C. Gibbs sampler

- We know

$$p_u(s) = \frac{1}{Z(u)} \exp \sum_{ij \in E} u_{ij}(s_i, s_j) = \frac{1}{Z(u)} \exp \langle \varphi(s), u \rangle$$

The marginal distributions $p_u(s_i, s_j) = \mu_{ij}(s_i, s_j)$ are given by $\mu = \mathbb{E}_u(\varphi)$

- Let $F : K^{|V|} \to \mathbb{R}^m$ be a random vector defined over the random field $S = \{S_i \mid i \in V\}$. How can we <u>estimate</u> its expectation $\mathbb{E}_u(F) = \sum_{s \in \mathcal{S}} p_u(s) F(s)$?

  1) generate an i.i.d. sample s^j, $j = 1, \ldots, \ell$ from $p_u(s)$
  2) estimate $\mathbb{E}_u(F) \approx \frac{1}{\ell} \sum_{j=1}^{\ell} F(s^j)$

- How to sample from $p_u(s)$?
  Design a homogeneous Markov chain with transition probability matrix $T(s \mid s')$, $s, s' \in K^{|V|}$ s.t.
  - the chain is irreducible and a-periodic, i.e. it has a unique stationary p.d. $p_*(s)$
  - ensure that $p_*(s) = p_u(s)$

<u>Practically</u>:

- Design a set of simpler transition matr. B_m, $m \in M$ s.t. $p_u(s)$ is stationary for all of them (but not necessarily unique)
- Compose T by

$$T = \prod_m B_m \quad \text{or} \quad T = \sum_{m \in M} \alpha_m B_m$$

- prove that T is irreducible and a-periodic, i.e. that T has a unique stationary distribution

<u>Gibbs sampler</u> Design $\mathcal{B}_i$, $i \in V$ by

$$\mathcal{B}_i(s|\tilde{s}) = \begin{cases} p_u(s_i|\tilde{s}_{V\setminus i}) = p_u(s_i|\tilde{s}_{N_i}) & \text{if } s_{V\setminus i} \equiv \tilde{s}_{V\setminus i} \\ 0 & \text{otherwise} \end{cases}$$

- stationarity of $p_u(s)$

$$\sum_{\tilde{s}\in K^{|V|}} \mathcal{B}_i(s|\tilde{s})\, p_u(\tilde{s}) = \sum_{k\in K} p_u(s_i|s_{V\setminus i})\, p_u(\tilde{s}_i = k, s_{V\setminus i})$$

$$= p_u(s_i|s_{V\setminus i})\, p_u(s_{V\setminus i}) = p_u(s)$$

- $T = \prod_{i\in V} \mathcal{B}_i$ and $T = \sum_{i\in V} \alpha_i \mathcal{B}_i$ have strictly positive matrix elements $\Rightarrow$ $p_u(s)$ is unique stationary distribution.

- $$p_u(s_i|s_{N_i}) = \frac{1}{Z_i(u)} \exp \sum_{j\in N_i} u_{ij}(s_i, s_j)$$
  is easy to compute!

<u>Remarks</u>

1) Gibbs sampler is easy to implement

$$s^{(0)} \xrightarrow{T} s^{(1)} \xrightarrow{T} \dots \rightarrow s^{(t)} \xrightarrow{T} s^{(t+1)}$$

$s^{(t)} \rightarrow s^{(t+1)}$: for each $i \in V$ resample s_i from $p_u(s_i|s_{N_i})$

2) Gibbs samplers are very slow

- How long shall we wait until the sequence $s^{(0)}, s^{(1)}, \ldots$ has come close to equilibrium?

- The successive realisations are correlated!

$$C_F(t) = \mathrm{Cov}(F_{t_0}, F_{t_0+t})$$
$$= \sum_{s,\tilde{s}} p_u(s) F(s)\, T^t(\tilde{s}|s)\, F(\tilde{s}) - \mathbb{E}_u^2(F)$$

$$\rho_F(t) = \frac{C_F(t)}{C_F(0)} \sim e^{-t/\tau}$$

This exponential decay can be slow!