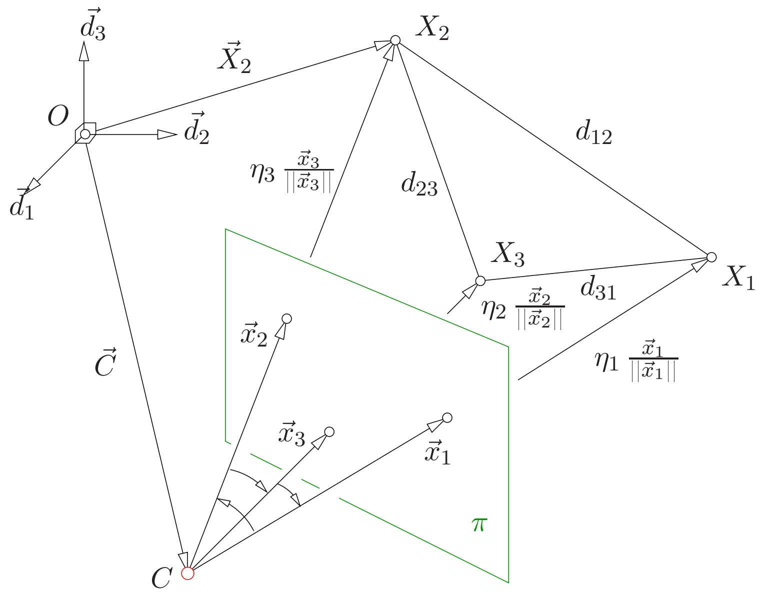


Figure 5.6: A calibrated camera pose can be computed from projections of three known points.

the camera from projections of three points. This is one of very classical problem. Its solution is known since ... when it has been solved by Grunert.

Figure 5.6 shows a camera with center C, which projects three points X_1, X_2 and X_3, represented by vectors $\vec{X}_{1\delta}$, $\vec{X}_{2\delta}$ and $\vec{X}_{3\delta}$ in (O, δ), into image points represented by $\vec{x}_{1\beta}$, $\vec{x}_{2\beta}$ and $\vec{x}_{3\beta}$.

§ 18 Classical formulation of the calibrated camera pose computation We introduce distances between pairs of points as

$$d_{12} = ||\vec{X}_{2\delta} - \vec{X}_{1\delta}||, \quad d_{23} = ||\vec{X}_{3\delta} - \vec{X}_{2\delta}||, \quad d_{31} = ||\vec{X}_{1\delta} - \vec{X}_{3\delta}|| \tag{5.64}$$

Since we see three different points, we know that all distances are non-zero.

Points X_1, X_2 and X_3 are in (C, γ) represented by vectors

$$\eta_i \frac{\vec{x}_{i\gamma}}{||\vec{x}_{i\gamma}||} = \eta_i \frac{\mathtt{K}^{-1}\vec{x}_{i\beta}}{||\mathtt{K}^{-1}\vec{x}_{i\beta}||}, \quad i = 1, 2, 3 \tag{5.65}$$

with η_i representing the distance from C to X_i. Distances η_i are non-zero since otherwise we could not see the points.

§ 19 Computing distances to camera center Calibrated perspective camera measures angles between projection rays

$$c_{ij} = \cos\angle(\vec{x}_i, \vec{x}_j) = \frac{\vec{x}_{i\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{j\beta}}{\|\mathtt{K}^{-1}\vec{x}_{i\beta}\|\|\mathtt{K}^{-1}\vec{x}_{j\beta}\|}, \quad i = 1, 2, 3, \ j = (i-1) \bmod 3 + 1 \tag{5.66}$$

Hence we have all quantities η_i, $\cos\angle(\vec{x}_i, \vec{x}_j)$ and d_{ij}, which we need to construct a set of equations using the rule of cosines $d_{ij}^2 = \eta_i^2 + \eta_j^2 - 2\,\eta_i\,\eta_j\,\cos\angle(\vec{x}_i, \vec{x}_j)$, i.e.

$$
\begin{aligned}
d_{12}^2 &= \eta_1^2 + \eta_2^2 - 2\,\eta_1\,\eta_2\,c_{12} && (5.67)\\
d_{23}^2 &= \eta_2^2 + \eta_3^2 - 2\,\eta_2\,\eta_3\,c_{23} && (5.68)\\
d_{31}^2 &= \eta_3^2 + \eta_1^2 - 2\,\eta_3\,\eta_1\,c_{31} && (5.69)
\end{aligned}
$$

with $c_{ij} = \cos\angle(\vec{x}_i, \vec{x}_j)$.

We have three quadratic equations in three variables. We shall solve this system by manipulating the three equations to generate one equation in one variable, solving it and then substituting back to get other variables.

§ 20 A classical (tedious) solution Let us first get two equations in two variables. Let us generate new equations by multiplying the left side of (5.67) and (5.69) right side of (5.68) and right side of (5.67) and (5.69) by left side of (5.68)

$$
\begin{aligned}
d_{12}^2\,(\eta_2^2 + \eta_3^2 - 2\,\eta_2\,\eta_3\,c_{23}) &= d_{23}^2\,(\eta_1^2 + \eta_2^2 - 2\,\eta_1\,\eta_2\,c_{12}) && (5.70)\\
d_{31}^2\,(\eta_2^2 + \eta_3^2 - 2\,\eta_2\,\eta_3\,c_{23}) &= d_{23}^2\,(\eta_3^2 + \eta_1^2 - 2\,\eta_3\,\eta_1\,c_{31}) && (5.71)
\end{aligned}
$$

We could have made three different choices which equation to use twice but since all $d_{ij} \neq 0$, and hence all sides of the equations are nonzero, all the choices are equally valid.

We have now two equations with three variables but since the equations are homogeneous, we will be able to reduce the number of variables to two by dividing equations by (e.g.) η_1^2 (which is non-zero) to get

$$
\begin{aligned}
d_{12}^2\left(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}\right) &= d_{23}^2\left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) && (5.72)\\
d_{31}^2\left(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}\right) &= d_{23}^2\left(1 + \eta_{13}^2 - 2\,\eta_{13}\,c_{31}\right) && (5.73)
\end{aligned}
$$

with $\eta_{12} = \frac{\eta_2}{\eta_1}$ and $\eta_{13} = \frac{\eta_3}{\eta_1}$. Notice that we have a simpler situation than before with only two quadratic equations in two variables. Let us proceed further towards a one equation in one variable.

We rearrange the terms to get homogeneous polynomials in η_{13} on the left and the rest on the right

$$
\begin{aligned}
d_{12}^2\,\eta_{13}^2 + (-2\,d_{12}^2\,\eta_{12}\,c_{23})\,\eta_{13} &= d_{23}^2\left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) - d_{12}^2\eta_{12}^2\\
(d_{31}^2 - d_{23}^2)\,\eta_{13}^2 + (2\,d_{23}^2\,c_{31} - 2\,d_{31}^2\,\eta_{12}\,c_{23})\,\eta_{13} &= d_{23}^2 - d_{31}^2\,\eta_{12}^2
\end{aligned}
$$

to get two quadratic equations

$$
\begin{aligned}
m_1\,\eta_{13}^2 + p_1\,\eta_{13} &= q_1 && (5.74)\\
m_2\,\eta_{13}^2 + p_2\,\eta_{13} &= q_2
\end{aligned}
$$

in η_{13} with

$$
\begin{aligned}
m_1 &= d_{12}^2 && (5.75)\\
p_1 &= -2\,d_{12}^2\,\eta_{12}\,c_{23} && (5.76)\\
q_1 &= d_{23}^2\left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) - d_{12}^2\eta_{12}^2 && (5.77)\\
m_2 &= d_{31}^2 - d_{23}^2 && (5.78)\\
p_2 &= 2\,d_{23}^2\,c_{31} - 2\,d_{31}^2\,\eta_{12}\,c_{23} && (5.79)\\
q_2 &= d_{23}^2 - d_{31}^2\,\eta_{12}^2 && (5.80)
\end{aligned}
$$

We have "hidden" the variable η_{12} in the new coefficients. We can now look upon Equations 5.74 as on a linear system

$$\begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix} \begin{bmatrix} \eta_{13}^2 \\ \eta_{13} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{5.81}$$

The matrix of the system (5.81) either is or is not singular.

