Lecture 8

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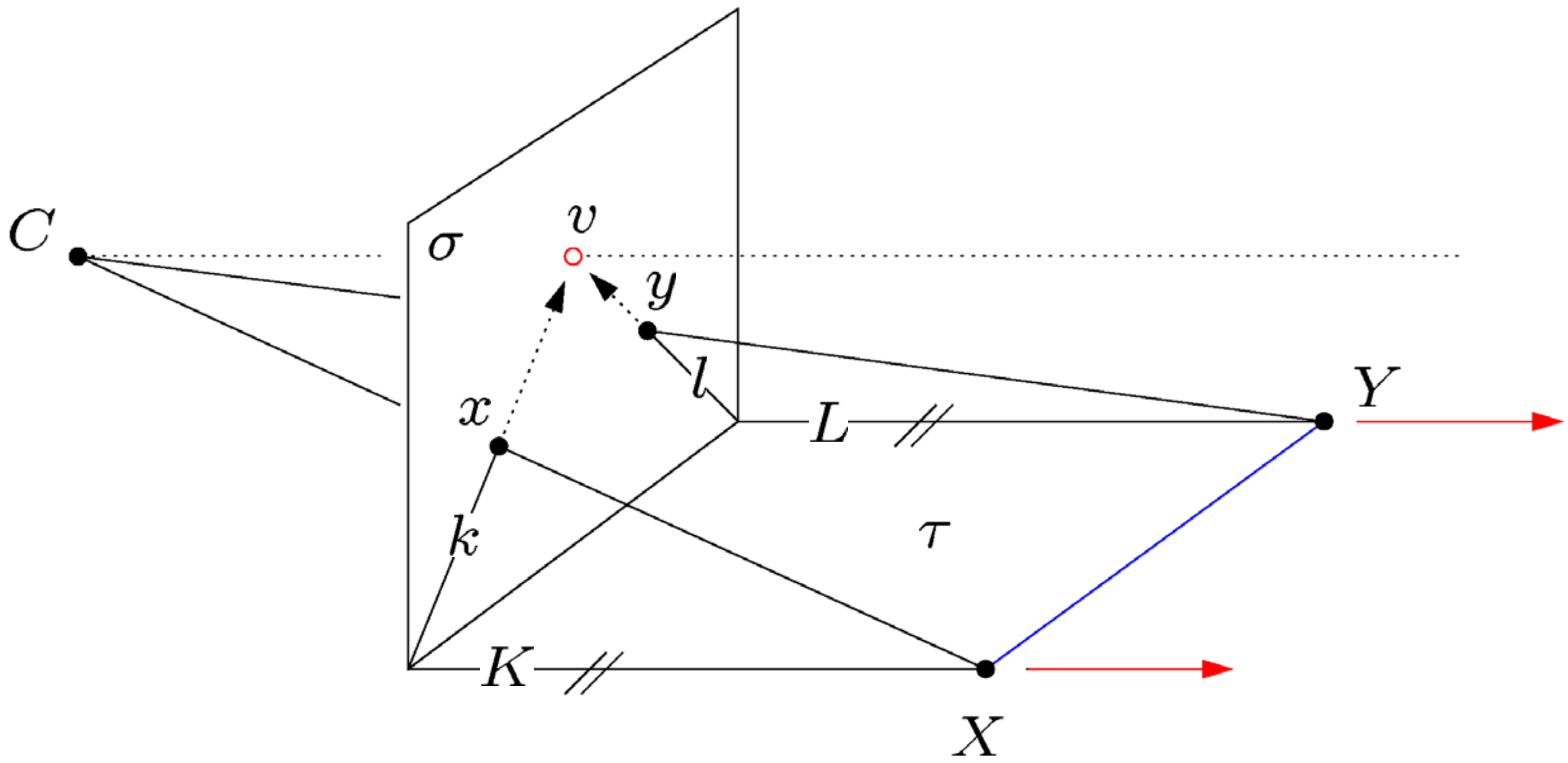
Vanishing point



When modeling perspective projection in the affine space with affine projection planes, we meet somewhat unpleasant situations. For instance, imagine a projection of two parallel lines K, L, which are in a plane τ in the space into the projection plane π through the center C.

The lines K, L project to image lines k, l. As we go with two points X, Y along the lines k, l away from the projection plane, their images x, y get closer and closer to the point v in the image but they do not reach point v. we shall call point v the *point of convergence* of lines K, L.

The point v is called the *vanishing point* (ubeznik in Czech).

