GRAPHICAL MARKOV MODELS (WS2016)
4. SEMINAR: ADDITIONAL HINTS

Assignment 1. None

Assignment 2. The non-trivial direction is Definition 1b ⇒ Definition 1a. Assume that

$$p(s) = \prod_{\{i,j\}\in E} g_{ij}(s_i, s_j)$$

where $T = (V, E)$ is an undirected tree and $s_i \in K$, $i \in V$ are random variables.

Choose any vertex $r \in V$ as root and denote the corresponding edge orientations by $\vec{E}_r$. Choose a leaf $l \in V$ and denote by m the only vertex connected to l by an edge. Prove that the conditional distribution $p(s_l \mid s_m)$ can be written as

$$p(s_l \mid s_m) = g_{ml}(s_m, s_l)\, b(s_m),$$

where b is some function which depends only on the value of s_m. Denote the sub-tree obtained by removing the node l by $T' = (V', E')$ and let s' be a realisation on this sub-tree. Deduce that

$$p(s') = a(s_m) \prod_{\{i,j\}\in E'} g_{ij}(s_i, s_j),$$

where a is some function which depends only on the value of s_m.

Apply this recursively and conclude that Definition 1a follows from Definition 1b.

Assignment 3. Notice that the function $x \log x$ is defined for non-negative $x \geqslant 0$ and is convex. Notice that there is a constraint for every edge $\{i, j\} \in E$ and any pair $k, k' \in K$. Consequently, there will be a Lagrange multiplier $\lambda_{ij}(k, k')$ for each of these constraints. Consider the minimisation of the Lagrange function w.r.t. the variables $p(s)$ and conclude that the optimal joint distribution is a product of factors, each on depending only on a pair of random variables (for edges of T).