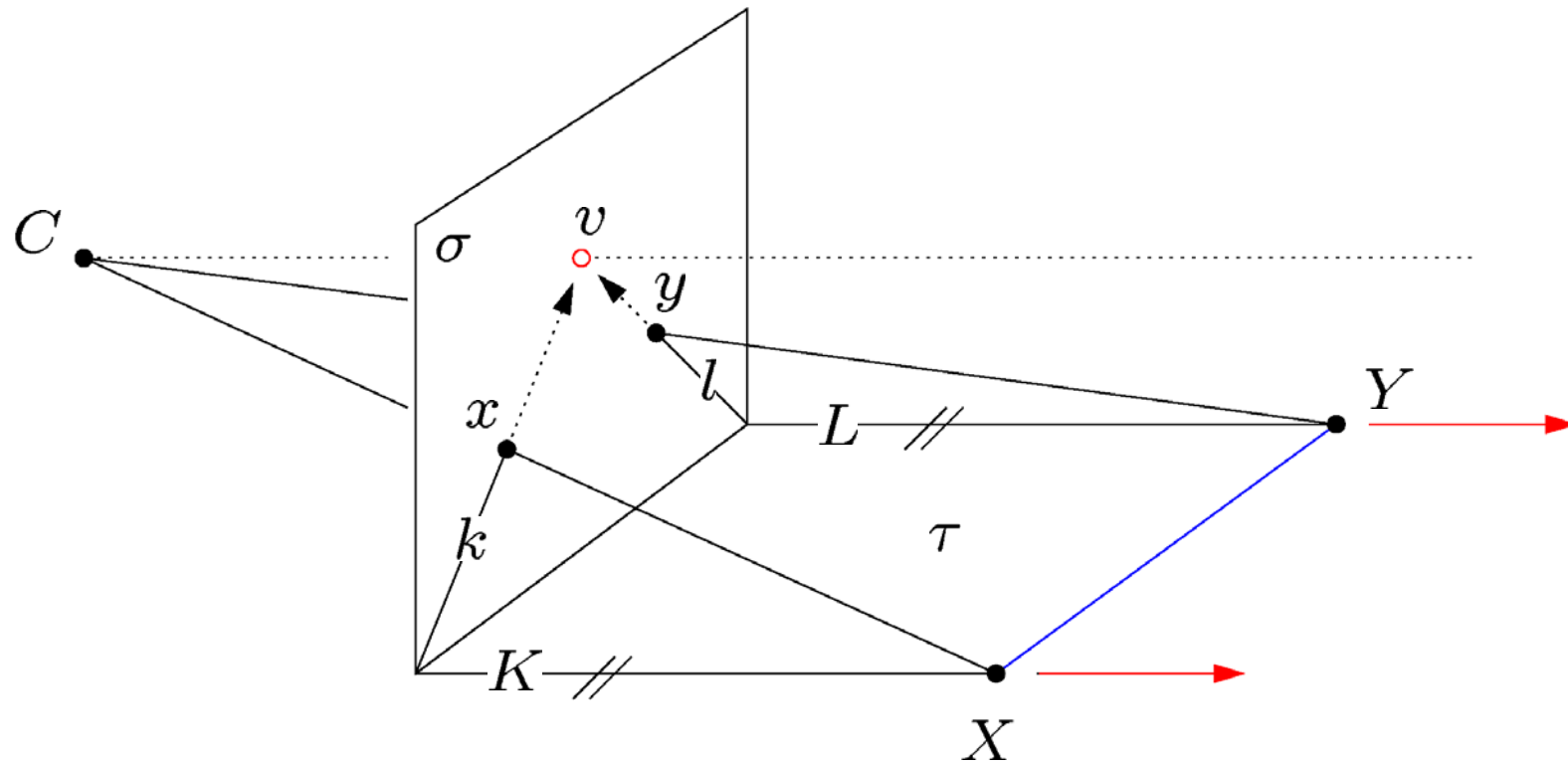


Lecture 8

Vanishing point

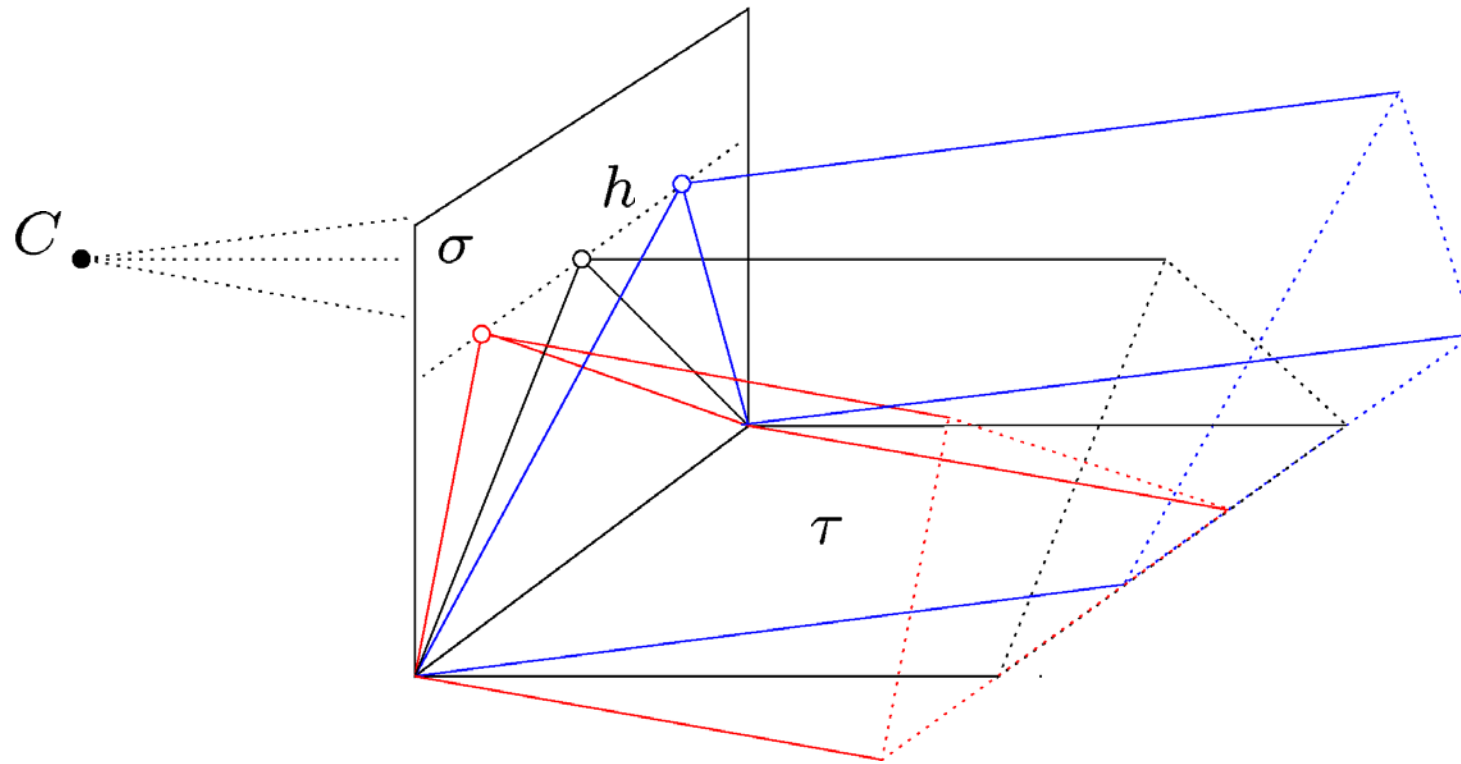


When modeling perspective projection in the affine space with affine projection planes, we meet somewhat unpleasant situations. For instance, imagine a projection of two parallel lines K, L , which are in a plane τ in the space into the projection plane π through the center C .

The lines K, L project to image lines k, l . As we go with two points X, Y along the lines k, l away from the projection plane, their images x, y get closer and closer to the point v in the image but they do not reach point v . We shall call point v the *point of convergence* of lines K, L .

The point v is called the *vanishing point* (ubezňik in Czech).

Horizon



If we take all sets of parallel lines in τ , each set with a different direction, then all the points of convergence in the image will fill a complete line h .

The line h is called the *horizon* (horizont in Czech) (or *vanishing line*).

Now, imagine that we project all points from τ to π using the affine geometrical projection model. Then, no point from τ will project to h . On the other hand, when projecting in the opposite direction, i.e. π to τ , line h has no image, i.e. it does not project anywhere to τ .