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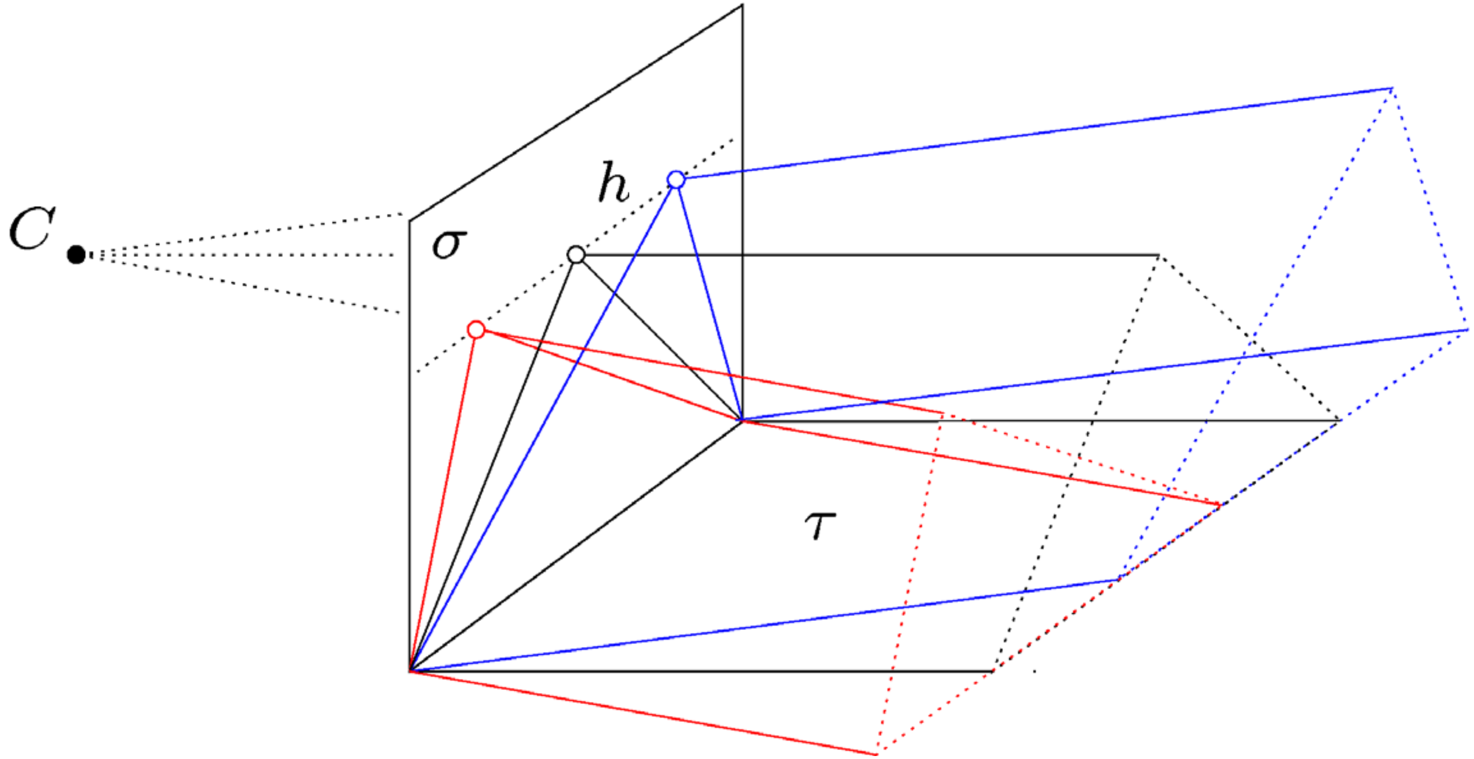
Horizon



If we take all sets of parallel lines in τ, each set with a different direction, then all the points of convergence in the image will fill a complete line h.

The line h is called the *horizon* (horizont in Czech) (or *vanishing line*).

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Now, imagine that we project all points from τ to π using the affine geometrical projection model. Then, no point from τ will project to h. On the other hand, when projecting in the opposite direction, i.e. π to τ, line h has no image, i.e. it does not project anywhere to τ.

When using the affine geometrical model with the projective plane to model the perspective projection (which is equivalent to the algebraic model in $\mathbb{R}^3$), all points of the projective plane τ (obtained as the projective completion of the affine plane τ) will have exactly one image in the projective plane π (obtained as the projective completion of the affine plane π) and vice versa.

This total symmetry is useful and beautiful.

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Internally Calibrated Camera

We say that a camera is internally calibrated if the camera coordinate system is constructed by using an orhogonal basis γ. In such a case, we can measure the angle betwen projection rays generated by vectors $\vec{x}$, $\vec{y}$ by the formula

$$\cos\measuredangle(\vec{x},\vec{y}) = \frac{\vec{x}_\gamma^\top \vec{y}_\gamma}{(\vec{x}_\gamma^\top \vec{x}_\gamma)^{\frac{1}{2}} (\vec{y}_\gamma^\top \vec{y}_\gamma)^{\frac{1}{2}}}$$

In general, the camera basis β, derived from the image basis and the projection center, is often not orthogonal. Then, we need to use

$$\vec{x}_\beta = \mathtt{K}\,\vec{x}_\gamma$$

and evaluate

$$\cos\measuredangle(\vec{x},\vec{y}) = \frac{\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{y}_\beta}{(\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_\beta^\top \omega\, \vec{y}_\beta}{(\vec{x}_\beta^\top \omega\, \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \omega\, \vec{y}_\beta)^{\frac{1}{2}}}$$

with

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

Once we have matrix ω, we can recover matrix $\mathtt{K}$ from it.

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Assuming

$$\mathtt{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

we get

$$\mathtt{K}^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{k_{11}} & \frac{-k_{12}}{k_{11}k_{22}} & \frac{k_{12}\,k_{23}-k_{13}\,k_{22}}{k_{11}\,k_{22}\,k_{23}} \\ 0 & \frac{1}{k_{22}} & \frac{-k_{23}}{k_{22}} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{K} = \begin{bmatrix} \frac{1}{m_{11}} & \frac{-m_{12}}{m_{11}m_{22}} & \frac{m_{12}\,m_{23}-m_{13}\,m_{22}}{m_{11}\,m_{22}\,m_{23}} \\ 0 & \frac{1}{m_{22}} & \frac{-m_{23}}{m_{22}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}\,m_{12} & m_{11}\,m_{13} \\ m_{11}\,m_{12} & m_{12}^2 + m_{22}^2 & m_{12}\,m_{13} + m_{22}\,m_{23} \\ m_{11}\,m_{13} & m_{12}\,m_{13} + m_{22}\,m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

which can be solved for $\mathtt{K}^{-1}$ up to the sign of the rows of $\mathtt{K}^{-1}$ as follows.