§ **21 Case A** If it is not singular, we can solve the system by Cramer's rule [5, 9, 3]

$$\eta_{13}^2 \left| \begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix} \right| = \left| \begin{bmatrix} q_1 & p_1 \\ q_2 & p_2 \end{bmatrix} \right| \tag{5.82}$$

$$\eta_{13} \left| \begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix} \right| = \left| \begin{bmatrix} m_1 & q_1 \\ m_2 & q_2 \end{bmatrix} \right| \tag{5.83}$$

giving

$$\eta_{13}^2 \, (m_1 \, p_2 - m_2 \, p_1) = q_1 \, p_2 - q_2 \, p_1 \tag{5.84}$$

$$\eta_{13} \, (m_1 \, p_2 - m_2 \, p_1) = m_1 \, q_2 - m_2 \, q_1 \tag{5.85}$$

Eliminating η_{13} (by squaring the second equation, multiplying the first one by $m_1 \, p_2 - m_2 \, p_1$, which is non-zero, and comparing the left hand sides) yields

$$(m_1 \, p_2 - m_2 \, p_1)\,(q_1 \, p_2 - q_2 \, p_1) = (m_1 \, q_2 - m_2 \, q_1)^2 \tag{5.86}$$

Substituting Formulas 5.75-5.80 into Equation 5.86 yields

$$0 = a_4 \, \eta_{12}^4 + a_3 \, \eta_{12}^3 + a_2 \, \eta_{12}^2 + a_1 \, \eta_{12} + a_0 \tag{5.87}$$

with coefficients

$$\begin{aligned} a_4 = {} & -d_{23}^8 - d_{12}^4 \, d_{23}^4 - d_{23}^4 \, d_{31}^4 - 2\, d_{12}^2 \, d_{23}^4 \, d_{31}^2 + 2\, d_{23}^6 \, d_{31}^2 + 2\, d_{12}^2 \, d_{23}^6 \\ & +4\, d_{12}^2 \, c_{23}^2 \, d_{23}^4 \, d_{31}^2 \end{aligned} \tag{5.88}$$

$$\begin{aligned} a_3 = {} & 4\, d_{12}^4 \, d_{23}^4 \, c_{31} \, c_{23} - 4\, d_{12}^2 \, d_{23}^6 \, c_{12} - 4\, d_{12}^2 \, c_{23} \, d_{23}^6 \, c_{31} + 4\, d_{23}^4 \, c_{12} \, d_{31}^4 \\ & +4\, d_{23}^8 \, c_{12} - 4\, d_{12}^2 \, d_{23}^4 \, c_{31} \, d_{31}^2 \, c_{23} - 8\, d_{12}^2 \, c_{23}^2 \, d_{23}^4 \, d_{31}^2 \, c_{12} - 8\, d_{23}^6 \, c_{12} \, d_{31}^2 \\ & +4\, d_{12}^2 \, d_{23}^4 \, c_{12} \, d_{31}^2 \end{aligned} \tag{5.89}$$

$$\begin{aligned} a_2 = {} & 8\, d_{23}^6 \, c_{12}^2 \, d_{31}^2 + 4\, d_{23}^6 \, d_{31}^2 - 2\, d_{23}^4 \, d_{31}^4 + 2\, d_{12}^4 \, d_{23}^4 - 4\, d_{12}^4 \, d_{23}^4 \, c_{31}^2 \\ & -4\, d_{23}^8 \, c_{12}^2 - 4\, d_{12}^4 \, c_{23}^2 \, d_{23}^4 - 2\, d_{23}^8 + 8\, d_{12}^2 \, c_{23} \, d_{23}^6 \, c_{31} \, c_{12} \\ & +4\, d_{12}^2 \, c_{23}^2 \, d_{23}^4 \, d_{31}^2 - 4\, d_{23}^4 \, c_{12}^2 \, d_{31}^4 + 4\, d_{12}^2 \, d_{23}^6 \, c_{31}^2 + 8\, d_{12}^2 \, d_{23}^4 \, c_{31} \, d_{31}^2 \, c_{23} \, c_{12} \end{aligned} \tag{5.90}$$

$$\begin{aligned} a_1 = {} & 4\, d_{23}^4 \, c_{12} \, d_{31}^4 + 4\, d_{12}^2 \, d_{23}^6 \, c_{12} + 4\, d_{23}^8 \, c_{12} - 4\, d_{12}^2 \, c_{23} \, d_{23}^6 \, c_{31} \\ & -8\, d_{12}^2 \, d_{23}^6 \, c_{31}^2 \, c_{12} - 4\, d_{12}^2 \, d_{23}^4 \, c_{31} \, d_{31}^2 \, c_{23} - 4\, d_{12}^2 \, d_{23}^4 \, c_{12} \, d_{31}^2 \\ & +4\, d_{12}^4 \, d_{23}^4 \, c_{31} \, c_{23} - 8\, d_{23}^6 \, c_{12} \, d_{31}^2 \end{aligned} \tag{5.91}$$

$$\begin{aligned} a_0 = {} & 2\, d_{23}^6 \, d_{31}^2 + 2\, d_{12}^2 \, d_{23}^4 \, d_{31}^2 - d_{23}^4 \, d_{31}^4 - d_{12}^4 \, d_{23}^4 + 4\, d_{12}^2 \, d_{23}^6 \, c_{31}^2 \\ & -d_{23}^8 - 2\, d_{12}^2 \, d_{23}^6 \end{aligned} \tag{5.92}$$

We will use computation of eigenvalues to find a numerical solution to Equation 5.87. Construct the companion matrix

$$\mathtt{C} = \begin{bmatrix} 0 & 0 & 0 & -\frac{a_0}{a_4} \\ 1 & 0 & 0 & -\frac{a_1}{a_4} \\ 0 & 1 & 0 & -\frac{a_2}{a_4} \\ 0 & 0 & 1 & -\frac{a_3}{a_4} \end{bmatrix} \tag{5.93}$$

and observe that

$$|\,\eta_{12}\,\mathtt{I} - \mathtt{C}\,| \quad = \quad \eta_{12}^4 + \frac{a_3}{a_4}\,\eta_{12}^3 + \frac{a_2}{a_4}\,\eta_{12}^2 + \frac{a_1}{a_4}\,\eta_{12} + \frac{a_0}{a_4} \tag{5.94}$$

Therefore, a numerical approximation of η_{12} can be obtained by computing, e.g., `>>eig(C)` in `Matlab`. Complex solutions are artifacts of the method and should not be further considered. For every real solution, we can then substitute back to Equation 5.85 to obtain the corresponding

$$\begin{aligned}\eta_{13} \quad &= \quad \frac{m_1\,q_2 - m_2\,q_1}{m_1\,p_2 - m_2\,p_1} \qquad (5.95)\\ &= \quad \frac{d_{12}^2\,(d_{23}^2 - d_{31}^2\,\eta_{12}^2) + (d_{23}^2 - d_{31}^2)\,(d_{23}^2\,(1+\eta_{12}^2 - 2\,\eta_{12}\,c_{12}) - d_{12}^2\,\eta_{12}^2)}{2\,d_{12}^2\,(d_{23}^2\,c_{31} - d_{31}^2\,c_{23}\,\eta_{12}) + 2\,(d_{31}^2 - d_{23}^2)\,d_{12}^2\,c_{23}\,\eta_{12}}\end{aligned}$$

