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# Perspective Camera

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**covered by**

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, 7.4, Example: 2.19

## ► Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	$n$	$O$
plane		$\pi, \varphi$

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^T, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as  $\mathbf{m} = (u, v)$ ,  $\mathbf{X} = (x, y, z)$ , etc.

- vectors are always meant to be columns  $\mathbf{x} \in \mathbb{R}^{n,1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^T, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^T, \quad \underline{\mathbf{n}}$$

'in-line' forms:  $\underline{\mathbf{m}} = (m_1, m_2, m_3)$ ,  $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$ , etc.

- matrices are  $\mathbf{Q} \in \mathbb{R}^{m,n}$

## ► Image Line

line in the plane

$$a u + b v + c = 0$$

corresponds to (homogeneous) vector

$$\underline{\mathbf{n}} \simeq (a, b, c)$$

and the equivalence class for  $\lambda \in \mathbb{R}, \lambda \neq 0$   $(\lambda a, \lambda b, \lambda c) \simeq (a, b, c)$

- the set of equivalence classes of vectors in  $\mathbb{R}^3 \setminus (0, 0, 0)$  forms the projective space  $\mathbb{P}^2$   
a set of rays
- standard representation for finite  $\underline{\mathbf{n}} = (n_1, n_2, n_3)$  is  $\lambda \underline{\mathbf{n}}$ , where  $\lambda = \frac{1}{\sqrt{n_1^2 + n_2^2}}$   
assuming  $n_1^2 + n_2^2 \neq 0$ ;  $\mathbf{1}$  is the unit, usually  $\mathbf{1} = 1$
- naming convention: a special entity is the **Ideal Line** (line at infinity)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$$

- I may sometimes wrongly use = instead of  $\simeq$ , help me chase the mistakes down

## ► Image Point

Point  $\mathbf{m} = (u, v)$  is incident on the line  $\underline{\mathbf{n}} = (a, b, c)$  iff

this works both ways!

$$a u + b v + c = 0$$

can be rewritten as (with scalar product):

$$(u, v, 1) \cdot (a, b, c) = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$$

point is also represented by a homogeneous vector

$$\underline{\mathbf{m}} \simeq (u, v, 1)$$

and the equivalence class for  $\lambda \in \mathbb{R}, \lambda \neq 0$  is

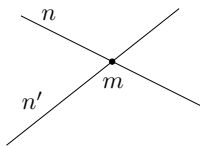
$$(m_1, m_2, m_3) = \lambda \underline{\mathbf{m}} \simeq \underline{\mathbf{m}}$$

- standard representation for finite point  $\underline{\mathbf{m}}$  is  $\lambda \underline{\mathbf{m}}$ , where  $\lambda = \frac{1}{m_3}$  assuming  $m_3 \neq 0$
- when  $\mathbf{1} = 1$  then units are pixels and  $\lambda \underline{\mathbf{m}} = (u, v, 1)$
- when  $\mathbf{1} = f$  then all components have a similar magnitude,  $f \sim$  image diagonal  
use  $\mathbf{1} = 1$  unless you know what you are doing;  
all entities participating in a formula must be expressed in the same units
- naming convention: **Ideal Point** (point at infinity)  $\underline{\mathbf{m}}_\infty \simeq (m_1, m_2, 0)$   
a proper member of  $\mathbb{P}^2$
- all such points lie on the ideal line  $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$ , ie.  $\underline{\mathbf{m}}_\infty^\top \underline{\mathbf{n}}_\infty = 0$

## ► Line Intersection and Point Join

The point of **intersection**  $m$  of image lines  $n$  and  $n'$ ,  $n \neq n'$  is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}'}$$



**proof:** If  $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}'}$  is the intersection point, it must be incident on both lines. Indeed,

$$\underline{\mathbf{n}}^T \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}'})}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}'^T} \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}'})}_{\underline{\mathbf{m}}} = 0$$

The **join**  $n$  of two image points  $m$  and  $m'$ ,  $m \neq m'$  is

$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}'}$$

Parallel lines intersect at the line at infinity  $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$

$$a u + b v + c = 0,$$

$$a u + b v + d = 0,$$

$$d \neq c$$

$$(a, b, c) \times (a, b, d) \simeq (b, -a, 0)$$

- all such intersections lie on the ideal line  $\underline{\mathbf{n}}_\infty$
- line at infinity represents a set of directions in plane