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$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}\,m_{12} & m_{11}\,m_{13} \\ m_{11}\,m_{12} & m_{12}^2 + m_{22}^2 & m_{12}\,m_{13} + m_{22}\,m_{23} \\ m_{11}\,m_{13} & m_{12}\,m_{13} + m_{22}\,m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

provides equations

$$\begin{aligned}
o_{11} = m_{11}^2 &\Rightarrow m_{11} = s_1\sqrt{o_{11}} \\
o_{12} = m_{11}\,m_{12} &\Rightarrow m_{12} = o_{12}/(s_1\sqrt{o_{11}}) = s_1\,o_{12}/\sqrt{o_{11}} \\
o_{13} = m_{11}\,m_{13} &\Rightarrow m_{13} = o_{13}/(s_1\sqrt{o_{11}}) = s_1\,o_{13}/\sqrt{o_{11}} \\
o_{22} = m_{12}^2 + m_{22}^2 &\Rightarrow m_{22} = s_2\sqrt{o_{22} - m_{12}^2} = s_2\sqrt{o_{22} - o_{12}^2/o_{11}} \\
o_{23} = m_{12}\,m_{13} + m_{22}\,m_{23} &\Rightarrow m_{23} = s_2\,(o_{23} - o_{12}\,o_{13}/o_{11})/\sqrt{o_{22} - o_{12}^2/o_{11}} \\
&\phantom{\Rightarrow m_{23}} = s_2\,(o_{11}\,o_{23} - o_{12}\,o_{13})/\sqrt{o_{11}^2 o_{22} - o_{11}\,o_{12}^2}
\end{aligned}$$

for $s_1 = \pm 1$ and $s_2 = \pm 1$.

Hence

$$\mathtt{K} = \begin{bmatrix} s_1\sqrt{o_{11}} & s_1\,o_{12}/\sqrt{o_{11}} & s_1\,o_{13}/\sqrt{o_{11}} \\ 0 & s_2\sqrt{o_{22} - o_{12}^2/o_{11}} & s_2\,(o_{23} - o_{12}\,o_{13}/o_{11})/\sqrt{o_{22} - o_{12}^2/o_{11}} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

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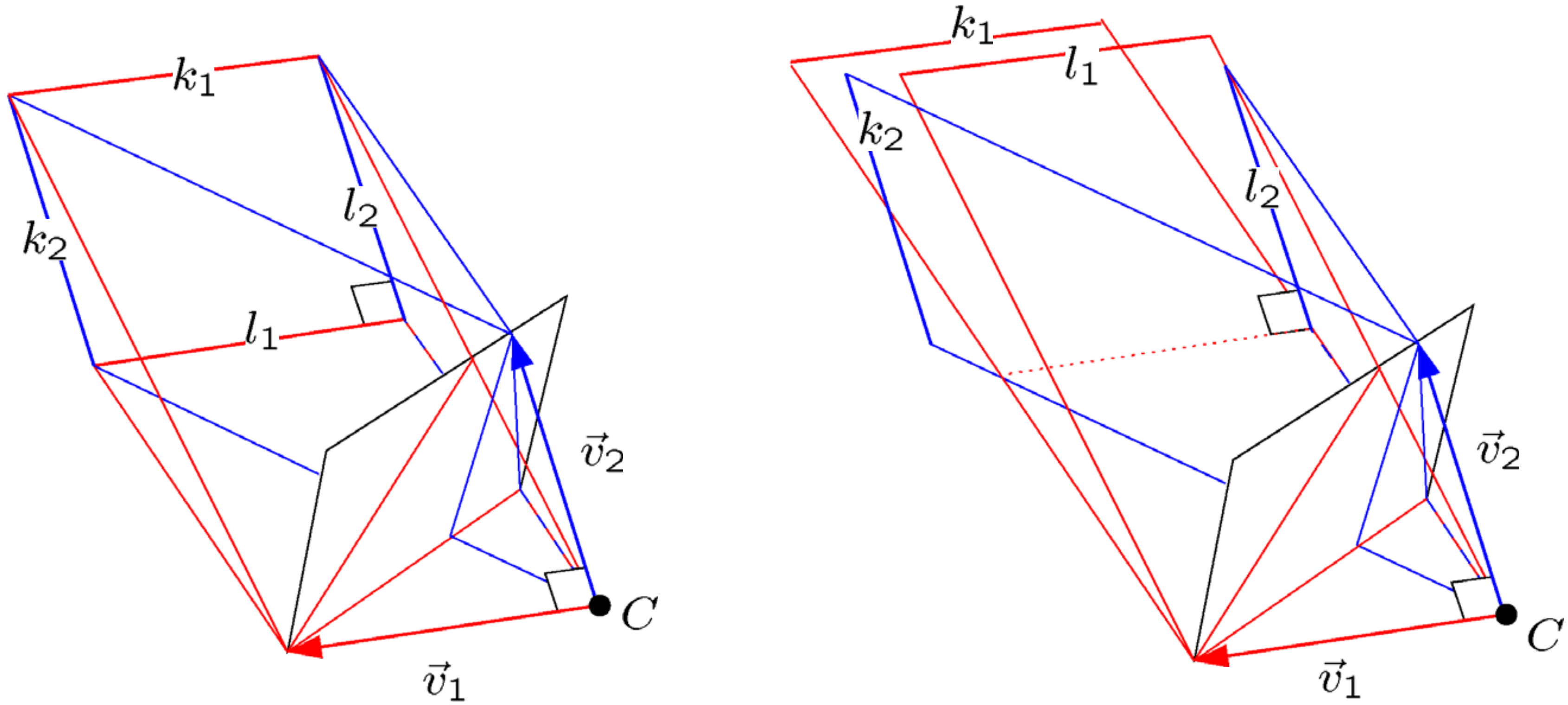
Camera calibration from perpendicular vanishing points