To get η_1, η_2 and η_3, we consider Equation 5.67, which can be rearranged as

$$d_{12}^2 = \eta_1^2\,(1+\eta_{12}^2 - 2\,\eta_{12}\,c_{12}) \tag{5.96}$$

and hence yields positive

$$\eta_1 \quad = \quad \frac{d_{12}}{\sqrt{1+\eta_{12}^2 - 2\,\eta_{12}\,c_{12}}} \tag{5.97}$$

$$\eta_2 \quad = \quad \eta_1\,\eta_{12} \tag{5.98}$$

$$\eta_3 \quad = \quad \eta_1\,\eta_{13} \tag{5.99}$$

§ **22 Case B** Let us now look at what happens when the matrix of the system (5.81) is singular. Then we have

$$m_1\,p_2 - m_2\,p_1 \quad = \quad 0 \tag{5.100}$$

$$-2\,d_{12}^2\,d_{23}^2\,(\eta_{12}\,c_{23} - c_{31}) \quad = \quad 0 \tag{5.101}$$

$$\eta_{12}\,c_{23} \quad = \quad c_{31} \tag{5.102}$$

§ **23 Case B1** When $c_{23} \neq 0$, then we get

$$\eta_{12} = \frac{c_{31}}{c_{23}} \tag{5.103}$$

Substituting it to Equations 5.72 we get

$$d_{12}^2\left((\frac{c_{31}}{c_{23}})^2 + \eta_{13}^2 - 2\,\frac{c_{31}}{c_{23}}\,\eta_{13}\,c_{23}\right) \quad = \quad d_{23}^2\left(1 + (\frac{c_{31}}{c_{23}})^2 - 2\,\frac{c_{31}}{c_{23}}\,c_{12}\right) \tag{5.104}$$

$$d_{12}^2\left(c_{31}^2 + c_{23}^2\,\eta_{13}^2 - 2\,c_{31}\,c_{23}^2\,\eta_{13}\right) \quad = \quad d_{23}^2\left(c_{23}^2 + c_{31}^2 - 2\,c_{31}\,c_{23}\,c_{12}\right) \tag{5.105}$$

and after some more manipulation obtain a quadratic equation

$$(d_{12}^2\,c_{23}^2)\,\eta_{13}^2 + (-2\,d_{12}^2\,c_{23}^2\,c_{31})\,\eta_{13} + d_{12}^2\,c_{31}^2 - d_{23}^2\,c_{23}^2 - d_{23}^2\,c_{31}^2 + 2\,d_{23}^2\,c_{12}\,c_{23}\,c_{31} = 0$$

in η_{13}. We get η_1, η_2 and η_3 from Equations 5.97, 5.98, 5.99.

§ **24 Case B2** When $c_{23} = 0$, then it follows from Equation 5.102 that $c_{31} = 0$ as well. Returning back to equations 5.72, 5.73 provides

$$d_{12}^2 \left(\eta_{12}^2 + \eta_{13}^2\right) = d_{23}^2 \left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) \tag{5.106}$$
$$d_{31}^2 \left(\eta_{12}^2 + \eta_{13}^2\right) = d_{23}^2 \left(1 + \eta_{13}^2\right) \tag{5.107}$$

Expressing η_{13} from Equation 5.107 gives

$$(d_{23}^2 - d_{31}^2)\,\eta_{13}^2 = d_{31}^2\,\eta_{12}^2 - d_{23}^2 \tag{5.108}$$

§ **25 Case B2.1** When $d_{23}^2 \neq d_{31}^2$, then we can write

$$\eta_{13}^2 = \frac{d_{31}^2\,\eta_{12}^2 - d_{23}^2}{d_{23}^2 - d_{31}^2} \tag{5.109}$$

to substitute it into Equation 5.106

$$d_{12}^2 \left(\eta_{12}^2 + \frac{d_{31}^2\,\eta_{12}^2 - d_{23}^2}{d_{23}^2 - d_{31}^2}\right) = d_{23}^2 \left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) \tag{5.110}$$

which we further manipulate to get quadratic equation in η_{12}

$$d_{23}^2 \left(d_{12}^2 - d_{23}^2 + d_{31}^2\right)\,\eta_{12}^2 \tag{5.111}$$
$$+(2\,c_{12}\,d_{23}^2\,(d_{23}^2 - d_{31}^2))\,\eta_{12} + d_{23}^2\,(d_{31}^2 - d_{12}^2 - d_{23}^2) = 0 \tag{5.112}$$

We get η_1, η_2 and η_3 from Equations 5.97, 5.98, 5.99.

§ **26 Case B2.2** Finally, when $d_{23}^2 = d_{31}^2$, then we get from Equation 5.108

$$\eta_{12} = 1 \tag{5.113}$$

and from Equation 5.106

$$\eta_{13}^2 = \frac{d_{23}^2}{d_{12}^2}\,(2 - 2\,c_{12}) - 1 \tag{5.114}$$

We get η_1, η_2 and η_3 from Equations 5.97, 5.98, 5.99.

§ **27 A modern (more elegant) solution** The classical solution is perfectly valid but it was quite tedious to derive it. Let us now present another, somewhat more elegant, solution, which exploits some of more recent results of algebraic geometry [14, 15].

Let us consider Equations 5.67, 5.68, 5.69 and proceed to Equations 5.72, 5.73, but this time with use all three pairs to get three equations in η_{12}, η_{13}

$$f_1 = d_{12}^2 \left(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}\right) - d_{23}^2 \left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) = 0 \tag{5.115}$$
$$f_2 = d_{31}^2 \left(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}\right) - d_{23}^2 \left(1 + \eta_{13}^2 - 2\,\eta_{13}\,c_{31}\right) = 0 \tag{5.116}$$
$$f_3 = d_{12}^2 \left(1 + \eta_{13}^2 - 2\,\eta_{13}\,c_{31}\right) - d_{31}^2 \left(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}\right) = 0 \tag{5.117}$$

It is known [14, 15] that the solution to a set of k algebraic equations

$$f_i(x_1, \dots, x_n) = 0, \quad i = 1 \dots, k$$

in n variables, can (provided it exists) be always obtained by deriving a polynomial $g(x_n) = 0$ in the last variable by the following procedure. If the system, does not have any solution, $g_n = 1$, i.e. a non-zero constant, leading to contradiction $1 = 0$.

First generate additional equations by multiplying all f_i by all possible monomials up to degree m

$$x_1, \ldots, x_n, x_1^2, x_1\, x_2, \ldots, x_n^2, x_1^3, x_1^2\, x_2, \ldots, x_n^m$$

to get equations

$$f_1 = 0, \ldots, f_n = 0,\; x_1 f_1 = 0, \ldots, x_n f_n = 0,\; x_1^2 f_1 = 0,\; x_1\, x_2 f_1 = 0, \ldots, x_n^m f_n = 0$$

The degree m needs to be chosen such that the next step yields the desired results. It is always possible to choose such m but it may sometimes be found only by using more and more monomials until the Gaussian elimination of the matrix of coefficients, which combine monomials, does not produce a row corresponding to an equation in x_n only. Let us demonstrate this process by solving our problem.