## ► Homography

Projective space  $\mathbb{P}^2$ : Vector space of dimension 3 excluding the zero vector,  $\mathbb{R}^3 \setminus (0, 0, 0)$   
but including 'points at infinity' and the 'line at infinity'

**Collineation:** Let  $\underline{x}_1, \underline{x}_2, \underline{x}_3$  be collinear points in  $\mathbb{P}^2$ . Bijection (1:1, onto)  $h: \mathbb{P}^2 \mapsto \mathbb{P}^2$  is a collineation iff  $h(\underline{x}_1), h(\underline{x}_2), h(\underline{x}_3)$  are collinear.

i.e.

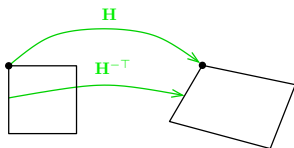
- collinear image points are mapped to collinear image points lines are mapped to lines
- concurrent image lines are mapped to concurrent image lines bijection!
- point-line incidence is preserved concurrent = intersecting at the same point

- a mapping  $h: \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a collineation iff there exists a non-singular  $3 \times 3$  matrix  $\mathbf{H}$  such that

$$h(\underline{x}) \simeq \mathbf{H} \underline{x} \quad \text{for all } \underline{x} \in \mathbb{P}^2$$

- homogeneous matrix representant:  $\det \mathbf{H} = 1$
- collineations form a group isomorphic to  $SO(3)$   
group of  $3 \times 3$  matrices with unit determinant and with matrix multiplication
- in this course we will use the term **homography** but mean collineation

## ► Mapping Points and Lines by Homography



$$\begin{aligned}\underline{\mathbf{m}}' &\simeq \mathbf{H} \underline{\mathbf{m}} && \text{image point} \\ \underline{\mathbf{n}}' &\simeq \mathbf{H}^{-\top} \underline{\mathbf{n}} && \text{image line}\end{aligned}$$

• incidence is preserved:  $(\underline{\mathbf{m}}')^\top \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^\top \mathbf{H}^\top \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

1. collineation has 8 DOF; it is given by 4 correspondences (points, lines) in a general position
2. extending pixel coordinates to homogeneous coordinates  $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
3. mapping by homography, eg.  $\underline{\mathbf{m}}' = \mathbf{H} \underline{\mathbf{m}}$
4. conversion of the result  $\underline{\mathbf{m}}' = (m'_1, m'_2, m'_3)$  to canonical coordinates (pixels):

$$u' = \frac{m'_1}{m'_3} \mathbf{1}, \quad v' = \frac{m'_2}{m'_3} \mathbf{1}$$

5. can use the unity for the homogeneous coordinate on one side of the equation only!



# Elementary Decomposition of a Homography

**Unique decompositions:**  $\mathbf{A} = \mathbf{A}_S \mathbf{A}_A \mathbf{A}_P \quad (= \mathbf{A}'_P \mathbf{A}'_A \mathbf{A}'_S)$

$$\mathbf{A}_S = \begin{bmatrix} s \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \quad \text{similarity}$$

$$\mathbf{A}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \quad \text{special affine}$$

$$\mathbf{A}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix} \quad \text{special projective}$$

$\mathbf{K}$  – upper triangular matrix with positive diagonal entries

$\mathbf{R}$  – orthogonal,  $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ ,  $\det \mathbf{R} = 1$

$s, w \in \mathbb{R}$ ,  $s > 0$ ,  $w \neq 0$

$$\mathbf{A} = \begin{bmatrix} s \mathbf{R} \mathbf{K} + \mathbf{t} \mathbf{v}^\top & w \mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

- must use 'skinny' QR decomposition, which is unique [Golub & van Loan 1996, Sec. 5.2.6]
- $\mathbf{A}_S$ ,  $\mathbf{A}_A$ ,  $\mathbf{A}_P$  are collineation subgroups  
(eg.  $\mathbf{K} = \mathbf{K}_1 \mathbf{K}_2$ ,  $\mathbf{K}^{-1}$ ,  $\mathbf{I}$  are all upper triangular with unit determinant, associativity holds)

