STRUCTURED MODEL LEARNING (SS2015)
3. SEMINAR

Assignment 1. The LP-relaxation of the task of searching the most probable realisation of a Gibbs random field on a graph (V, E) (see lecture, sec. 3) reads as

$$\sum_{\{ij\}\in E}\sum_{s_i,s_j\in K}\mu_{ij}(s_i,s_j)u_{ij}(s_i,s_j)\to\max_{\mu}$$

$$\text{s.t.}\ \sum_{s_i,s_j\in K}\mu_{ij}(s_i,s_j)=1\quad\forall\{i,j\}\in E$$

$$\sum_{s_j\in K}\mu_{ij}(s_i,s_j)=\sum_{s_l\in K}\mu_{il}(s_i,s_l)\quad\forall i,j,l\colon\{i,j\},\{i,l\}\in E,\quad\forall s_i\in K$$

$$\mu>0$$

Notice that both, $u_{ij}(s_i, s_j)$ and $u_{ji}(s_j, s_i)$ denote the same function. Construct the dual task. Verify that the variables of the dual task are reparametrisations of the GRF. Interpret the dual task as minimisation of an upper bound of the primal objective w.r.t. to reparametrisations.

Assignment 2. Consider a K-valued Gibbs random field on a tree (V, E). Its joint probability distribution is given by

$$p_u(s)=\frac{1}{Z(u)}\exp\Big[\sum_{\{ij\}\in E}u_{ij}(s_i,s_j)\Big].$$

a) Find an algorithm for computing the pairwise marginal distributions $p(s_i, s_j)$ for all edges $\{i, j\}$ of the tree.
b) Can you interpret this algorithm in terms of reparametrisations?
c) Find a "distributed" version of your algorithm, such that it can be generalised for graphs with loops.

Assignment 3. Fill in the details for deriving the update step of the mean field approximation (see sec. 4.A of the lecture).

Assignment 4. Consider a binary valued Gibbs random field

$$p_u(s)=\frac{1}{Z(u)}\exp\Big[\sum_{i\in V}u_i(s_i,s_{i+1})\Big].$$

defined on a ring (V, E) with $V = \{0, 1, 2, \dots, n-1\}$ and $E = \big\{\{i, i+1 \pmod n\} \mid i \in V\big\}$. Notice, that the expression $i + 1$ in s_{i+1} is consequently to be taken modulo n. The random variables of the field $S = \{S_i \mid i \in V\}$ are binary valued. We may assume $s_i = 0, 1$ for all vertices $i \in V$. The functions u_{ij} are defined by

$$u(s_i,s_{i+1})=\begin{cases}\alpha_i & \text{if } s_i=s_{i+1},\\ -\alpha_i & \text{otherwise,}\end{cases}$$

with given reals $\alpha_i \in \mathbb{R}$.
a) Find an algorithm for computing the marginal probabilities $p(s_i, s_{i+1})$ of this random field in polynomial time.
b) Choose your favorite programming language and implement the algorithm found by you in a).
c) Implement a Gibbs sampler for the GRF. Check experimentally, how large should a sample of realisations be (assuming a contiguous seqence of realisations generated by the sampler) if we want to estimate the marginals with 1% precision. Consider in particular the two situations $\alpha_i = 0.3, \forall i \in V$ and $\alpha_i = 2.0, \forall i \in V$ for a ring with $n = 5$.