We use the following four monomials of maximal degree two

$$\eta_{12},\; \eta_{13},\; \eta_{12}\, \eta_{13},\; \eta_{12}^2$$

Notice that we did not include the second degree monomial η_{13}^2 since it turns out that equations generated by that monomial are not necessary. We obtain $15 = 3+4\,3$ equations

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \eta_{12}\, f_1 \\ \eta_{12}\, f_2 \\ \eta_{12}\, f_3 \\ \eta_{13}\, f_1 \\ \eta_{13}\, f_2 \\ \eta_{13}\, f_3 \\ \eta_{12}\, \eta_{13}\, f_1 \\ \eta_{12}\, \eta_{13}\, f_2 \\ \eta_{12}\, \eta_{13}\, f_3 \\ \eta_{12}^2\, f_1 \\ \eta_{12}^2\, f_2 \\ \eta_{12}^2\, f_3 \end{bmatrix} = \mathtt{M} \begin{bmatrix} \eta_{12}\, \eta_{13}^3 \\ \eta_{13}^3 \\ \eta_{12}^2\, \eta_{13}^2 \\ \eta_{13}^2\, \eta_{12} \\ \eta_{13}^2 \\ \eta_{12}^3\, \eta_{13} \\ \eta_{13}\, \eta_{12}^2 \\ \eta_{13}\, \eta_{12} \\ \eta_{13} \\ \eta_{12}^4 \\ \eta_{12}^3 \\ \eta_{12}^2 \\ \eta_{12} \\ 1 \end{bmatrix} = \mathtt{M}\,\mathtt{m} = 0 \tag{5.118}$$

with

$$\mathtt{M} = \begin{bmatrix}
0 & 0 & 0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 \\
0 & 0 & 0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 \\
0 & 0 & 0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 \\
0 & 0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 & 0 \\
0 & 0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 & 0 \\
0 & 0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 & 0 \\
0 & m_1 & 0 & -m_7 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 & 0 & 0 & 0 \\
0 & m_5 & 0 & m_9 & -m_{10} & 0 & -m_3 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -m_1 & 0 & 0 & m_{11} & 0 & m_3 & -m_{12} & m_6 & 0 & 0 & 0 & 0 & 0 \\
m_1 & 0 & -m_7 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_5 & 0 & m_9 & -m_{10} & 0 & -m_3 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-m_1 & 0 & 0 & m_{11} & 0 & m_3 & -m_{12} & m_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 \\
0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 & 0 & 0 \\
0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 & 0 & 0
\end{bmatrix}$$

and

$$\begin{array}{lllllllllll}
m_1 & = & d_{12}^2 & m_4 & = & d_{12}^2 - d_{23}^2 & m_7 & = & 2\, d_{12}^2\, c_{23} & m_{10} & = & 2\, d_{23}^2\, c_{31} \\
m_2 & = & d_{23}^2 & m_5 & = & d_{23}^2 - d_{31}^2 & m_8 & = & 2\, d_{23}^2\, c_{12} & m_{11} & = & 2\, d_{12}^2\, c_{31} \\
m_3 & = & d_{31}^2 & m_6 & = & d_{31}^2 - d_{12}^2 & m_9 & = & 2\, d_{31}^2\, c_{23} & m_{12} & = & 2\, d_{31}^2\, c_{12}
\end{array} \tag{5.119}$$

Matrix $\mathtt{M}$ holds coefficients and vector $\mathtt{m}$ holds monomials.

Notice that in the last five monomials on the right in Equation 5.118, there appears only η_{12}. We have deliberately ordered monomials in this way. Next, we perform Gaussian elimination (with pivoting) of matrix $\mathtt{M}$ and get new matrix $\mathtt{M}'$.

One can verify that that the 10^{th} row of $\mathtt{M}'$ has the first nine elements equal to zero. Therefore

$$\mathtt{M}'_{10,:}\,\mathtt{m} = 0$$

is a polynomial only in η_{12}. In fact, it is exactly a non-zero multiple of polynomials obtained in cases A, B1, B2.1 and B2.2 above.

Discussion of the cases happens in the Gaussian elimination with pivoting, which avoids dividing by elements close to zero. The resulting polynomial may be of degree four (case A) but will come out of a lower degree in other cases.

§ 28 Computing camera center coordinates To compute the coordinates of the camera center, we shall use the fact that the center must lie on the intersection of three spheres with radii η_1, η_2, η_3 and with centers X_1, X_2, X_3.

To simplify the computation, we will choose a special coordinate system, in which the computation will be easier. We place the origin of the new coordinate system to point X_1 and construct new basis τ, in which vectors $\vec{Y}_{1\tau}$, $\vec{Y}_{2\tau}$, $\vec{Y}_{3\tau}$, representing X_1, X_2, X_3 in the new coordinates system (X_1, τ) will be particularly simple. We require

$$\vec{Y}_{1\tau} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{Y}_{2\tau} = \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}, \quad \vec{Y}_{3\tau} = \begin{bmatrix} q \\ r \\ 0 \end{bmatrix} \tag{5.120}$$

To fully specify τ, we will require it to be orthonormal. We shall see that this requirement, altogether with Equation 5.120, fix τ uniquely.

Vectors $\vec{Y}_{i\tau}$ are related to vectors $\vec{X}_{i\delta}$ as

$$\vec{Y}_{i\tau} = \mathtt{S}\,(\vec{X}_{i\delta} - \vec{X}_{1\delta}), \quad i = 1, 2, 3 \tag{5.121}$$

where $\mathtt{S}$ is the the matrix transforming coordinates of a vector from basis δ to basis τ.

Since both δ and τ are orthonormal, there holds $\mathtt{S}^\top \mathtt{S} = \mathtt{I}$, $|\mathtt{S}| = 1$. We see that $\vec{Y}_{i\tau}$ becomes automatically zero vector by the choice of the origin. Let us now construct $\mathtt{S}$ and determine p, q and r.

Exploiting the required coordinates we get

$$\begin{bmatrix} \vec{Y}_{1\tau} & \vec{Y}_{2\tau} & \vec{Y}_{3\tau} \end{bmatrix} = \mathtt{S} \begin{bmatrix} \mathtt{0} & \vec{X}_{2\delta} - \vec{X}_{1\delta} & \vec{X}_{3\delta} - \vec{X}_{1\delta} \end{bmatrix} \tag{5.122}$$

$$\mathtt{S}^\top \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathtt{0} & \vec{X}_{2\delta} - \vec{X}_{1\delta} & \vec{X}_{3\delta} - \vec{X}_{1\delta} \end{bmatrix} \tag{5.123}$$

$$\begin{bmatrix} \mathtt{s}_1 & \mathtt{s}_2 & \mathtt{s}_3 \end{bmatrix} \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathtt{0} & \vec{X}_{2\delta} - \vec{X}_{1\delta} & \vec{X}_{3\delta} - \vec{X}_{1\delta} \end{bmatrix} \tag{5.124}$$

where column matrices $\mathtt{s}_1$, $\mathtt{s}_2$, $\mathtt{s}_3$ denote columns of $\mathtt{S}^\top$. Passing to vector equations delivers

$$p\,\mathtt{s}_1 = \vec{X}_{2\delta} - \vec{X}_{1\delta}, \quad q\,\mathtt{s}_1 + r\,\mathtt{s}_2 = \vec{X}_{3\delta} - \vec{X}_{1\delta} \tag{5.125}$$

Since we require $\|\mathtt{s}_1\| = 1$, we get

$$p = \|\vec{X}_{2\delta} - \vec{X}_{1\delta}\|, \quad \mathtt{s}_1 = \frac{\vec{X}_{2\delta} - \vec{X}_{1\delta}}{\|\vec{X}_{2\delta} - \vec{X}_{1\delta}\|} \tag{5.126}$$

Next we use $\mathtt{s}_1^\top \mathtt{s}_2 = 0$ and $\|\mathtt{s}_1\| = \|\mathtt{s}_2\| = 1$ to get q and r

$$\begin{aligned}
\mathtt{s}_1^\top (q\,\mathtt{s}_1 + r\,\mathtt{s}_2) &= \mathtt{s}_1^\top (\vec{X}_{3\delta} - \vec{X}_{1\delta}) && (5.127)\\
q &= \mathtt{s}_1^\top (\vec{X}_{3\delta} - \vec{X}_{1\delta}) && (5.128)\\
r\,\mathtt{s}_2 &= \vec{X}_{3\delta} - \vec{X}_{1\delta} - q\,\mathtt{s}_1 && (5.129)\\
r &= \|\vec{X}_{3\delta} - \vec{X}_{1\delta}) - q\,\mathtt{s}_1\| && (5.130)
\end{aligned}$$

and we get $\mathtt{s}_2$ as

$$\mathtt{s}_2 = \frac{\vec{X}_{3\delta} - \vec{X}_{1\delta} - q\,\mathtt{s}_1}{\|\vec{X}_{3\delta} - \vec{X}_{1\delta} - q\,\mathtt{s}_1\|} \tag{5.131}$$

Finally, we compute

$$\mathtt{s}_3 = \mathtt{s}_1 \times \mathtt{s}_2 \tag{5.132}$$

Matrix $\mathtt{S}$ has been determined and hence we can get $\vec{Y}_{1\tau}$, $\vec{Y}_{2\tau}$, $\vec{Y}_{3\tau}$, in the form prescribed by (5.120).

