# GRAPHICAL MARKOV MODELS (WS2014)
# 2. SEMINAR

We consider the following Markov chain model for all subsequent assignments. The p.d. for sequences $s = (s_1, \dots, s_n)$ of length $n$ with states $s_i \in K$ is given by:

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1}).$$

The transition probabilities $p(s_i \mid s_{i-1})$ and the p.d. $p(s_1)$ for the state of the first element are assumed to be known.

**Assignment 1.**
**a)** Suppose that the marginal p.d.s $p(s_i)$ for the states of the $i$-th element of the sequence are known for all $i = 2, \dots, n$. Then it is easy to calculate all "inverse" transition probabilities $p(s_{i-1} \mid s_i)$. How?
**b)** Describe an efficient algorithm for calculating $p(s_i)$ for all $i = 2, \dots, n$.

**Assignment 2.** Suppose that there is a special state $k^* \in K$. We want to know how often this state appears on average in a sequence generated by the model. Describe an efficient method for the calculation of this average.
*Hint:* You may use the fact that the mean value of a sum of random variables is equal to the sum of their means. The number of occurrences of the state $k^*$ in a sequence $s$ can be obviously written in the form

$$\delta_{s_1 k^*} + \delta_{s_2 k^*} + \ldots + \delta_{s_n k^*}$$

where $\delta_{ij}$ denotes the Kronecker delta.

**Assignment 3.** Let $A \subset K$ be a subset of states and let $\mathcal{A} = A^n$ denote the set of all sequences $s$ with $s_i \in A$ for all $i = 1, \dots, n$. Find an efficient algorithm for calculating the probability $p(\mathcal{A})$ of the event $\mathcal{A}$.

**Assignment 4.** Consider a fully specified HMM for sequence pairs $(x, s)$ of length $n$ given by

$$p(x, s) = p(s_1) \cdot \prod_{i=2}^{n} p(s_i \mid s_{i-1}) \cdot \prod_{i=1}^{n} p(x_i \mid s_i).$$

Let $k^* \in K$ be a fixed, special hidden state. Let us call a subset of positions $I \subset \{1, 2, \dots, n\}$ a *segmentation* and associate a subset of hidden state sequences with each segmentation:

$$\mathcal{S}(I) = \big\{ s \in K^n \bigm| s_i = k^*, \forall i \in I \text{ and } s_i \in K \setminus k^*, \forall i \notin I \big\}.$$

**a)** Find an efficient algorithm for calculating $p(x, \mathcal{S}(I))$.
**b**$^{**}$**)** Suppose we want to find the most probable segmentation given an observed feature sequence $x$

$$I^* = \arg\max_{I} p(x, \mathcal{S}(I)).$$

Deduce an efficient algorithm for this task.

**Assignment 5.** Let $x$ be a gray value image of size $n \times m$, where $x_{ij}$ denotes the gray value of the pixel with coordinates $(i, j)$. The task is to segment such images into an upper and lower part by a boundary represented as a sequence of height values $s_j \in \{1, 2, \ldots, n\}$ for all $j = 1, 2, \ldots, m$.

The prior probability for boundaries is assumed to be a homogeneous Markov chain such that $p(s_j \mid s_{j-1}) = 0$ if $|s_j - s_{j-1}| > 1$. The appearance model for columns $\vec{x}_j$ given the boundary value $s_j$, is assumed to be conditional independent

$$p(\vec{x}_j \mid s_j) = \prod_{i \leqslant s_j} p_1(x_{ij}) \cdot \prod_{i > s_j} p_2(x_{ij}),$$

where $p_1()$ and $p_2()$ are two pd-s for gray values.

**a)** Deduce an efficient algorithm for determining the most probable boundary.

**b**$^{**}$**)** Suppose that the loss function $c(s, s')$ for incorrectly recognised boundaries is defined by

$$c(s, s') = \sum_{j=1}^{m} (s_j - s'_j)^2.$$

Formulate the segmentation task for this case. Deduce an efficient inference algorithm.