When using the affine geometrical model with the projective plane to model the perspective projection (which is equivalent to the algebraic model in \mathbb{R}^3), all points of the projective plane τ (obtained as the projective completion of the affine plane τ) will have exactly one image in the projective plane π (obtained as the projective completion of the affine plane π) and vice versa.

This total symmetry is useful and beautiful.

Internally Calibrated Camera

We say that a camera is internally calibrated if the camera coordinate system is constructed by using an orthogonal basis γ . In such a case, we can measure the angle between projection rays generated by vectors \vec{x} , \vec{y} by the formula

$$\cos \angle(\vec{x}, \vec{y}) = \frac{\vec{x}_\gamma^\top \vec{y}_\gamma}{(\vec{x}_\gamma^\top \vec{x}_\gamma)^{\frac{1}{2}} (\vec{y}_\gamma^\top \vec{y}_\gamma)^{\frac{1}{2}}}$$

In general, the camera basis β , derived from the image basis and the projection center, is often not orthogonal. Then, we need to use

$$\vec{x}_\beta = K \vec{x}_\gamma$$

and evaluate

$$\cos \angle(\vec{x}, \vec{y}) = \frac{\vec{x}_\beta^\top K^{-\top} K^{-1} \vec{y}_\beta}{(\vec{x}_\beta^\top K^{-\top} K^{-1} \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top K^{-\top} K^{-1} \vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_\beta^\top \omega \vec{y}_\beta}{(\vec{x}_\beta^\top \omega \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \omega \vec{y}_\beta)^{\frac{1}{2}}}$$

with

$$\omega = K^{-\top} K^{-1}$$

Once we have matrix ω , we can recover matrix K from it.