Let us now compute vector $\vec{D}_\tau$, which represents the camera center C in (X_1, τ). Introduce coordinates of

$$\vec{D}_\tau = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5.133}$$

to formulate that C is the intersection of three spheres with radii η_1, η_2, η_3 and with centers X_1, X_2, X_3 represented by $\vec{Y}_{1\tau}$, $\vec{Y}_{2\tau}$, $\vec{Y}_{3\tau}$

$$\begin{aligned}
x^2 + y^2 + z^2 &= \eta_1^2 && (5.134)\\
(x-p)^2 + y^2 + z^2 &= \eta_2^2 && (5.135)\\
(x-q)^2 + (y-r)^2 + z^2 &= \eta_3^2 && (5.136)
\end{aligned}$$

Subtracting Equation 5.134 from Equation 5.135 yields

$$\begin{aligned}
0 &= (x-p)^2 - x^2 + \eta_1^2 - \eta_2^2 && (5.137)\\
0 &= -2\,p\,x + p^2 + \eta_1^2 - \eta_2^2 && (5.138)\\
x &= \frac{p^2 + \eta_1^2 - \eta_2^2}{2\,p} && (5.139)
\end{aligned}$$

Subtracting Equation 5.135 from Equation 5.136 yields

$$\begin{aligned}
0 &= (x-r)^2 - y^2 - \eta_3^2 + \eta_2^2 + (x-q)^2 - (x-p)^2 && (5.140)\\
0 &= -2\,r\;y + r^2 - \eta_3^2 + \eta_2^2 + 2\,(p-q)\,x + q^2 - p^2 && (5.141)\\
y &= \frac{r^2 + q^2 - p^2 - \eta_3^2 + \eta_2^2 + 2\,(p-q)\,x}{2\,r} && (5.142)
\end{aligned}$$

Finally, we get z from Equation 5.134

$$z = \pm\sqrt{\eta_1^2 - x^2 - y^2} \tag{5.143}$$

which finally gives us $\vec{D}_\tau$. To get vector $\vec{C}_\delta$ representing camera center C in the world coordinate system (O, δ), we use Equation 5.121 to arrive at

$$\vec{C}_\delta = \mathtt{S}^\top \vec{D}_\tau + \vec{X}_{1\delta} \tag{5.144}$$

§ **29 Computing camera orientation** Camera orientation can be now recovered by relating direction vectors of rays, which we recover in (C, ϵ), to the same direction vectors measured in (C, δ). We know from Equation 5.44 that in general

$$\vec{y}_\epsilon = \mathtt{R}\,\vec{y}_\delta \tag{5.145}$$

We construct unit ray direction vectors in δ as follows

$$\vec{y}_{i\delta} = \frac{\vec{X}_{i\delta} - \vec{C}_{i\delta}}{\|\vec{X}_{i\delta} - \vec{C}_{i\delta}\|}, \quad i = 1, 2, 3 \tag{5.146}$$

To get unit $\vec{y}_{i\epsilon}$, we use the relationship between bases β, γ, and ϵ

$$\vec{y}_{i\epsilon} = \frac{\vec{x}_{i\epsilon}}{\|\vec{x}_{i\epsilon}\|} = \frac{f\,\vec{x}_{i\gamma}}{\|f\,\vec{x}_{i\gamma}\|} = \frac{\vec{x}_{i\gamma}}{\|\vec{x}_{i\gamma}\|} = \frac{\mathtt{K}^{-1}\,\vec{x}_{i\beta}}{\|\mathtt{K}^{-1}\,\vec{x}_{i\beta}\|} \tag{5.147}$$

Therefore, we get

$$\frac{\mathtt{K}^{-1}\,\vec{x}_{i\beta}}{\|\mathtt{K}^{-1}\,\vec{x}_{i\beta}\|} = \mathtt{R}\,\frac{\vec{X}_{i\delta} - \vec{C}_\delta}{\|\vec{X}_{i\delta} - \vec{C}_\delta\|} \quad i = 1, 2, 3 \tag{5.148}$$

which leads to the final equation determining $\mathtt{R}$

$$\mathtt{R} = \left[\begin{array}{ccc} \frac{\mathtt{K}^{-1}\,\vec{x}_{1\beta}}{\|\mathtt{K}^{-1}\,\vec{x}_{1\beta}\|} & \frac{\mathtt{K}^{-1}\,\vec{x}_{2\beta}}{\|\mathtt{K}^{-1}\,\vec{x}_{2\beta}\|} & \frac{\mathtt{K}^{-1}\,\vec{x}_{3\beta}}{\|\mathtt{K}^{-1}\,\vec{x}_{3\beta}\|} \end{array}\right] \left[\begin{array}{ccc} \frac{\vec{X}_{1\delta}-\vec{C}_\delta}{\|\vec{X}_{1\delta}-\vec{C}_\delta\|} & \frac{\vec{X}_{2\delta}-\vec{C}_\delta}{\|\vec{X}2\delta-\vec{C}_\delta\|} & \frac{\vec{X}_{3\delta}-\vec{C}_\delta}{\|\vec{X}_{3\delta}-\vec{C}_\delta\|} \end{array}\right]^{-1} \tag{5.149}$$

6 Homography

We shall next investigate the relationship between projections of 3D points by two perspective cameras into two images.

In general, the projections depend on the shape of the scene and camera poses and this relationship may be very difficult to describe. However, there are two very important situations when the relationship can be given in a form of a special image transform, the *homography*.

Let us first consider the situation when two (different) cameras *share a common projection center*. That means, the cameras may have different coordinate systems, different orientations but *must* have the same projection center. This situation often arises when photographing with a camera rotating around its projection center, e.g., when taking images for constructing a panorama capturing wide view angle. We shall see that the corresponding projections will be related by a homography.

Next, we shall look at a different situation when the cameras are unconstrained, i.e. they can be anywhere in the space and with completely different poses and coordinate systems, but 3D points are forced to lie in a single plane not containing the camera centers. This situation arises, e.g., when photographing a flat screen, a poster or a board from different viewpoints. Again, the corresponding projections of the points in the plane (but not the projections of the points out of the plane) will be related by a homography.

6.1 Homography between images with the same center

Let us consider two perspective cameras with identical projection centers $C = C'$, which project point X from space to their respective projection planes π and π', Figure 6.1. We introduce image coordinate systems (o, α) with $\alpha = [\vec{b}_1, \vec{b}_2]$ in π and (o', α') with $\alpha' = [\vec{b}'_1, \vec{b}'_2]$ in π' and use them to construct the corresponding camera coordinate systems (C, β) with $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3 = \overrightarrow{Co}]$ and (C, β') with $\beta' = [\vec{b}'_1, \vec{b}'_2, \vec{b}'_3 = \overrightarrow{Co'}]$.

Point X is projected to image points along the projection ray, which is intersected with π and π'. The projection of X in π is represented by vector $\vec{u}_\alpha = [u, v]^\top$. The projection of X in π' is represented by vector $\vec{u}'_{\alpha'} = [u', v']^\top$.

