2. Random fields (undirected graphical models)

- A K-valued random field S is a collection $\{S_i \mid i \in V\}$ of K-valued random variables

$i \in V$ can be: pixels, object parts, ...

$k \in K$ can be: colours, segment labels, depth values, poses of object parts, ...

- As usual, $s \in K^{|V|} \hat{=} \mathcal{S}$ denotes a realisation $s_i \in K$, $i \in V$

- $p \in \mathcal{P}$ denotes a joint p.d. $p : \mathcal{S} \to \mathbb{R}_+$

Questions:

1) What is the dimension of $\mathcal{P}$?, Which $p \in \mathcal{P}$ is the simplest?

2) We fix marginal distributions $p(s_i) = \mu_i(s_i)$ for each S_i. This decreases the dimension of $\mathcal{P}$ by ...? Which p is the simplest?

3) We equip V with a graph structure (V, E) and fix pairwise marginal distributions $p(s_i, s_j) = \mu_{ij}(s_i, s_j)$ for all edges $\{i,j\} \in E$. This decreases the dimension of $\mathcal{P}$ by ...? Which p is the simplest?

4) We equip V with a hypergraph structure and ...
.....

LA: Exponential families

- $S = \{S_i \mid i \in V\}$ is a K-valued random field
- $\Phi: \mathcal{S} = K^{|V|} \to \mathbb{R}^n$ is a random vector

Consider the task

$$\inf_p \Big\{ \sum_{s\in\mathcal{S}} p(s) \log p(s) \;\Big|\; \mathbb{E}_p(\Phi) = \mu, \sum_{s\in\mathcal{S}} p(s) = 1 \Big\}$$

Lagrange function

$$L(p) = \sum_{s\in\mathcal{S}} p(s) \log p(s) - \langle u, \mathbb{E}_p(\Phi) - \mu \rangle - \lambda \Big[\sum_{s\in\mathcal{S}} p(s) - 1 \Big]$$

$$\cdots\to \qquad p_u(s) = \exp\big[\langle u, \varphi(s) \rangle - \log Z(u) \big]$$

Assign. 3 in Seminar 1 $\Rightarrow$

If there $\exists$ a strictly positive $\bar{p}$ s.t. $\mathbb{E}_{\bar{p}}(\Phi) = \mu$ then p_u is a unique solution

Questions

(1) Which $\mu \in \mathbb{R}^n$ qualify?

$$\mathbb{E}_{\bar{p}}(\Phi) = \sum_{s\in\mathcal{S}} \bar{p}(s)\, \varphi(s) = \mu, \quad \bar{p}(s) > 0 \;\forall\, s \in \mathcal{S} \iff$$

μ lies in the interior of $\operatorname{conv} \Phi(\mathcal{S})$ where $\Phi(\mathcal{S}) = \{\varphi(s) \mid s \in \mathcal{S}\}$

(2) $u \mapsto \mu$? Yes, because

$$\nabla \log Z(u) = \nabla \log \sum_{s\in\mathcal{S}} e^{\langle u, \varphi(s) \rangle} =$$

$$= \frac{1}{Z(u)} \sum_{s\in\mathcal{S}} e^{\langle u, \varphi(s) \rangle} \varphi(s) = \mathbb{E}_u(\Phi) = \mu$$

(3) $\mu \mapsto u$? (provided that $\mu \in \operatorname{int}(\operatorname{conv}(\Phi(\mathcal{S})))$)

No, not always. p_u is unique, but not u!

$$\langle u, \varphi(s)\rangle - \log Z(u) = \langle \tilde{u}, \varphi(s)\rangle - \log Z(\tilde{u}) \quad \forall s \in \mathcal{S}$$

$\hookrightarrow$ $$\langle u - \tilde{u}, \varphi(s)\rangle = \mathrm{const}_s \quad \forall s \in \mathcal{S}$$

$$\{ u \in \mathbb{R}^n \mid \langle u, \varphi(s)\rangle = \mathrm{const}_s \ \forall s \in \mathcal{S} \} = L^\perp$$

$$\operatorname{aff}(\Phi(\mathcal{S})) = \varphi(s_0) - \underbrace{\operatorname{span}(\Phi(\mathcal{S}) - \varphi(s_0))}_{L}$$

for any $s_0 \in \mathcal{S}$

Suppose we write

$\operatorname{aff} \Phi(\mathcal{S}) = \{ u \in \mathbb{R}^n \mid Au = b \}$ where $A : \mathbb{R}^n \to \mathbb{R}^m$

then $L = \operatorname{Ker} A$ and $L^\perp = \operatorname{Im} A^T$

Answer:

- if $\operatorname{aff} \Phi(\mathcal{S}) = \mathbb{R}^n \Rightarrow \mu \mapsto u$ is a mapping
- if $\operatorname{aff} \Phi(\mathcal{S}) = \{ u \in \mathbb{R}^n \mid Au = b \} \neq \mathbb{R}^n \Rightarrow$ we can reparametrise u by

$$u \to u + A^T \psi, \quad \psi \in \mathbb{R}^m$$

and have

$$\langle u + A^T\psi, \varphi(s)\rangle = \langle u, \varphi(s)\rangle + \langle \psi, b\rangle$$
$$\log Z(u + A^T\psi) = \log Z(u) + \langle \psi, b\rangle$$