Assuming

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

we get

$$K^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k_{11}} & \frac{-k_{12}}{k_{11}k_{22}} & \frac{k_{12}k_{23}-k_{13}k_{22}}{k_{11}k_{22}k_{23}} \\ 0 & \frac{1}{k_{22}} & \frac{-k_{23}}{k_{22}} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{1}{m_{11}} & \frac{-m_{12}}{m_{11}m_{22}} & \frac{m_{12}m_{23}-m_{13}m_{22}}{m_{11}m_{22}m_{23}} \\ 0 & \frac{1}{m_{22}} & \frac{-m_{23}}{m_{22}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega = K^{-T}K^{-1}$$

$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}m_{12} & m_{11}m_{13} \\ m_{11}m_{12} & m_{12}^2 + m_{22}^2 & m_{12}m_{13} + m_{22}m_{23} \\ m_{11}m_{13} & m_{12}m_{13} + m_{22}m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

which can be solved for K^{-1} up to the sign of the rows of K^{-1} as follows.

$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11} m_{12} & m_{11} m_{13} \\ m_{11} m_{12} & m_{12}^2 + m_{22}^2 & m_{12} m_{13} + m_{22} m_{23} \\ m_{11} m_{13} & m_{12} m_{13} + m_{22} m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

provides equations

$$o_{11} = m_{11}^2 \Rightarrow m_{11} = s_1 \sqrt{o_{11}}$$

$$o_{12} = m_{11} m_{12} \Rightarrow m_{12} = o_{12} / (s_1 \sqrt{o_{11}}) = s_1 o_{12} / \sqrt{o_{11}}$$

$$o_{13} = m_{11} m_{13} \Rightarrow m_{13} = o_{13} / (s_1 \sqrt{o_{11}}) = s_1 o_{13} / \sqrt{o_{11}}$$

$$o_{22} = m_{12}^2 + m_{22}^2 \Rightarrow m_{22} = s_2 \sqrt{o_{22} - m_{12}^2} = s_2 \sqrt{o_{22} - o_{12}^2 / o_{11}}$$

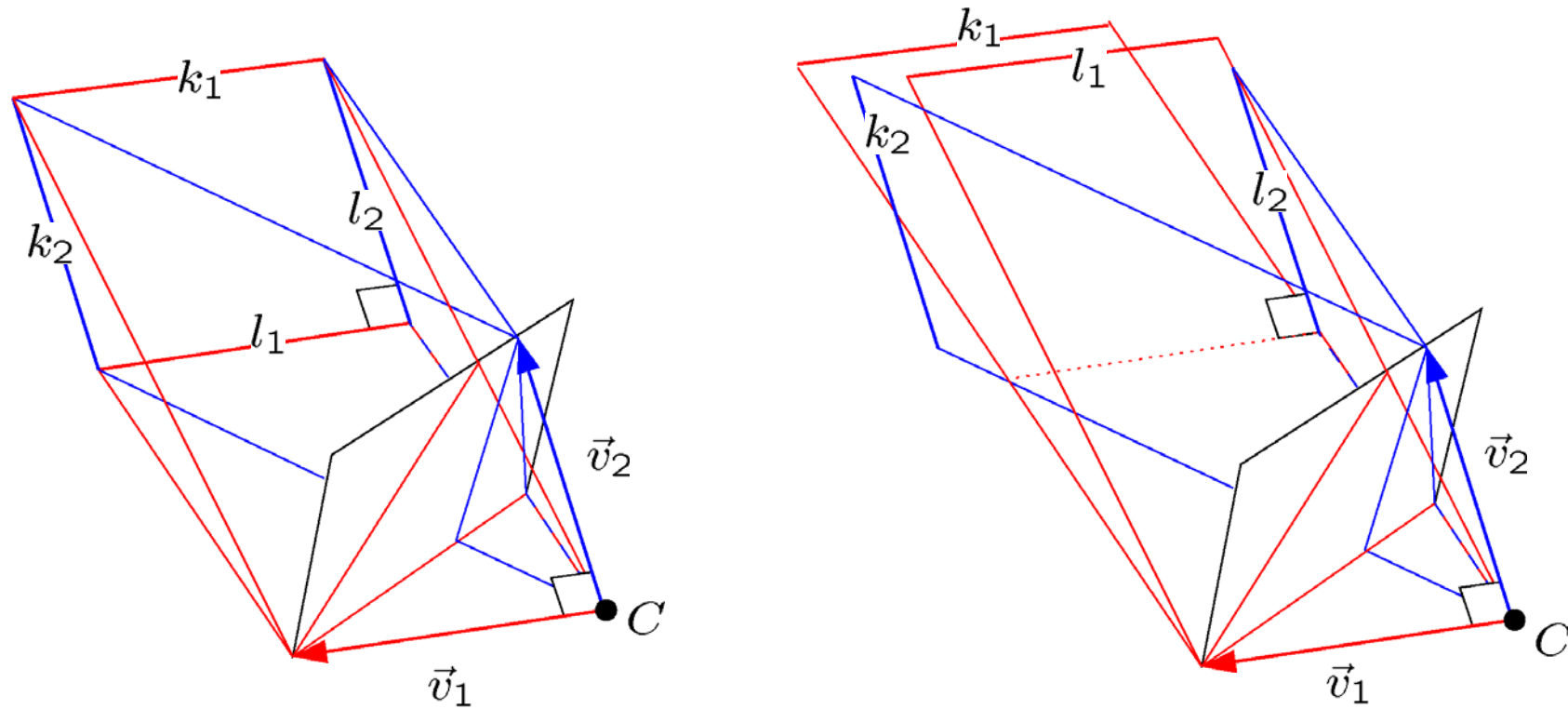
$$\begin{aligned} o_{23} = m_{12} m_{13} + m_{22} m_{23} &\Rightarrow m_{23} = s_2 (o_{23} - o_{12} o_{13} / o_{11}) / \sqrt{o_{22} - o_{12}^2 / o_{11}} \\ &= s_2 (o_{11} o_{23} - o_{12} o_{13}) / \sqrt{o_{11}^2 o_{22} - o_{11} o_{12}^2} \end{aligned}$$