Vectors $\vec{x}$ and $\vec{x}'$ are two direction vectors of the same ray and hence *are linearly dependent*. Since they are both non-zero for $X \neq C$, their linear dependence is equivalent with

$$\exists \lambda \in \mathbb{R} : \lambda \vec{x}' = \vec{x}$$

To arrive at the relationship between the available coordinates of vectors $\vec{x}$ and

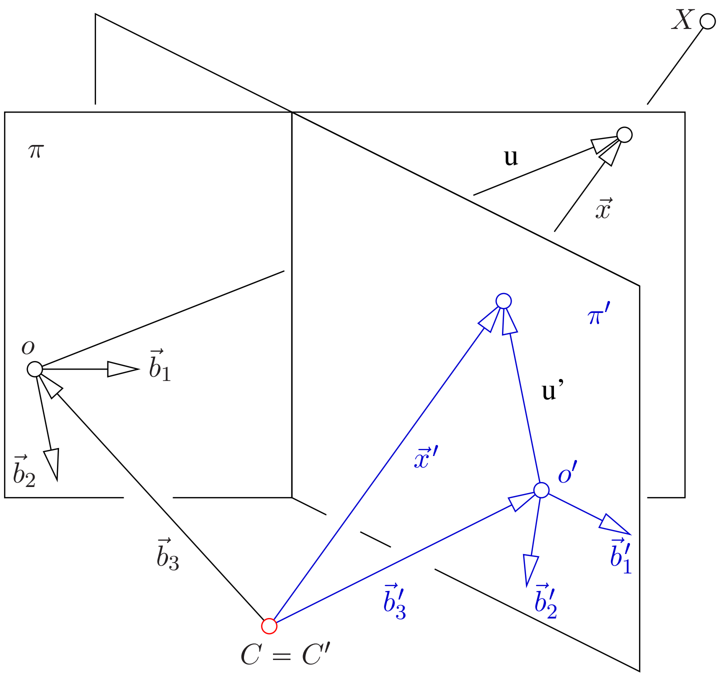


Figure 6.1: Cameras share a projections center. Image projections are related by a homography.

$\vec{x}'$, we shall now pass from vectors to their coordinates. There holds

$$\begin{aligned}\lambda\,\vec{x}' &= \vec{x}\\ \lambda\,\vec{x}'_{\beta'} &= \vec{x}_{\beta'}\\ \lambda\,\vec{x}'_{\beta'} &= \mathtt{H}\,\vec{x}_{\beta'}\end{aligned}$$

for some 3×3 real matrix $\mathtt{H}$ with $\operatorname{rank}\mathtt{H} = 3$, which transforms coordinates of a vector from basis β to basis β'.

Considering the choices of camera coordinate systems, we see that

$$\begin{aligned}\lambda\,\vec{x}'_{\beta'} &= \mathtt{H}\,\vec{x}_{\beta'}\\ \lambda\begin{bmatrix}u'\\ v'\\ 1\end{bmatrix} &= \mathtt{H}\begin{bmatrix}u\\ v\\ 1\end{bmatrix}\end{aligned}$$

We obtained an interesting relationship. The above equations tell us that the image projections are related by a transformation, which is (up to C) *independent* of the position of points X in space. This is the consequence of having $C = C'$.

6.2 Homography between images of a plane

6.2.1 Image of a plane

Let study the relationship between the coordinates of 3D points X, which all lie in a plane σ. Coordinates of points X are measured in a coordinate system (O, δ) with

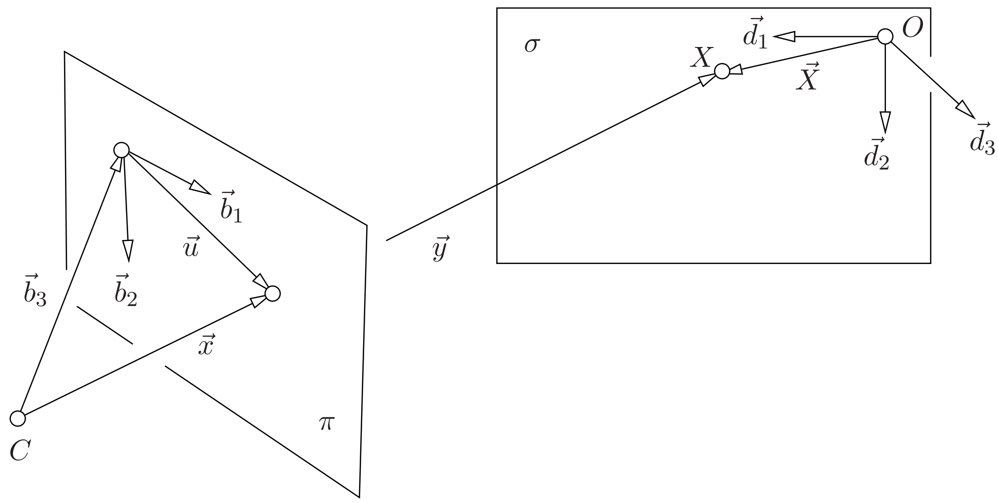


Figure 6.2: All 3D points are in a single plane. Coordinates in the plane and in the image are related by a homography.

$\delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$. Vectors $\vec{d}_1, \vec{d}_2$ span plane σ and therefore

$$\vec{X}_\delta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

for some real x, y.

The points X are projected by a perspective camera with projection matrix $\mathtt{P}$ into image coordinates $\vec{u}_\alpha = [u, v]^\top$, w.r.t. an image coordinate system (o, α) with $\alpha = [\vec{b}_1, \vec{b}_2]$. The corresponding camera coordinate system is (C, β) with $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$.

To find the relationship between the coordinates of $\vec{X}_\delta$ and $\vec{u}_\alpha$, we project points X by $\mathtt{P}$ into projections $\vec{x}_\beta$ as

$$\eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \eta\,\vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{H}\,\vec{y}_\tau$$

where $\mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3, \mathtt{p}_4$ are columns of $\mathtt{P}$.

Notice that 3×1 matrix $\vec{y}_\tau = [x, y, 1]^\top$ represents point X in the coordinate system (C, τ) with the basis $\tau = (\vec{d}_1, \vec{d}_2, \vec{d}_4)$, where the $\vec{d}_4 = \overrightarrow{CO}$ is the vector assigned to the pair of points (C, O). If point C is not in σ, then vectors $\vec{d}_1, \vec{d}_2, \vec{d}_4$ are independent and hence form a basis. Therefore, matrix

$$\mathtt{H} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix}$$

represents a change of coordinates and has rank 3.

When we think about pair (C, σ) as about a camera that shares projection center with camera (C, π) and imagine that points X are all (accidentally) in the image plane σ, we see that we recovered relationship between cameras sharing projection center.