We will now show how to calibrate the camera by fining the matrix $\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$ from at least three vanishing points in general position.

Let us have two pairs of parallel lines in space, such that they are also orthogonal, i.e. let k_1 be parallel with l_1 and k_2 be parallel with l_2 and at the same time let k_1 be orthogonal to k_2 and l_1 be otrthogonal to l_2.

This, for instance, happens when lines k_1, l_2, k_2, l_2 form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

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Let lines k_1, l_1, k_2, l_2 be represented by the corresponding vetors $\vec{k_1}_\beta, \vec{l_1}_\beta, \vec{k_2}_\beta, \vec{l_2}_\beta$ in the camera coordinates system with (in general non-orthogonal) basis β.

Parallel lines k_1 and l_1, resp. k_2 and l_2, generate vanishing points

$$\begin{aligned} \vec{v}_{1\beta} &= \vec{k_1}_\beta \times \vec{l_1}_\beta \\ \vec{v}_{2\beta} &= \vec{k_2}_\beta \times \vec{l_2}_\beta \end{aligned}$$

Vector $\vec{v}_1$ is a direction vector of the line through C, which is parallel with line l_1. Vector $\vec{v}_2$ is a direction vector of the line through C, which is parallel with line l_2. Lines l_1 and l_2 are perpendicular. Threfore, vector $\vec{v}_1$ is perpendicular to vector $\vec{v}_2$.

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The perpendicularity of $\vec{v}_1$ to $\vec{v}_2$ is, in the camera orthogonal basis δ, modeled by

$$\vec{v}_{1\gamma}^{\top}\,\vec{v}_{2\gamma} = 0$$

We therefore get

$$\begin{aligned}\vec{v}_{1\beta}^{\top}\,\mathtt{K}^{-1}\mathtt{K}^{-1}\vec{v}_{2\beta} &= 0\\ \vec{v}_{1\beta}^{\top}\,\omega\,\vec{v}_{2\beta} &= 0\end{aligned}$$

which is a linear homogeneous equation on ω.

There are 6 unknowns in ω and hence we need 5 pairs of perpendicular vanishing points spaning $\mathbb{R}^3$ to recover the one-dimensional space of matrices ω.

Often, when working with digital cameras, we can assume that pixels are square and hence the canonical choice of coordinates in the image plane, i.e. setting the corners of a pixel to $(0,0)^{\top}$, $(1,0)^{\top}$, $(0,1)^{\top}$, $(1,1)^{\top}$, gives an orthogonal basis in the image coordinate system and consequently, the camera coordinate system basis β has the first two vectors orthogonal. This leads to a more special $\mathtt{K}$ matrix, which is then as

$$\mathtt{K} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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The corresponding

$$\omega = \frac{1}{k_{11}^2} \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix}$$

then provides equation

$$\begin{aligned} \vec{v}_{1\beta}^\top \omega \, \vec{v}_{2\beta} &= 0 \\ \begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & o_1 \\ 0 & 1 & o_2 \\ o_1 & o_2 & o_3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} &= 0 \\ \begin{bmatrix} v_{23} v_{11} + v_{21} v_{13} & v_{23} v_{12} + v_{22} v_{13} & v_{23} v_{13} \end{bmatrix} \begin{bmatrix} o_1 \\ o_2 \\ o_3 \end{bmatrix} &= -(v_{21} v_{11} + v_{22} v_{12}) \end{aligned}$$

Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute o_1, o_2, o_3 and then

$$\begin{aligned} k_{13} &= -o_1 \\ k_{23} &= -o_2 \\ k_{11} &= \sqrt{o_3 - k_{13}^2 - k_{23}^2} \end{aligned}$$