for $s_1 = \pm 1$ and $s_2 = \pm 1$.

Hence

$$K = \begin{bmatrix} s_1 \sqrt{o_{11}} & s_1 o_{12} / \sqrt{o_{11}} & s_1 o_{13} / \sqrt{o_{11}} \\ 0 & s_2 \sqrt{o_{22} - o_{12}^2 / o_{11}} & s_2 (o_{23} - o_{12} o_{13} / o_{11}) / \sqrt{o_{22} - o_{12}^2 / o_{11}} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Camera calibration from perpendicular vanishing points



We will now show how to calibrate the camera by finding the matrix $\omega = K^{-T}K^{-1}$ from at least three vanishing points in general position.

Let us have two pairs of parallel lines in space, such that they are also orthogonal, i.e. let k_1 be parallel with l_1 and k_2 be parallel with l_2 and at the same time let k_1 be orthogonal to k_2 and l_1 be orthogonal to l_2 .

This, for instance, happens when lines k_1, l_2, k_2, l_1 form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

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Let lines k_1, l_1, k_2, l_2 be represented by the corresponding vectors $\vec{k}_{1\beta}, \vec{l}_{1\beta}, \vec{k}_{2\beta}, \vec{l}_{2\beta}$ in the camera coordinates system with (in general non-orthogonal) basis β .

Parallel lines k_1 and l_1 , resp. k_2 and l_2 , generate vanishing points

$$\begin{aligned}\vec{v}_{1\beta} &= \vec{k}_{1\beta} \times \vec{l}_{1\beta} \\ \vec{v}_{2\beta} &= \vec{k}_{2\beta} \times \vec{l}_{2\beta}\end{aligned}$$

Vector \vec{v}_1 is a direction vector of the line through C , which is parallel with line l_1 . Vector \vec{v}_2 is a direction vector of the line through C , which is parallel with line l_2 . Lines l_1 and l_2 are perpendicular. Therefore, vector \vec{v}_1 is perpendicular to vector \vec{v}_2 .

The perpendicularity of \vec{v}_1 to \vec{v}_2 is, in the camera orthogonal basis δ , modeled by

$$\vec{v}_{1\gamma}^\top \vec{v}_{2\gamma} = 0$$

We therefore get

$$\begin{aligned}\vec{v}_{1\beta}^\top \mathbf{K}^{-1} \mathbf{K}^{-1} \vec{v}_{2\beta} &= 0 \\ \vec{v}_{1\beta}^\top \omega \vec{v}_{2\beta} &= 0\end{aligned}$$

which is a linear homogeneous equation on ω .

There are 6 unknowns in ω and hence we need 5 pairs of perpendicular vanishing points spanning \mathbb{R}^3 to recover the one-dimensional space of matrices ω .