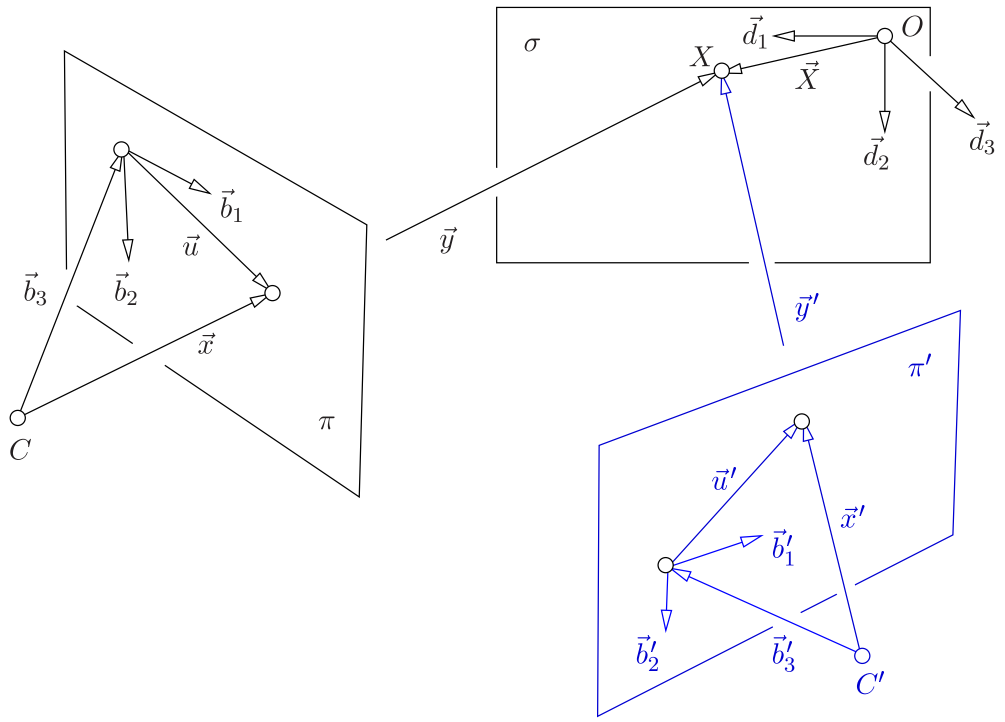


Figure 6.3: All 3D points are in a single plane. Two images of the points are related by a homography.

6.2.2 Two images of a plane

We shall now consider the situation when all points in the scene are in a single plane. Then, as we shall see, the projections of the 3D points which are in the plane are again related by a homography even when the camera centers *are located at possibly different points in the space.*

Let us consider a plane σ and two perspective cameras with (in general different) projection centers C and C', which *do not lie in* σ and corresponding projection matrices $\mathtt{P}$ and $\mathtt{P}'$

$$\begin{aligned}\mathtt{P} &= \begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4\end{bmatrix}\\ \mathtt{P}' &= \begin{bmatrix}\mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_3 & \mathtt{p}'_4\end{bmatrix}\end{aligned}$$

where $\mathtt{p}_i \in \mathbb{R}^3$ and $\mathtt{p}'_i \in \mathbb{R}^3$, $i = 1, \ldots, 4$ stand for the columns of $\mathtt{P}$, $\mathtt{P}'$.

We establish coordinates systems (O, δ), (C, β) and (C', β') in the standard way, see Figure 6.3 to get

$$\vec{X}_\delta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

for some real x, y.

Point $X \in \sigma$ is projected to the cameras as

$$\begin{aligned}
\eta\, \vec{x}_\beta &= \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}\, \vec{y}_\tau \\
\eta'\, \vec{y}'_\tau &= \mathtt{P}' \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_3 & \mathtt{p}'_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}'\, \vec{y}'_{\tau'}
\end{aligned}$$

for some $\lambda, \lambda' \in \mathbb{R}\backslash\{0\}$ and two new coordinate systems (C, τ) with $\tau = (\vec{d}_1, \vec{d}_2, \vec{d}_4)$, where the $\vec{d}_4 = \overrightarrow{CO}$ and (C', τ') with $\tau' = (\vec{d}_1, \vec{d}_2, \vec{d}'_4)$, where the $\vec{d}'_4 = \overrightarrow{CO'}$.

We see that there are two different vectors $\vec{y}$ and $\vec{y}'$, which appear on the right hand side of the equations but they are in different bases, i.e. as $\vec{y}_\tau$ and $\vec{y}'_{\tau'}$

$$\begin{aligned}
\eta\, \vec{x}_\beta &= \mathtt{G}\, \vec{y}_\tau \\
\eta' \vec{x}'_{\beta'} &= \mathtt{G}' \vec{y}'_{\tau'}
\end{aligned}$$

with $\mathtt{G} = [\mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_4]$ and $\mathtt{G}' = [\mathtt{p}'_1, \mathtt{p}'_2, \mathtt{p}'_4]$.

Coordinate systems (C, τ) and (C', τ') are so special that

$$\vec{y}_\tau = \vec{y}'_{\tau'}$$

for all points in σ and therefore, considering that $C \notin \sigma$ and $C' \notin \sigma$, we get

$$\eta' \vec{x}'_{\beta'} = \mathtt{G}'\, \mathtt{G}^{-1} \eta\, \vec{x}_\beta$$

which we can write

$$\xi\, \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta \tag{6.1}$$

for $\xi = \frac{\eta'}{\eta}$ and $\mathtt{H} = \mathtt{G}'\, \mathtt{G}^{-1}$. Clearly, $\mathtt{H} \in \mathbb{R}^{3\times3}$, $\operatorname{rank} \mathtt{H} = 3$.

We could interpret this situation also in such a way that two images of a (flat) image are related by the homography, which is a combination of the homographies relating the image to its images.

6.3 Spherical image

Consider a camera rotating around a center C and collecting n images all around such that every ray from C is captured in some image. We can choose one camera, e.g. the first one, and relate all other cameras to it as

$$\lambda_i\, \vec{x}_{\beta_1} = \mathtt{H}_i\, \vec{x}_{\beta_i}, \quad i = 1, \ldots, n$$

Since all vectors $\vec{x}$ were captured, there inevitably will appear a vector with coordinates

$$\vec{x}_{\beta_1} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Such vector does not represent any point in the affine projection plane π_1 of the first camera because it does not have the third coordinate equal to one. To be able to represent rays in all directions, we have to introduce *spherical image*, which is the set of all unit vectors in $\mathbb{R}^3$ (also called *omnidirectional image*). We sometimes use only a subset of the sphere, typically a cylinder, to capture *panoramic image*. In such a case, we can remap pixels onto such cylinder and then unwarp the cylinder into a plane. Notice however, that in such a representation, straight lines in space do not project to straight lines in images.

All equations we have developed so far work also for vectors with last zero coordinate. We shall see later that there is yet another important representation, which is somewhere between the affine projection plane and full a spherical image, called *projective plane*.

6.4 Homography

Let us summarize the findings related to homography to see where it appears.

Let us encounter one of the following situations

1. **Two images with one projection center** Let $[u, v]^\top$ and $[u', v']$ be coordinates of the projections of 3D points into two images by two perspective cameras with identical projection centers;

2. **Image of a plane**. Let $[u, v]^\top$ be coordinates of 3D points all in one plane π, w.r.t. a coordinate system in σ and $[u', v']^\top$ coordinates of their projections by a perspective cameras with projection center not in the plane σ;

3. **Two images of a plane** Let $[u, v]^\top$ and $[u', v']$ be coordinates of the projections of 3D points all in one plane σ, into two images by two perspective cameras with projection centers not in σ;

then there holds

$$\exists\, \mathtt{H} \in \mathbb{R}^{3\times 3},\ \mathrm{rank}\, \mathtt{H} = 3,\ \text{so that}\ \forall\, [u, v]^\top \overset{corr}{\leftrightarrow} [u', v']^\top\ \exists \lambda \in \mathbb{R} :\ \lambda \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathtt{H} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{6.2}$$

where $w = w' = 1$ for perspective images and may be general for spherical images.

In all three cases, coordinates of points are related by a homography.

We have used linear algebra to derive the relationship between the coordinates of image points in the above form. The homography can be also represented in a different way.

To see that, we shall eliminate λ as follows

$$\lambda \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \mathtt{H} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\lambda u' &= h_{11}\,u + h_{12}\,v + h_{13}\\
\lambda v' &= h_{21}\,u + h_{22}\,v + h_{23}\\
\lambda 1 &= h_{31}\,u + h_{32}\,v + h_{33}\\
u' &= \frac{h_{11}\,u + h_{12}\,v + h_{13}}{h_{31}\,u + h_{32}\,v + h_{33}}\\
v' &= \frac{h_{21}\,u + h_{22}\,v + h_{23}}{h_{31}\,u + h_{32}\,v + h_{33}}
\end{aligned}$$

We see that mapping h obtained as

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = h\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \frac{h_{11}\,u+h_{12}\,v+h_{13}}{h_{31}\,u+h_{32}\,v+h_{33}} \\ \frac{h_{21}\,u+h_{22}\,v+h_{23}}{h_{31}\,u+h_{32}\,v+h_{33}} \end{bmatrix}$$

is a mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$ but *it is not linear*! It contains fractions of *affine* functions.