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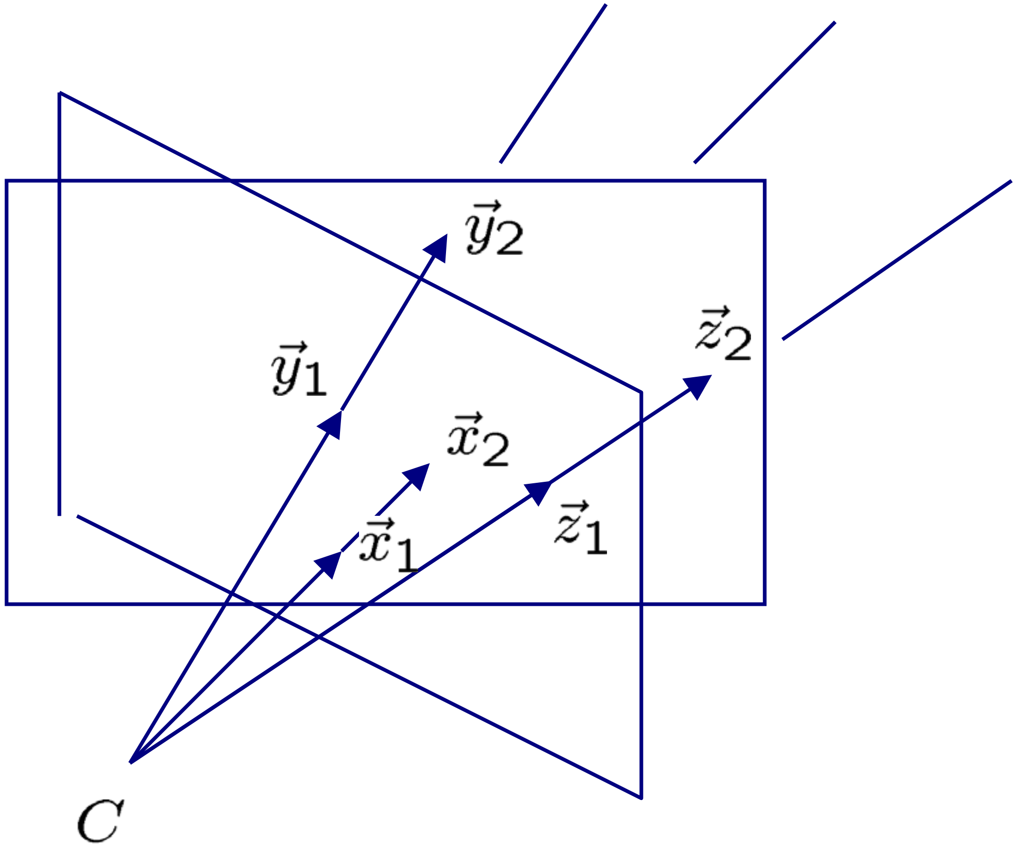
Camera calibration from rotation



Camera rotates

$$\cos \measuredangle(\vec{x},\vec{y}) = \frac{\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{y}_\beta}{(\vec{x}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_\beta)^{\frac{1}{2}}(\vec{y}_\beta^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_\beta^\top \omega\, \vec{y}_\beta}{(\vec{x}_\beta^\top \omega\, \vec{x}_\beta)^{\frac{1}{2}}(\vec{y}_\beta^\top \omega\, \vec{y}_\beta)^{\frac{1}{2}}}$$

with

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

Once we have matrix ω, we can recover matrix $\mathtt{K}$ from it.

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Rays are measured twice in different bases and producing different direction vectors but the angle between the corresponding vectors are equal:

$$\frac{\vec{x}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta}}{(\vec{x}_{1\beta}{}^\top \omega\, \vec{x}_{1\beta})^{\frac{1}{2}}(\vec{y}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta})^{\frac{1}{2}}} = \cos\measuredangle(\vec{x}_1, \vec{y}_1) = \cos\measuredangle(\vec{x}_2, \vec{y}_2) = \frac{\vec{x}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta}}{(\vec{x}_{2\beta}{}^\top \omega\, \vec{x}_{2\beta})^{\frac{1}{2}}(\vec{y}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta})^{\frac{1}{2}}}$$

Squaring the above we get

$$\frac{(\vec{x}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta})^2}{(\vec{x}_{1\beta}{}^\top \omega\, \vec{x}_{1\beta})(\vec{y}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta})} = \frac{(\vec{x}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta})^2}{(\vec{x}_{2\beta}{}^\top \omega\, \vec{x}_{2\beta})(\vec{y}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta})}$$

$$(\vec{x}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta})^2(\vec{x}_{2\beta}{}^\top \omega\, \vec{x}_{2\beta})(\vec{y}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta}) = (\vec{x}_{2\beta}{}^\top \omega\, \vec{y}_{2\beta})^2(\vec{x}_{1\beta}{}^\top \omega\, \vec{x}_{1\beta})(\vec{y}_{1\beta}{}^\top \omega\, \vec{y}_{1\beta})$$

which is a quartic equation in elements of ω.

When nothing is known about ω, we need 5 pairs to get five equations of degree four in five variables. Five pairs can be constructed from four points. Six pairs can be chosen from four points but only five among them are independent since the direction of the fourth ray is detemined by specifyting angles to two other rays.

Solving this system is, however, very difficult. There might be $4^5 = 2^{10} =$ 1024 solutions in general. When we assume $k_{12} = 0$ and $k_{11} = k_{12}$ and hence have

$$\omega = \begin{bmatrix} 1 & 0 & o_1 \\ 0 & 1 & o_2 \\ o_1 & o_2 & o_3 \end{bmatrix}$$

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we need only three pairs from three points. We get then three equations of degree four in three varaibles. Even that is not easy to solve.

To make the problem simpler, we will relax the contraints on ω when estimating it from data and use the constraints only when computing ω from the homography.