Often, when working with digital cameras, we can assume that pixels are square and hence the canonical choice of coordinates in the image plane, i.e. setting the corners of a pixel to $(0,0)^\top$, $(1,0)^\top$, $(0,1)^\top$, $(1,1)^\top$, gives an orthogonal basis in the image coordinate system and consequently, the camera coordinate system basis β has the first two vectors orthogonal. This leads to a more special \mathbf{K} matrix, which is then as

$$\mathbf{K} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The corresponding

$$\omega = \frac{1}{k_{11}^2} \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix}$$

then provides equation

$$\vec{v}_{1\beta}^\top \omega \vec{v}_{2\beta} = 0$$

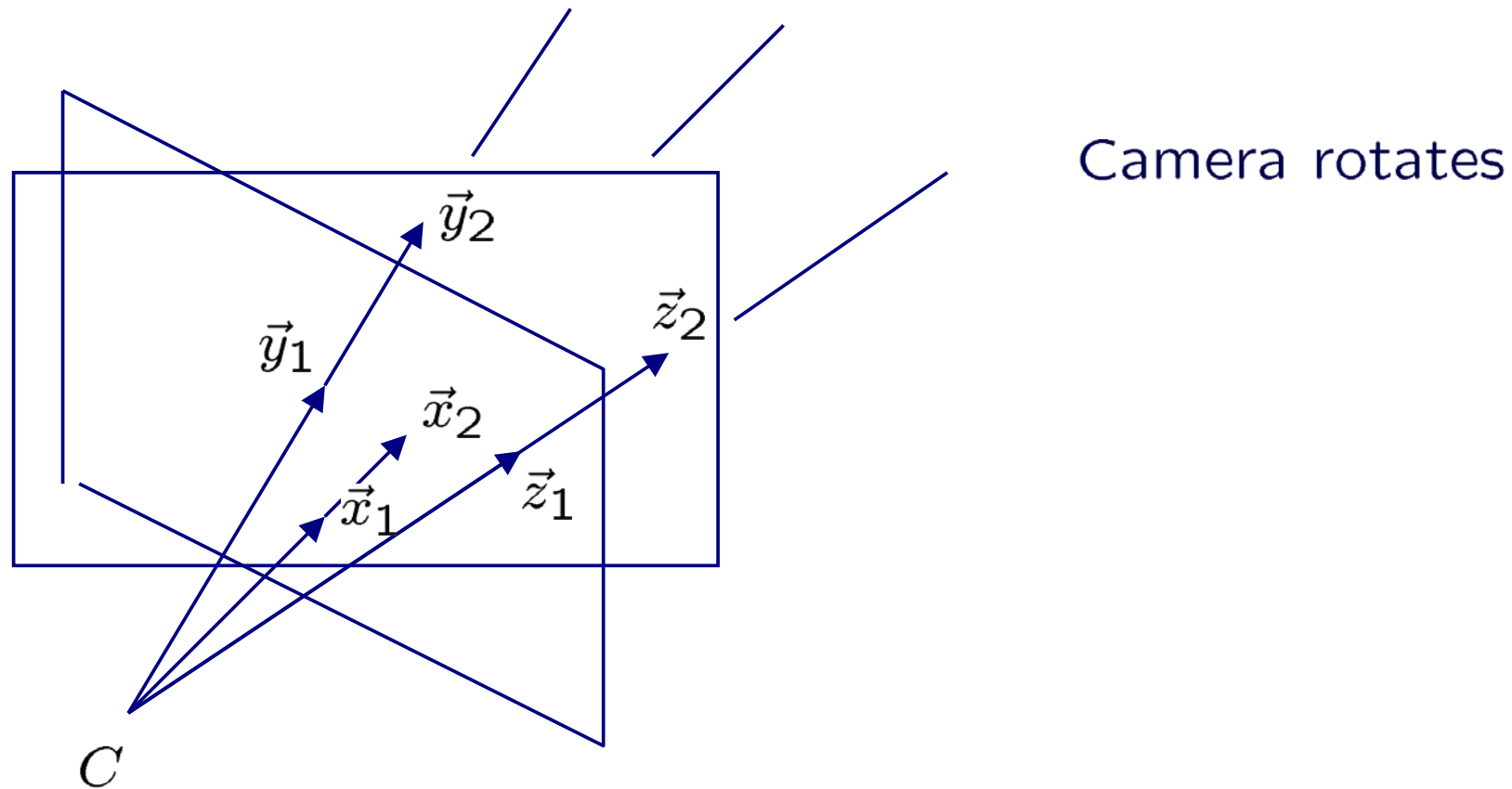
$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & o_1 \\ 0 & 1 & o_2 \\ o_1 & o_2 & o_3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$\begin{bmatrix} v_{23} v_{11} + v_{21} v_{13} & v_{23} v_{12} + v_{22} v_{13} & v_{23} v_{13} \end{bmatrix} \begin{bmatrix} o_1 \\ o_2 \\ o_3 \end{bmatrix} = -(v_{21} v_{11} + v_{22} v_{12})$$

Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute o_1, o_2, o_3 and then

$$\begin{aligned} k_{13} &= -o_1 \\ k_{23} &= -o_2 \\ k_{11} &= \sqrt{o_3 - k_{13}^2 - k_{23}^2} \end{aligned}$$

Camera calibration from rotation - via correspondences



$$\cos \angle(\vec{x}, \vec{y}) = \frac{\vec{x}_\beta^\top \mathbf{K}^{-\top} \mathbf{K}^{-1} \vec{y}_\beta}{(\vec{x}_\beta^\top \mathbf{K}^{-\top} \mathbf{K}^{-1} \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \mathbf{K}^{-\top} \mathbf{K}^{-1} \vec{y}_\beta)^{\frac{1}{2}}} = \frac{\vec{x}_\beta^\top \omega \vec{y}_\beta}{(\vec{x}_\beta^\top \omega \vec{x}_\beta)^{\frac{1}{2}} (\vec{y}_\beta^\top \omega \vec{y}_\beta)^{\frac{1}{2}}}$$

with

$$\omega = \mathbf{K}^{-\top} \mathbf{K}^{-1}$$

Once we have matrix ω , we can recover matrix \mathbf{K} from it.

Rays are measured twice in different bases and producing different direction vectors but the angle between the corresponding vectors are equal:

$$\frac{\vec{x}_{1\beta}^\top \omega \vec{y}_{1\beta}}{(\vec{x}_{1\beta}^\top \omega \vec{x}_{1\beta})^{\frac{1}{2}} (\vec{y}_{1\beta}^\top \omega \vec{y}_{1\beta})^{\frac{1}{2}}} = \cos \angle(\vec{x}_1, \vec{y}_1) = \cos \angle(\vec{x}_2, \vec{y}_2) = \frac{\vec{x}_{2\beta}^\top \omega \vec{y}_{2\beta}}{(\vec{x}_{2\beta}^\top \omega \vec{x}_{2\beta})^{\frac{1}{2}} (\vec{y}_{2\beta}^\top \omega \vec{y}_{2\beta})^{\frac{1}{2}}}$$

Squaring the above we get

$$\frac{(\vec{x}_{1\beta}^\top \omega \vec{y}_{1\beta})^2}{(\vec{x}_{1\beta}^\top \omega \vec{x}_{1\beta})(\vec{y}_{1\beta}^\top \omega \vec{y}_{1\beta})} = \frac{(\vec{x}_{2\beta}^\top \omega \vec{y}_{2\beta})^2}{(\vec{x}_{2\beta}^\top \omega \vec{x}_{2\beta})(\vec{y}_{2\beta}^\top \omega \vec{y}_{2\beta})}$$

$$(\vec{x}_{1\beta}^\top \omega \vec{y}_{1\beta})^2 (\vec{x}_{2\beta}^\top \omega \vec{x}_{2\beta})(\vec{y}_{2\beta}^\top \omega \vec{y}_{2\beta}) = (\vec{x}_{2\beta}^\top \omega \vec{y}_{2\beta})^2 (\vec{x}_{1\beta}^\top \omega \vec{x}_{1\beta})(\vec{y}_{1\beta}^\top \omega \vec{y}_{1\beta})$$

which is a quartic equation in elements of ω .

When nothing is known about ω , we need 5 pairs to get five equations of degree four in five variables. Five pairs can be constructed from four points. Six pairs can be chosen from four points but only five among them are independent since the direction of the fourth ray is determined by specifying angles to two other rays.

Solving this system is, however, very difficult. There might be $4^5 = 2^{10} = 1024$ solutions in general. When we assume $k_{12} = 0$ and $k_{11} = k_{12}$ and hence have

$$\omega = \begin{bmatrix} 1 & 0 & o_1 \\ 0 & 1 & o_2 \\ o_1 & o_2 & o_3 \end{bmatrix}$$

we need only three pairs from three points. We get then three equations of degree four in three variables. Even that is not easy to solve.

Camera calibration from rotation - via homographies - general K

To make the problem simpler, we will relax the constraints on ω when estimating it from data and use the constraints only when computing ω from the homography.

Consider having the homography between images, i.e.

$$\lambda_{ij} \vec{x}_{j\beta} = H_{ij} \vec{x}_{i\beta'} = K_j R_j R_i^\top K_i^{-1} \vec{x}_{i\beta'} = K_j R_{ij}^\top K_i^{-1} \vec{x}_{i\beta'}$$

for all i, j . Therefore

$$\begin{aligned} H_{ij} &= K_j R_{ij} K_i^{-1} \\ K_j^{-1} H_{ij} K_i &= R_{ij} \end{aligned}$$

which implies

$$\begin{aligned} (K_j^{-1} H_{ij} K_i)^\top (K_j^{-1} H_{ij} K_i) &= I \\ H_{ij}^\top K_i^{-\top} K_j^{-1} H_{ij} &= K_i^{-\top} K_j^{-1} \end{aligned}$$

Let us now assume that $K_i = K$ for all i . Then we get

$$\begin{aligned} H_{ij}^\top K^{-\top} K^{-1} H_{ij} &= K^{-\top} K^{-1} \\ H_{ij}^\top \omega H_{ij} &= \omega \end{aligned}$$

which are linear equations in elements of ω .