On the other hand, we can understand the homography as a linear mapping in certain sense. However, it is not a linear mapping in the natural sense in which we really use it.

Matrix $\mathtt{H}$ is a matrix and in this sense it represents a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$. However, we are not interested in the individual vectors in $\mathbb{R}^3$ but in complete one-dimensional subspaces, which correspond to the direction vectors representing projection rays.

Notice that λ can accommodate for any change of the length of $\begin{bmatrix} u & v & 1 \end{bmatrix}^\top$ (except for making it zero) since it can be split into ξ, ξ' and used as

$$\begin{aligned}
\xi' \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} &= \mathtt{H}\,\xi \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}\\
\mathtt{x}' &= \mathtt{H}\,\mathtt{x}
\end{aligned}$$

We can now think about $\mathtt{x}$ and $\mathtt{x}'$ as about one-dimensional subspaces of $\mathbb{R}^3$ generated by $\vec{x}$ and $\vec{x}'$. The "equation"

$$\mathtt{x}' = \mathtt{H}\,\mathtt{x}$$

then actually means

$$\exists \vec{x} \in \mathtt{x} \text{ and } \exists \vec{x}' \in \mathtt{x}' \text{ such that } \vec{x}' = \mathtt{H}\,\vec{x}$$

Thus the homography can be seen as a mapping between one-dimensional subspaces of $\mathbb{R}^3$. While $\mathbb{R}^3$ itself is a linear space, the set of its one-dimensional subspaces, in the way we use them, *is not* a linear space and therefore the homography is not a linear mapping although it is represented by a matrix $\mathtt{H}$, which is used to multiply vectors.

It is also important to notice the true relationship between homographies and 3×3 real matrices. Any 3×3 real matrix of rank 3 represents a homography but many *different* matrices represent the same homography. Let's see why.

Let us consider $\mathtt{H} \in \mathbb{R}^{3\times 3}$ and $\mathtt{G} \in \mathbb{R}^{3\times 3}$ such that $\mathtt{H} = \tau\,\mathtt{G}$ for some $\tau \neq 0$. We can write

$$\begin{aligned}
\xi'\,\vec{x}' &= \mathtt{H}\,\vec{x}\\
\xi\,\xi'\,\vec{x}' &= \xi\,\mathtt{H}\,\vec{x}\\
\xi\,\xi'\,\vec{x}' &= \mathtt{G}\,\vec{x}\\
\lambda'\,\vec{x}' &= \mathtt{G}\,\vec{x}
\end{aligned}$$

We see that $\mathtt{H}$ and $\mathtt{G}$ represent the *same homography*. Indeed, two matrices related by a non-zero multiple represent the same homography. Hence, it suggests itself to associate homographies with one-dimensional subspaces of 3×3 matrices.

6.5 Image of image of image

Let us now consider an interesting and classical question. What happens when we take a photograph of a photograph of a planar painting? What is the realtionship of the second photograph to the original planar painting?

6.6 Computing homography from image matches

Let us turn to the computational aspect of the homography relationship between images. Our goal is to find the homograph mapping from a few pairs of corresponding image points. We shall see that this problem leads to solving a system of linear equations.

6.6.1 General perspective cameras

Our goal is to find matrix $\mathtt{H}$ in Equation 6.2 without assuming any knowledge about cameras. Let us introduce symbols for rows of homography $\mathtt{H}$

$$\mathtt{H} = \begin{bmatrix} \mathtt{h}_1^\top \\ \mathtt{h}_2^\top \\ \mathtt{h}_3^\top \end{bmatrix}$$

and rewrite the above matrix Equation 6.2 as

$$\begin{aligned}
\lambda\,u' &= \mathtt{h}_1^\top \mathtt{x}\\
\lambda\,v' &= \mathtt{h}_2^\top \mathtt{x}\\
\lambda &= \mathtt{h}_3^\top \mathtt{x}
\end{aligned}$$

Eliminate λ from the first two equations using the third one

$$\begin{aligned}
(\mathtt{h}_3^\top \mathtt{x})\,u' &= \mathtt{h}_1^\top \mathtt{x}\\
(\mathtt{h}_3^\top \mathtt{x})\,v' &= \mathtt{h}_2^\top \mathtt{x}
\end{aligned}$$

move all to the left hand side and reshape it using $\mathtt{x}^\top \mathtt{y} = \mathtt{y}^\top \mathtt{x}$

$$\begin{aligned} \mathtt{x}^\top \mathtt{h}_1 - (u' \mathtt{x}^\top)\, \mathtt{h}_3 &= 0 \\ \mathtt{x}^\top \mathtt{h}_2 - (v' \mathtt{x}^\top)\, \mathtt{h}_3 &= 0 \end{aligned}$$

Introduce notation

$$\mathtt{h} = \begin{bmatrix} \mathtt{h}_1^\top & \mathtt{h}_2^\top & \mathtt{h}_3^\top \end{bmatrix}^\top$$

and express the above two equations in a matrix form

$$\begin{bmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \end{bmatrix} \mathtt{h} = 0$$

Every correspondence $[u, v]^\top \overset{corr}{\leftrightarrow} [u', v']^\top$ brings two rows to a matrix

$$\begin{aligned} \begin{bmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \\ & & & & \vdots & & & & \end{bmatrix} \mathtt{h} &= 0 \\ \\ \mathtt{A} \qquad\qquad \mathtt{h} &= 0 \end{aligned}$$

If $\xi\, \mathtt{G} = \mathtt{H}$, $\xi \neq 0$ then both $\mathtt{G}, \mathtt{H}$ represent the same homography. We are therefore looking for one-dimensional subspaces of 3×3 matrices of rank 3. Each such subspace determines one homography. Also note that the zero matrix, $\mathtt{0}$, does not represent an interesting mapping.

We need therefore at least 4 correspondences in general position to obtain rank 8 matrix

$$\begin{aligned} \begin{bmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \\ & & & & \vdots & & & & \end{bmatrix} \mathtt{h} &= 0 \\ \mathtt{A} \qquad\qquad \mathtt{h} &= 0 \end{aligned}$$

By general position we mean that the matrix $\mathtt{A}$ must have *rank* 8 to provide a single one-dimensional subspace of its solutions. This happens when no 3 out of the 4 points are on the same line.

Notice that $\mathtt{A}$ can be written in the form

$$\mathtt{A} = \begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1'u_1 & -u_1'v_1 & -u_1' \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u_2'u_2 & -u_2'v_2 & -u_2' \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v_1'u_1 & -v_1'v_1 & -v_1' \\ 0 & 0 & 0 & u_2 & v_2 & 1 & -v_2'u_2 & -v_2'v_2 & -v_2' \\ & & & & \vdots & & & & \end{bmatrix}$$

which can be rewritten more concisely as

$$\mathtt{A} = \begin{bmatrix} \mathtt{x}_1^\top & \mathtt{0} & -u_1'\,\mathtt{x}_1^\top \\ \mathtt{x}_2^\top & \mathtt{0} & -u_2'\,\mathtt{x}_2^\top \\ & \vdots & \\ \mathtt{0} & \mathtt{x}_1^\top & -v_1'\,\mathtt{x}_1^\top \\ \mathtt{0} & \mathtt{x}_2^\top & -v_2'\,\mathtt{x}_2^\top \\ & \vdots & \end{bmatrix}$$

6.6.2 Calibrated cameras

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