Consider having the homography between images, i.e.

$$\lambda_{ij}\,\vec{x}_{j\beta} = \mathtt{H}_{ij}\,\vec{x}_{i\beta'} = \mathtt{K}_j\,\mathtt{R}_j\,\mathtt{R}_i^\top \mathtt{K}_i^{-1}\,\vec{x}_{i\beta'} = \mathtt{K}_j\,\mathtt{R}_{ij}^\top \mathtt{K}_i^{-1}\,\vec{x}_{i\beta'}$$

for all i, j. Therefore

$$\begin{aligned} \mathtt{H}_{ij} &= \mathtt{K}_j\,\mathtt{R}_{ij}^\top \mathtt{K}_i^{-1} \\ \mathtt{K}_j^{-1}\mathtt{H}_{ij}\,\mathtt{K}_i &= \mathtt{R}_{ij}^\top \end{aligned}$$

which implies

$$\left(\mathtt{K}_j^{-1}\mathtt{H}_{ij}\,\mathtt{K}_i\right)^\top \left(\mathtt{K}_j^{-1}\mathtt{H}_{ij}\,\mathtt{K}_i\right) = \mathtt{I}$$

$$\mathtt{H}_{ij}^\top\,\mathtt{K}_i^{-\top}\mathtt{K}_j^{-1}\mathtt{H}_{ij} = \mathtt{K}_i^{-\top}\mathtt{K}_j^{-1}$$

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Let us now assume that $\mathtt{K}_i = \mathtt{K}$ for all i. Then we get

$$\mathtt{H}_{ij}^\top \mathtt{K}^{-\top} \mathtt{K}^{-1} \mathtt{H}_{ij} = \mathtt{K}^{-\top} \mathtt{K}^{-1}$$

$$\mathtt{H}_{ij}^\top \omega \, \mathtt{H}_{ij} = \omega$$

which are linear equations in elements of ω. These equatione holds only for $\mathtt{H}_{ij}$ recovered with all constraints follewing from the problem. i.e.

$$\det \mathtt{H}_{ij} = \det \mathtt{K} \, \mathtt{R}_{ij} \, \mathtt{K}^{-1} = \det \mathtt{K} \, \mathtt{K}^{-1} = 1$$

When we look at eigevalues of $\mathtt{H}_{ij}$, we see that

$$\mathtt{H}_{ij} \, \vec{x} = \lambda \, \vec{x}$$

$$\mathtt{K} \, \mathtt{R}_{ij} \, \mathtt{K}^{-1} \, \vec{x} = \lambda \, \vec{x}$$

$$\mathtt{R}_{ij} \, \mathtt{K}^{-1} \, \vec{x} = \lambda \, \mathtt{K}^{-1} \, \vec{x}$$

$$\mathtt{R}_{ij} \, \vec{y} = \lambda \, \vec{y}$$

and therefore eigenvalues of $\mathtt{H}$ are 1, $a + j\,b$ and $a - j\,b$ with real a, b such that $a^2 + b^2 = 1$. We also see that

$$\vec{x} = \mathtt{K} \, \vec{y}$$

and therefore, the eigenvector of $\mathtt{H}$ corresponding to eigenvalue 1 is the direction vector of the rotation axis in basis β of the camera coordinate system.

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Let us now look back at computing ω from $\mathtt{H}$. Having a matrix $\mathtt{G} = \xi\,\mathtt{H}$, we obtain $\mathtt{H}$ as

$$\mathtt{H} = \frac{1}{(\det \mathtt{G})^{\frac{1}{3}}}\mathtt{G}$$

We next expand previous reationship between $\mathtt{H}$ and ω

$$\mathtt{H}_{ij}^\top\,\omega\,\mathtt{H}_{ij} = \omega$$

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}^\top \begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & 1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & 1 \end{bmatrix}$$

which gives an overdetermined system of nine linear equations for five variables $o_{11}, o_{12}, o_{13}, o_{22}, o_{23}$, e.g.

$$h_{11}\,(h_{11}\,o_{11}+h_{21}\,o_{12}+h_{31}\,o_{13})+h_{21}\,(h_{11}\,o_{12}+h_{21}\,o_{22}+h_{31}\,o_{23})+h_{31}\,(h_{11}\,o_{13}+h_{21}\,o_{23}+h_{31}) = o_{11}$$