These equatione holds only for H_{ij} recovered with all constraints following from the problem. i.e.

$$\det H_{ij} = \det (K R_{ij} K^{-1}) = \det (K K^{-1}) = 1$$

When we look at eigevalues of H_{ij} , we see that

$$H_{ij} \vec{x} = \lambda \vec{x}$$

$$K R_{ij} K^{-1} \vec{x} = \lambda \vec{x}$$

$$R_{ij} K^{-1} \vec{x} = \lambda K^{-1} \vec{x}$$

$$R_{ij} \vec{y} = \lambda \vec{y}$$

and therefore eigenvalues of H are 1 , $a + j b$ and $a - j b$ with real a, b such that $a^2 + b^2 = 1$. We also see that

$$\vec{x} = K \vec{y}$$

and therefore, the eigenvector of H corresponding to eigenvalue 1 is the direction vector of the rotation axis in basis β of the camera coordinate system.

Let us now look back at computing ω from H . Having a matrix $G = \xi H$, we obtain H as

$$H = \frac{1}{(\det G)^{\frac{1}{3}}} G$$

We next expand previous relationship between H and ω

$$H_{ij}^{\top} \omega H_{ij} = \omega$$

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}^{\top} \begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix}$$

which gives a system of six linear equations for six variables $o_{11}, o_{12}, o_{13}, o_{22}, o_{23}, o_{33}$

$$\begin{aligned} 0 &= (h_{11}^2 - 1) o_{11} + 2 h_{11} h_{21} o_{12} + 2 h_{11} h_{31} o_{13} + h_{21}^2 o_{22} + 2 h_{21} h_{31} o_{23} + h_{31}^2 o_{33} \\ 0 &= h_{11} h_{12} o_{11} + (h_{12} h_{21} + h_{11} h_{22} - 1) o_{12} + (h_{12} h_{31} + h_{11} h_{32}) o_{13} + h_{21} h_{22} o_{22} \\ &\quad + (h_{22} h_{31} + h_{21} h_{32}) o_{23} + h_{32} h_{31} o_{33} \\ 0 &= h_{11} h_{13} o_{11} + (h_{13} h_{21} + h_{11} h_{23}) o_{12} + (h_{13} h_{31} + h_{11} h_{33} - 1) o_{13} + h_{23} h_{21} o_{22} \\ &\quad + (h_{23} h_{31} + h_{21} h_{33}) o_{23} + h_{31} h_{33} o_{33} \\ 0 &= h_{12}^2 o_{11} + 2 h_{12} h_{22} o_{12} + 2 h_{12} h_{32} o_{13} + (h_{22}^2 - 1) o_{22} + 2 h_{22} h_{32} o_{23} + h_{32}^2 o_{33} \\ 0 &= h_{13} h_{12} o_{11} + (h_{13} h_{22} + h_{12} h_{23}) o_{12} + (h_{12} h_{33} + h_{13} h_{32}) o_{13} + h_{22} h_{23} o_{22} \\ &\quad + (h_{23} h_{32} - 1 + h_{22} h_{33}) o_{23} + h_{32} h_{33} o_{33} \\ 0 &= h_{13}^2 o_{11} + 2 h_{13} h_{23} o_{12} + 2 h_{13} h_{33} o_{13} + h_{23}^2 o_{22} + 2 h_{23} h_{33} o_{23} + (h_{33}^2 - 1) o_{33} \end{aligned}$$

and which can be rewritten as

$$\begin{bmatrix} h_{11}^2 - 1 & 2 h_{11} h_{21} & 2 h_{11} h_{31} & h_{21}^2 & 2 h_{21} h_{31} & h_{31}^2 \\ h_{11} h_{12} & h_{12} h_{21} + h_{11} h_{22} - 1 & h_{12} h_{31} + h_{11} h_{32} & h_{21} h_{22} & h_{22} h_{31} + h_{21} h_{32} & h_{31} h_{32} \\ h_{11} h_{13} & h_{13} h_{21} + h_{11} h_{23} & h_{13} h_{31} + h_{11} h_{33} - 1 & h_{21} h_{23} & h_{23} h_{31} + h_{21} h_{33} & h_{31} h_{33} \\ h_{12}^2 & 2 h_{12} h_{22} & 2 h_{12} h_{32} & h_{22}^2 - 1 & 2 h_{22} h_{32} & h_{32}^2 \\ h_{12} h_{13} & h_{13} h_{22} + h_{12} h_{23} & h_{12} h_{33} + h_{13} h_{32} & h_{22} h_{23} & h_{23} h_{32} - 1 + h_{22} h_{33} & h_{32} h_{33} \\ h_{13}^2 & 2 h_{13} h_{23} & 2 h_{13} h_{33} & h_{23}^2 & 2 h_{23} h_{33} & h_{33}^2 - 1 \end{bmatrix} \begin{bmatrix} o_{11} \\ o_{12} \\ o_{13} \\ o_{22} \\ o_{23} \\ o_{33} \end{bmatrix} = 0$$

