STRUCTURED MODEL LEARNING (SS2015)
1. SEMINAR

Assignment 1. Prove the following functions $x \in \mathbb{R} \mapsto f(x)$ are convex and calculate ∂f:

(a) $|x|$

(b) $\delta_{\mathbb{R}_+}$

(c) $\begin{cases} -\sqrt{x} & \text{if } x \geqslant 0 \\ +\infty & \text{otherwise} \end{cases}$

Assignment 2. Calculate $\partial\|.\|$ and $\partial\|.\|_1$.

Assignment 3. * (**Maximum entropy**) Define a convex function $h\colon \mathbb{R} \to (-\infty, +\infty]$ by

$$h(u) = \begin{cases} u \log u - u & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ +\infty & \text{if } u < 0 \end{cases}$$

and a convex function $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ by

$$f(x) = \sum_{i=1}^{n} h(x_i).$$

Suppose $\hat{x}$ lies in the interior of $\mathbb{R}^n_+$.

(a) Prove f is strictly convex on $\mathbb{R}^n_+$ with compact level sets.

(b) Prove $f'(x; \hat{x} - x) = -\infty$ for any point x on the boundary of $\mathbb{R}^n_+$.

(c) Suppose the map $G\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear with $G\hat{x} = b$. Prove for any vector c in $\mathbb{R}^n$ that the problem

$$\inf\big\{ f(x) + \langle c, x\rangle \;\big|\; Gx = b, x \in \mathbb{R}^n \big\}$$

has a unique solution $\bar{x}$ lying in $\mathbb{R}^n_{++}$.

(d) Prove that some vector λ in $\mathbb{R}^m$ satisfies $\nabla f(\bar{x}) = G^*\lambda - c$, and deduce $x_i = \exp(G^*\lambda - c)_i$.

Assignment 4. Calculate f^* and check $f = f^{**}$ for the following convex functions $f\colon \mathbb{R} \to \mathbb{R}$: e^x, $\log(1 + e^x)$ and $\sqrt{1 + x^2}$.

Assignment 5. * (**Maximum entropy example**)

(a) Prove the function $g\colon \mathbb{E} \to (-\infty, +\infty]$ defined for points $a^1, a^2, \ldots, a^m$ in $\mathbb{E}$ by

$$g(z) = \inf_{x \in \mathbb{R}^m} \Big\{ \sum_i \exp^*(x_i) \;\Big|\; \sum_i x_i = 1,\; \sum_i x_i a^i = z \Big\}$$

is convex.

(b) For any point $y \in R^m$, prove

$$g^*(y) = \sup_{x \in \mathbb{R}^m} \Big\{ \sum_i \big(x_i \left\langle a^i, y \right\rangle - \exp^*(x_i) \big) \;\Big|\; \sum_i x_i = 1 \Big\}.$$

(c) Apply Assignment 3 to deduce the conjugacy formula

$$g^*(y) = 1 + \log\Big(\sum_i \exp\langle a^i, y\rangle\Big).$$

(d) Compute the conjugate of the function of $x \in \mathbb{R}^m$,

$$\begin{cases} \sum_i \exp^*(x_i) & \text{if } \sum_i x_i = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Assignment 6. Prove the Fenchel-Young inequality.

Assignment 7. Consider Fenchel duality for linear constraints and deduce duality theorems for the following separable problems

$$\inf\Big\{\sum_{i=1}^{n} p(x_i) \;\Big|\; Ax = b\Big\},$$

where the map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, $b \in \mathbb{R}^m$, and the function $p\colon \mathbb{R} \to (-\infty, +\infty]$ is convex, defined as follows

(a) (**Nearest point in polyhedrons**) $p(t) = t^2/2$ with domain $\mathbb{R}_+$.
(b) (**Analytic centre**) $p(t) = -\log t$ with domain $\mathbb{R}_{++}$.
(c) (**Maximum entropy**) $p = \exp^*$.

Assignment 8. With the notation of Lecture 1, section B., prove that the formula for the subdifferential of a max-function

$$\partial g(\bar{x}) = \bigcup\Big\{\partial\big(\sum_{i\in I} \lambda_i g_i\big)(\bar{x}) \;\Big|\; \lambda \in \mathbb{R}^I\,,\ \sum_{i\in I} \lambda_i = 1\Big\}$$

simplifies to

$$\partial g(\bar{x}) = \operatorname{conv} \bigcup_{i\in I} \partial g_i(\bar{x})$$

if the calculus rule for subdifferentials holds for the functions g_i, $i \in I$ at point $\bar{x}$.

Assignment 9. (**Examples of duals**) Calculate the Lagrangian dual problem for the following problems (for given vectors $a^1, a2, \ldots, a^m$, and c in $\mathbb{R}^n$).

(a) The linear program

$$\inf_{x\in\mathbb{R}^n} \big\{\langle c, x\rangle \;\big|\; \big\langle a^i, x\big\rangle \leqslant b_i \text{ for } i = 1, 2, \ldots, m\big\}.$$

(b) Another linear program

$$\inf_{x\in\mathbb{R}^n} \big\{\langle c, x\rangle + \delta_{\mathbb{R}^n_+}(x) \;\big|\; \big\langle a^i, x\big\rangle \leqslant b_i \text{ for } i = 1, 2, \ldots, m\big\}.$$

(c) The quadratic program (for $C \in \mathbb{S}^n_{++}$, i.e. C positive definite)

$$\inf_{x\in\mathbb{R}^n} \Big\{\frac{\langle x, Cx\rangle}{2} \;\Big|\; \big\langle a^i, x\big\rangle \leqslant b_i \text{ for } i = 1, 2, \ldots, m\Big\}.$$

(d) The penalised linear program

$$\inf_{x\in\mathbb{R}^n} \big\{\langle c, x\rangle + \epsilon \operatorname{lb}(x) \;\Big|\; \big\langle a^i, x\big\rangle \leqslant b_i \text{ for } i = 1, 2, \ldots, m\Big\}.$$