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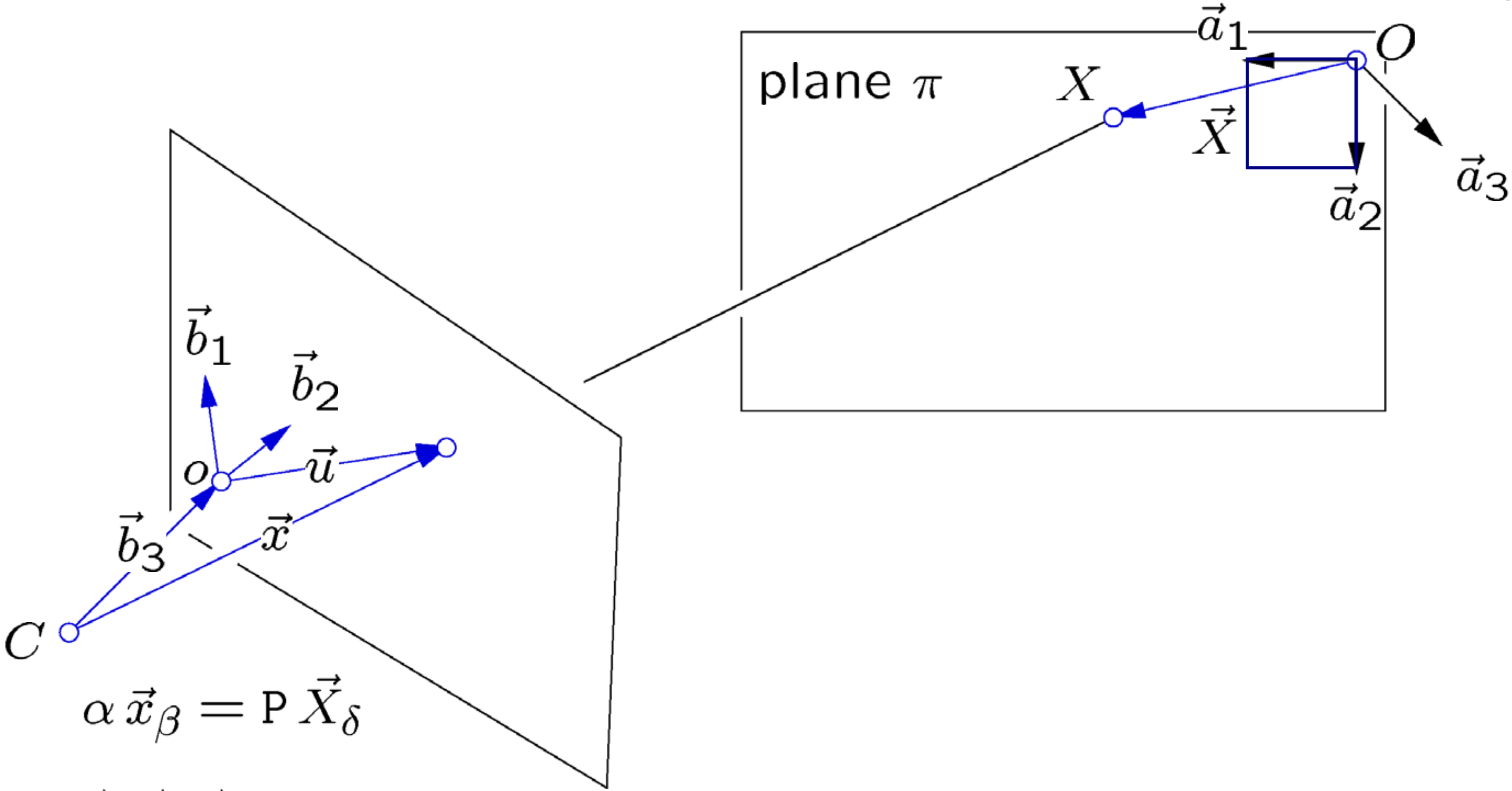
Camera calibration from homography to a "metric plane"

World coordinate system

$$\gamma = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$$
$$W = (O, \delta)$$



$$\alpha\, \vec{x}_\beta = \mathrm{P}\, \vec{X}_\delta$$

$\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$
$S = (C, \beta)$
Camera coordinate system

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Camera calibration from homography to a "metric plane"

Let us recall the relationship between the coordinates of points X, which all lie in a plane π and are measured in a coordinate system $(O, \vec{a}_1, \vec{a}_2)$ in π. The points X are projected by a perspective camera with projection matrix $\mathtt{P}$ into image coordinates (u, v), w.r.t. an image coordinate system $(o, \vec{b}_1, \vec{b}_2)$. The corresponding camera coordinate sysytem is (C, β) with $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$.

Points X are projected by a perspective camera with a projection matrix $\mathtt{P}$ into projections $\vec{x}_\beta$ as

$$\alpha\,\vec{x}_\beta = \mathtt{P}\,\mathbf{X}_\gamma = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}\begin{bmatrix}x\\ y\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_4\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix} = \mathtt{H}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}$$

where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ are columns of $\mathtt{P}$. Recall that the columns of $\mathtt{P}$ can be writen as

$$\begin{aligned}\mathtt{P} &= \left[\mathtt{A}\,|\,-\mathtt{A}\,\vec{C}_\delta\right]\\ &= \begin{bmatrix}\vec{a}_{1\beta} & \vec{a}_{2\beta} & \vec{a}_{3\beta} & -\vec{C}_\beta\end{bmatrix}\end{aligned}$$

and therefore we get

$$\begin{aligned}\mathbf{h}_1 = \mathbf{p}_1 &= \vec{a}_{1\beta}\\ \mathbf{h}_2 = \mathbf{p}_2 &= \vec{a}_{2\beta}\\ \mathbf{h}_3 = \mathbf{p}_4 &= -\vec{C}_\beta\end{aligned}$$

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Now imagine, that we are observing a suqare with 4 corner points X_1, X_2, X_3 and X_4 in the plane π and we construct the coordinate system in π by assigning coordinates to the corners as

$$\begin{aligned}
\vec{X}_{1\gamma} &= \begin{bmatrix}0 & 0 & 0\end{bmatrix} \\
\vec{a}_{1\gamma} = \vec{X}_{2\gamma} &= \begin{bmatrix}1 & 0 & 0\end{bmatrix} \\
\vec{a}_{2\gamma} = \vec{X}_{3\gamma} &= \begin{bmatrix}0 & 1 & 0\end{bmatrix} \\
\vec{X}_{4\gamma} &= \begin{bmatrix}1 & 1 & 0\end{bmatrix}
\end{aligned}$$

By this construction, the angle measured by the formula

$$\cos\measuredangle(\vec{X}_1, \vec{X}_2) = \frac{\vec{X}_{1\delta}^\top \vec{X}_{2\delta}}{(\vec{X}_{1\delta}^\top \vec{X}_{1\delta})^{\frac{1}{2}} (\vec{X}_{2\delta}^\top \vec{X}_{2\delta})^{\frac{1}{2}}}$$

corresponds to the angle measured by a ruler and a compass.