# Homography Subgroups

group	DOF	matrix	invariant properties
projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	incidence, concurrency, colinearity, cross-ratio, convex hull, order of contact (intersection, tangency, inflection), tangent discontinuities and cusps.
affine	6	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> parallelism, ratio of areas, ratio of lengths on parallel lines, linear combinations of vectors (e.g. midpoints), line at infinity $\underline{n}_\infty$ (not pointwise)
similarity	4	$\begin{bmatrix} s \cos \phi & s \sin \phi & t_x \\ -s \sin \phi & s \cos \phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> ratio of lengths, angle, the circular points $I = (1, i, 0)$ , $J = (1, -i, 0)$ .
Euclidean	3	$\begin{bmatrix} \cos \phi & \sin \phi & t_x \\ -\sin \phi & \cos \phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	<u>all above plus:</u> length, area

# Some Homographic Tasters

**Rectification of camera rotation:** Slides 60 (geometry), 122 (homography estimation)

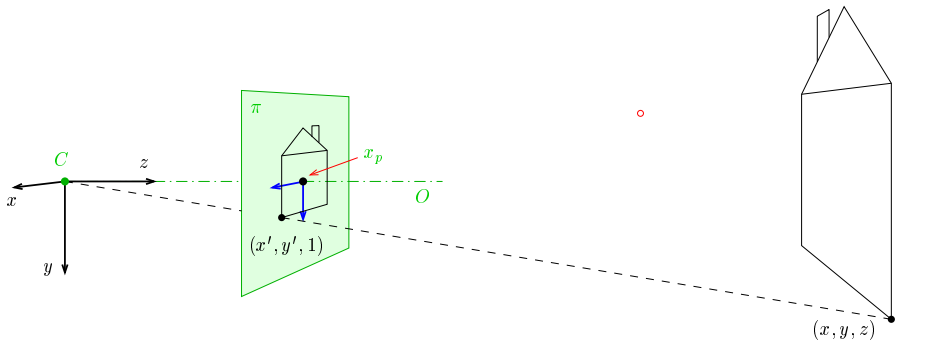


**Homographic Mouse for Visual Odometry:** Slide TBD

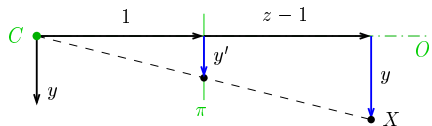


illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

## ► Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. right-handed canonical coordinate system  $(x, y, z)$
2. origin = center of projection  $C$
3. image plane  $\pi$  at unit distance from  $C$
4. optical axis  $O$  is perpendicular to  $\pi$
5. principal point  $x_p$ : intersection of  $O$  and  $\pi$
6. in this picture we are looking 'down the street'
7. **perspective camera is given by  $C$  and  $\pi$**



projected point in the natural image coordinate system:

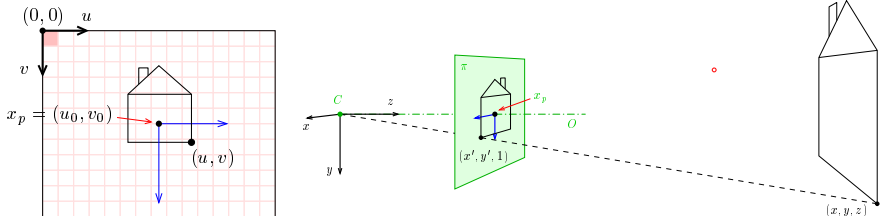
$$\frac{y'}{1} = y' = \frac{y}{1 + z - 1} = \frac{y}{z}, \quad x' = \frac{x}{z}$$

## ► Natural and Canonical Image Coordinate Systems

projected point **in canonical camera**

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix}^\top = \begin{bmatrix} \frac{x}{z} & \frac{y}{z} & 1 \end{bmatrix}^\top = \frac{1}{z} \begin{bmatrix} x & y & z \end{bmatrix}^\top \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \underline{\mathbf{X}}$$

projected point **in scanned image** notice the chimney!



$$\begin{aligned} u &= f \frac{x}{z} + u_0 \\ v &= f \frac{y}{z} + v_0 \end{aligned} \quad \frac{1}{z} \begin{bmatrix} f x + z u_0 \\ f y + z v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K} \mathbf{P}_0 \underline{\mathbf{X}} = \mathbf{P} \underline{\mathbf{X}}$$

- 'calibration' matrix  $\mathbf{K}$  transforms canonical camera  $\mathbf{P}_0$  to standard projective camera  $\mathbf{P}$

Thank You



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