Notice that this is a system of homogeneous equations and therefore we will recover ω only up to a non-zero multiple t . In general, the six equations will have only the trivial solution since H will not satisfy all required constraints. Therefore, we will use only five equations to compute a basis Q of the one-dimensional space of solutions. We will then look for (an approximation of) $\omega = tQ$ by solving for parameter t .

It can be verified that for

$$\omega = \begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix}$$

there holds

$$\left(o_{23}^2 - \frac{o_{13}^2 o_{12}^2}{o_{11}^2} - \left(o_{22} - \frac{o_{12}^2}{o_{11}} \right) \left(o_{33} - \frac{o_{13}^2}{o_{11}} - 1 \right) \right)^2 - 4 \frac{o_{13}^2 o_{12}^2}{o_{11}^2} \left(o_{22} - \frac{o_{12}^2}{o_{11}} \right) \left(o_{33} - \frac{o_{13}^2}{o_{11}} - 1 \right) = 0$$

Now, using

$$\begin{bmatrix} o_{11} & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \omega = tQ = \begin{bmatrix} tq_{11} & tq_{12} & tq_{13} \\ tq_{12} & tq_{22} & tq_{23} \\ tq_{13} & tq_{23} & tq_{33} \end{bmatrix}$$

we get

$$\begin{aligned} 0 &= (-2q_{23}^2q_{11}^2q_{22}q_{33} + q_{22}^2q_{33}^2q_{11}^2 + 2q_{23}^2q_{11}q_{12}^2q_{33} \\ &\quad - 2q_{22}^2q_{33}q_{11}q_{13}^2 + q_{12}^4q_{33}^2 + 2q_{23}^2q_{11}q_{22}q_{13}^2 + q_{22}^2q_{13}^4 \\ &\quad - 4q_{23}^2q_{13}^2q_{12}^2 + 2q_{13}^2q_{12}^2q_{22}q_{33} - 2q_{22}^2q_{33}^2q_{11}q_{12}^2 \\ &\quad + q_{23}^4q_{11}^2)t^4 + (-2q_{23}^2q_{11}q_{12}^2 + 2q_{22}^2q_{13}^2q_{11} - 2q_{13}^2q_{12}^2q_{22} \\ &\quad - 2q_{12}^4q_{33} - 2q_{22}^2q_{33}q_{11}^2 + 2q_{23}^2q_{11}^2q_{22} + 4q_{22}q_{33}q_{11}q_{12}^2)t^3 \\ &\quad + (-2q_{22}q_{11}q_{12}^2 + q_{22}^2q_{11}^2 + q_{12}^4)t^2 \end{aligned}$$

which is a polynomial in t

$$0 = at^4 + bt^3 + ct^2$$

with

$$\begin{aligned} a &= -2q_{23}^2q_{11}^2q_{22}q_{33} + q_{22}^2q_{33}^2q_{11}^2 + 2q_{23}^2q_{11}q_{12}^2q_{33} \\ &\quad - 2q_{22}^2q_{33}q_{11}q_{13}^2 + q_{12}^4q_{33}^2 + 2q_{23}^2q_{11}q_{22}q_{13}^2 + q_{22}^2q_{13}^4 \\ &\quad - 4q_{23}^2q_{13}^2q_{12}^2 + 2q_{13}^2q_{12}^2q_{22}q_{33} - 2q_{22}^2q_{33}^2q_{11}q_{12}^2 + q_{23}^4q_{11}^2 \\ b &= -2q_{23}^2q_{11}q_{12}^2 + 2q_{22}^2q_{13}^2q_{11} - 2q_{13}^2q_{12}^2q_{22} \\ &\quad - 2q_{12}^4q_{33} - 2q_{22}^2q_{33}q_{11}^2 + 2q_{23}^2q_{11}^2q_{22} + 4q_{22}q_{33}q_{11}q_{12}^2 \\ c &= -2q_{22}q_{11}q_{12}^2 + q_{22}^2q_{11}^2 + q_{12}^4 \end{aligned}$$

Since we are interested only in $t \neq 0$, we can solve

$$0 = at^2 + bt + c$$

to get $\omega = tQ$.