I C Fenchel conjugate & duality

Definition 3 The Fenchel conjugate of a function $h: \mathbb{E} \to [-\infty, +\infty]$ is the function $h^*: \mathbb{E} \to [-\infty, +\infty]$ defined by

$$h^*(\varphi) = \sup_{x \in \mathbb{E}} \{ \langle \varphi, x \rangle - h(x) \}. \qquad \square$$

The function h^* is convex and if the domain of h is nonempty then h^* never takes the value $-\infty$.

Example 3

a) $$\exp^*(t) = \begin{cases} t \log t - t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ +\infty & \text{otherwise} \end{cases}$$

b) the function $lb: \mathbb{R}^n \to (-\infty, +\infty]$ defined by

$$lb(x) = \begin{cases} -\sum_{i=1}^{n} \log x_i & \text{if } x \in \mathbb{R}^n_{++} \\ +\infty & \text{otherwise} \end{cases}$$

satisfies

$$lb^*(x) = lb(-x) - n \quad \forall x \in \mathbb{R}^n$$

Fenchel-Young inequality: Any points $\varphi, x \in \mathbb{E}$ in the domain of a function $h: \mathbb{E} \to [-\infty, +\infty]$ satisfy the inequality

$$h(x) + h^*(\varphi) \geq \langle \varphi, x \rangle.$$

Equality holds if and only if $\varphi \in \partial h(x)$

Fenchel duality & convex calculus

For given functions $f: \mathbb{E} \to (-\infty, +\infty]$ and $g: \mathbb{Y} \to (-\infty, +\infty]$ and a linear map $A: \mathbb{E} \to \mathbb{Y}$ let

$$p = \inf_{x \in \mathbb{E}} \{ f(x) + g(Ax) \}$$

$$d = \sup_{\varphi \in \mathbb{Y}} \{ -f^*(A^*\varphi) - g^*(-\varphi) \}$$

then $p \geq d$ (weak duality). if f and g are convex and satisfy the condition

$$0 \in \operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f)$$

then $p = d$ (strong duality) and the supremum is atained if finite

At any point $x \in \mathbb{E}$ the calculus rule

$$\partial (f + g \circ A)(x) \supset \partial f(x) + A^* \partial g(Ax)$$

holds, with equality if f and g are convex and the above condition holds.

Example 4 (Fenchel duality for linear constraints)

Given an convex function $f: \mathbb{E} \to (-\infty, +\infty]$ and a linear map $A: \mathbb{E} \to \mathbb{Y}$, the strong duality equality

$$\inf_{x \in \mathbb{E}} \{ f(x) \mid Ax = b \} = \sup_{\varphi \in \mathbb{Y}} \{ \langle b, \varphi \rangle - f^*(A^*\varphi) \}$$

holds if b belongs to $\operatorname{int}(A \operatorname{dom} f)$

D: Lagrangian duality

We consider the convex program

$$\inf\{f(x) \mid g_i(x) \le 0,\ i=1,2,..,m,\ x \in \mathbb{E}\}, \qquad (1)$$

where functions $f, g_i : \mathbb{E} \to (-\infty, +\infty]$ are convex and satisfy $\emptyset \neq \operatorname{dom} f \subset \bigcap_i \operatorname{dom} g_i$

Denote the vector with components $g_i(x)$ by $g(x)$

- The Lagrange function $L : \mathbb{E} \times \mathbb{R}^m_+ \to (-\infty, +\infty]$ is defined by

$$L(x;\lambda) = f(x) + \lambda^T g(x)$$

- A Lagrange multiplier vector $\bar{\lambda} \in \mathbb{R}^m_+$ for a feasible $\bar{x} \in \mathbb{E}$:
  - $\bar{x}$ minimises $L(\cdot\,;\bar{\lambda})$ over $\mathbb{E}$ and
  - $\bar{\lambda}$ satisfies complementary slackness $\bar{\lambda}_i = 0$ whenever $g_i(\bar{x}) < 0$

- Slater condition: There exists $\bar{x}$ in $\operatorname{dom} f$ with $g_i(\bar{x}) < 0$ for $i = 1, 2, .., m$. (2)

<u>Theorem 2</u> (Lagrangian necessary conditions)

Suppose $\bar{x} \in \operatorname{dom} f$ is optimal for the convex program and that the Slater condition holds. Then there is a Lagrange multiplier vector for $\bar{x}$.

Notice that the Lagrangian encapsulates all the information of the primal problem (1):

$$\sup_{\lambda \in \mathbb{R}^m_+} L(x;\lambda) = \begin{cases} f(x) & \text{if } x \text{ is feasible} \\ +\infty & \text{otherwise} \end{cases}$$

Denoting the optimal value of (1) by p we could rewrite (1) in the following form

$$p = \inf_{x \in \mathbb{E}} \sup_{\lambda \in \mathbb{R}^m_+} L(x;\lambda)$$

This makes it natural to consider the problem

$$d = \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in \mathbb{E}} L(x;\lambda)$$

This is the dual problem. It consists in maximising over $\lambda \in \mathbb{R}^m_+$ the dual function

$$\varphi(\lambda) = \inf_{x \in \mathbb{E}} L(x;\lambda)$$

Weak duality: $p \geq d$ (, φ is concave)

Theorem 3 (strong duality)

- If the problem (1) is convex and the Slater condition holds $\Rightarrow$ $p = d$, d is attained if finite
- If the problem (1) is convex and $\hat{\lambda}_0 f + \hat{\lambda}^T g$ has compact level sets for some $\hat{\lambda}_0 \geq 0$, $\hat{\lambda} \in \mathbb{R}^m_+$ $\Rightarrow$ $p = d$, p is attained if finite

E. Minimising a difference of convex functions, DCA

Let $g, h : \mathbb{E} \to (-\infty, +\infty]$ be convex (and lsc).
Consider the task

$$\inf \{ g(x) - h(x) \mid x \in \mathbb{E} \} \qquad (3)$$

We have

$$\begin{aligned}
p &= \inf_{x \in \mathbb{E}} \{ g(x) - h(x) \} \\
&= \inf_{x \in \mathbb{E}} \{ g(x) - \sup_{y \in \mathbb{E}} \{ \langle x, y \rangle - h^*(y) \} \} \\
&= \inf_{y \in \mathbb{E}} \inf_{x \in \mathbb{E}} \{ g(x) - \langle x, y \rangle + h^*(y) \} \\
&= \inf_{y \in \mathbb{E}} \{ h^*(y) - g^*(y) \}
\end{aligned}$$

The task

$$\inf \{ h^*(y) - g^*(y) \mid y \in \mathbb{E} \} \qquad (4)$$

is the DC-dual of (3).

DC-Algorithms construct two sequences $\{x^k\}, \{y^k\}$ with decreasing values of the primal and dual objective: $y^k \in \partial h(x^k)$, $x^{k+1} \in \partial g^*(y^k)$

Explanation: use affine minorisation of h and g^*

- $\inf_x \{ g(x) - h(x) \} \leq \inf_x \{ g(x) - [h(x^k) + \langle x - x^k, y^k \rangle] \}$
  where $y^k \in \partial h(x^k)$. Solve rhs $\to$ $x^{k+1} \in \partial g^*(y^k)$

- $\inf_y \{ h^*(y) - g^*(y) \} \leq \inf_y \{ h^*(y) - [g^*(y^k) + \langle y - y^k, x^{k+1} \rangle] \}$
  where $x^{k+1} \in \partial g^*(y^k)$. Solve rhs $\to$ $y^{k+1} \in \partial h(x^{k+1})$