We see that we get two constraints on $\vec{a}_{1\delta}$, $\vec{a}_{2\delta}$

$$\begin{aligned}
\vec{a}_{1\delta}^\top \vec{a}_{2\delta} &= 0 \\
\vec{a}_{1\delta}^\top \vec{a}_{1\delta} - \vec{a}_{2\delta}^\top \vec{a}_{2\delta} &= 0
\end{aligned}$$

which lead to

$$\begin{aligned}
\vec{a}_{1\beta}^\top \mathtt{K}^{-\top} \mathtt{K}^{-1} \vec{a}_{2\beta} &= 0 \\
\vec{a}_{1\beta}^\top \mathtt{K}^{-\top} \mathtt{K}^{-1} \vec{a}_{1\beta} - \vec{a}_{2\beta}^\top \mathtt{K}^{-\top} \mathtt{K}^{-1} \vec{a}_{2\beta} &= 0
\end{aligned}$$

by using $\vec{a}_{i\beta} = \mathtt{K}\,\mathtt{R}\,\vec{a}_{i\gamma}$ for $i = 1, 2$, and $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$.

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These are two linear equations

$$\begin{aligned} \vec{a}_{1\beta}^\top \omega \vec{a}_{2\beta} &= 0 \\ \vec{a}_{1\beta}^\top \omega \vec{a}_{1\beta} - \vec{a}_{2\beta}^\top \omega \vec{a}_{2\beta} &= 0 \end{aligned}$$

on ω in terms of estimated $\lambda \mathtt{H}$

$$\begin{aligned} \mathbf{h}_1^\top \omega \mathbf{h}_2 &= 0 \\ \mathbf{h}_1^\top \omega \mathbf{h}_1 - \mathbf{h}_2^\top \omega \mathbf{h}_2 &= 0 \end{aligned}$$

Every square provides 2 equations and therefore 3 squares in planes in general positions suffice to calibrate full $\mathtt{K}$ matrix and two such squares suffice to calibrate $\mathtt{K}$ when pixels are square.

To calibrate the camera, we first assign coordinates to the corners of the square as above, then find the homography $\mathtt{H}$ from the plane to the image

$$\alpha_i \vec{x}_{i\beta} = \mathtt{H} \vec{X}_{i\gamma}$$

for $\alpha_i = 1, \ldots, 4$ and finally use columns of $\mathtt{H}$ the find ω.

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Line coordinates under homography

Let us now investigate the behaviour of homogeneous coordinates of lines in projective plane mapped by a homography.

Let us have two points represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$. We now map the points, represented by vectors $\vec{x}_\beta$, $\vec{y}_\beta$, by a homogprahy, represented by matrix $\mathtt{H}$, to points represented by vectors $\vec{x'}_\beta$, $\vec{y'}_\beta$ such that $\exists\,\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1, \lambda_2 \neq 0$

$$\begin{aligned} \lambda_1\, \vec{x'}_\beta &= \mathtt{H}\, \vec{x}_\beta \\ \lambda_2\, \vec{y'}_\beta &= \mathtt{H}\, \vec{y}_\beta \end{aligned}$$

Homogeneous coordinates $\vec{p}_\beta$ of the line passing through points represeted by $\vec{x}_\beta$, $\vec{y}_\beta$ and homogeneous coordinates $\vec{p'}_\beta$ of the line passing through points represeted by $\vec{x'}_\beta$, $\vec{y'}_\beta$ are obtained by solving the linear systems

$$\begin{aligned} \vec{p}_\beta^\top\, \vec{x}_\beta = 0 \qquad & \vec{p'}_\beta^\top\, \vec{x'}_\beta = 0 \\ \vec{p}_\beta^\top\, \vec{y}_\beta = 0 \qquad & \vec{p'}_\beta^\top\, \vec{y'}_\beta = 0 \end{aligned}$$

Substituting into the equations above, we get

$$\begin{aligned} \lambda_1\, \vec{p}_\beta^\top\, \mathtt{H}^{-1}\, \vec{x'}_\beta &= 0 \quad \Leftrightarrow \quad \vec{p}_\beta^\top\, \mathtt{H}^{-1}\, \vec{x'}_\beta = 0 \\ \lambda_2\, \vec{p}_\beta^\top\, \mathtt{H}^{-1}\, \vec{y'}_\beta &= 0 \quad \Leftrightarrow \quad \vec{p}_\beta^\top\, \mathtt{H}^{-1}\, \vec{y'}_\beta = 0 \end{aligned}$$

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We see that $\vec{p'}_\beta$ and $\mathtt{H}^{-\top}\vec{p}_\beta$ are solutions of the same set of homogeneous equations. If $\vec{x}_\beta$, $\vec{y}_\beta$ are independent, then there is $\lambda \in \mathbb{R}$ such that

$$\lambda\,\vec{p'}_\beta = \mathtt{H}^{-\top}\,\vec{p}_\beta$$

since the solution space is one-dimensional.

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