B. Graphical models (undirected)

- $S = \{S_i \mid i \in V\}$ is a K-valued random field
- (V, E) is an undirected graph
- Fix pairwise marginal distributions $p(s_i, s_j) = \mu_{ij}(s_i, s_j)$ for all edges $\{i,j\} \in E$

Under these constraints: search joint p.d. with maximal entropy ($\approx$ minimal information)

Everything said in A. applies here because: Any marginal probability $p(s_i = k, s_j = k')$ can be seen as the expectation of a random variable

Hence: If $\mu_{ij}(s_i, s_j)$ defines a valid set of marginal distributions, then the task has a unique solution

$$p_u(s) = \frac{1}{Z(u)} \exp\Big[\sum_{\{i,j\}\in E} u_{ij}(s_i, s_j)\Big]. \qquad (*)$$

However, the potentials $u_{ij} : K^2 \to \mathbb{R}$ are defined up to reparametrisations.

The distribution p_u is a Gibbs random field and factorises over the edges of the graph (V, E)

<u>Problems</u>: All tasks

a) given potentials $u_{ij}: K^2 \to \mathbb{R}$ $\forall \{ij\} \in E$, compute

- $Z(u) = \sum'_{s \in \mathcal{S}} \exp\left[\sum'_{ij \in E} u_{ij}(s_i, s_j)\right]$ and/or

- marginal prob's $p_u(s_i)$, $p_u(s_i, s_j)$,

b) check whether $\mu_{ij}: K^2 \to \mathbb{R}_+$ $\forall \{ij\} \in E$ represent a valid system of pairwise marginals,

c) given a valid system of marginals $\mu_{ij}: K^2 \to \mathbb{R}_+$ $\forall \{ij\} \in E$, compute the potentials u_{ij}

d) given potentials u_{ij} find the most probable ~~configura~~ realisations

$$\arg\max_{s \in \mathcal{S}} p_u(s)$$

are <u>NP-hard</u>. Polynomial time complexity algorithms exist if (V, E) is acyclic.

3. The most probable realisation of a GRF

- $S = \{S_i \mid i \in V\}$ is a K-valued random field
- (V, E) is an undirected graph
- The joint distribution $p(S)$ is a Gibbs random field

Task: Find

$$\operatorname*{argmax}_{s \in \mathcal{S}} \frac{1}{Z(u)} \exp \sum_{ij \in E} u_{ij}(s_i, s_j) =$$

$$= \operatorname*{argmax}_{s \in \mathcal{S}} \sum_{ij \in E} u_{ij}(s_i, s_j)$$

Remark 1 The task is solvable in polynomial time if K is completely ordered and all functions $-u_{ij} : K^2 \to \mathbb{R}$ are submodular w.r.t. the ordering. ■

Abstract view on the task:

$$\operatorname*{argmax}_{s \in \mathcal{S}} \langle u, \varphi(s) \rangle \qquad \text{discrete task}$$

relax it to

$$\operatorname*{argmax}_{p \in \mathcal{P}} \sum_{s \in \mathcal{S}} p(s) \langle u, \varphi(s) \rangle = \operatorname*{argmax}_{p \in \mathcal{P}} \langle u, \mathbb{E}_p(\Phi) \rangle$$

Notice that for a GRF $\mathbb{E}_p(\Phi) \hat{=}$ pairwise marginals of p

The task reads

$$\langle u, \mu \rangle \to \max_{\mu}$$

$$\text{s.t.} \quad \mu \in \operatorname{conv} \Phi(\mathcal{S})$$

It is a linear optimisation task, but $\operatorname{conv} \Phi(\mathcal{S})$ is „complicated" Thus, relax the task to a simpler polytope $L \supset \operatorname{conv} \Phi(\mathcal{S})$

Back to GRFs

$$\sum_{ij \in E} \sum_{s_i, s_j \in K} \mu_{ij}(s_i, s_j)\, u_{ij}(s_i, s_j) \to \max_{\mu}$$

$$\text{s.t.} \quad \sum_{s_i, s_j} \mu_{ij}(s_i, s_j) = 1 \qquad \forall \{ij\} \in E$$

$$\sum_{s_j} \mu_{ij}(s_i, s_j) = \sum_{s_e} \mu_{ie}(s_i, s_e) \qquad \forall i, j, \ell : \{i, j\}, \{i, \ell\} \in E, \; \forall s_i \in K$$

$$\mu > 0$$

What have we done from abstract perspective:

$$\langle u, \mu \rangle \to \max_{\mu}$$

$$\text{s.t.} \quad \mu \in \left[\operatorname{aff} \Phi(\mathcal{S})\right] \cap \mathbb{R}^n_+$$

The constraint can be written as (see sec. 2)

$$\begin{cases} A\mu = b \\ \mu \geq 0 \end{cases}$$

where $\operatorname{aff} \Phi(S) = \{ \mu \in \mathbb{R}^n \mid A\mu = b \}$

The relaxed task reads

$$\langle u, \mu \rangle \to \max_{\mu}$$

s.t.
$$A\mu = b$$
$$\mu \geq 0$$

Its dual task is

$$\langle b, \psi \rangle \to \min_{\psi}$$

s.t. $A^T \psi$

Remember: ψ and $A^T\psi$ describe reparametrisations!

„Translate" this back to the domain of GRFs $\to$ assignment for seminar