GRAPHICAL MARKOV MODELS (WS2013)
2. SEMINAR

We consider the following Markov chain model for all subsequent assignments. The p.d. for sequences $s = (s_1, \ldots, s_n)$ of length n with states $s_i \in K$ is given by:

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1}).$$

The transition probabilities $p(s_i \mid s_{i-1})$ and the p.d. $p(s_1)$ for the state of the first element are assumed to be known.

Assignment 1.
a) Suppose that the marginal p.d.s $p(s_i)$ for the states of the i-th element of the sequence are known for all $i = 2, \ldots, n$. Then it is easy to calculate all "inverse" transition probabilities $p(s_{i-1} \mid s_i)$. How?
b) Describe an efficient algorithm for calculating $p(s_i)$ for all $i = 2, \ldots, n$.

Assignment 2. Suppose that there is a special state $k^* \in K$. We want to know how often this state appears on average in a sequence generated by the model. Describe an efficient method for the calculation of this average.
Hint: You may use the fact that the mean value of a sum of random variables is equal to the sum of their means. The number of occurrences of the state k^* in a sequence s can be obviously written in the form

$$\delta_{s_1 k^*} + \delta_{s_2 k^*} + \ldots + \delta_{s_n k^*}$$

where δ_{ij} denotes the Kronecker delta.

Assignment 3. Let $A \subset K$ be a subset of states and let $\mathcal{A} = A^n$ denote the set of all sequences s with $s_i \in A$ for all $i = 1, \ldots, n$. Find an efficient algorithm for calculating the probability $p(\mathcal{A})$ of the event $\mathcal{A}$.

Assignment 4. Consider a fully specified HMM for sequence pairs (x, s) of length n given by

$$p(x, s) = p(s_1) \cdot \prod_{i=2}^{n} p(s_i \mid s_{i-1}) \cdot \prod_{i=1}^{n} p(x_i \mid s_i).$$

Let $k^* \in K$ be a fixed, special hidden state. Let us call a subset of positions $I \subset \{1, 2, \ldots, n\}$ a *segmentation* and associate a subset of hidden state sequences with each segmentation:

$$\mathcal{S}(I) = \big\{ s \in K^n \;\big|\; s_i = k^*, \forall i \in I \text{ and } s_i \in K \setminus k^*, \forall i \notin I \big\}.$$

a) Find an efficient algorithm for calculating $p(x, \mathcal{S}(I))$.
b****)** Suppose we want to find the most probable segmentation given an observed feature sequence x

$$I^* = \arg\max_I p(x, \mathcal{S}(I)).$$

Deduce an efficient algorithm for this task.

Assignment 5. Let x be a gray value image of size $n \times m$, where x_{ij} denotes the gray value of the pixel with coordinates (i, j). The task is to segment such images into an upper and lower part by a boundary represented as a sequence of height values $s_j \in \{1, 2, \ldots, n\}$ for all $j = 1, 2, \ldots, m$.

The prior probability for boundaries is assumed to be a homogeneous Markov chain such that $p(s_j \mid s_{j-1}) = 0$ if $|s_j - s_{j-1}| > 1$. The appearance model for columns $\vec{x}_j$ given the boundary value s_j, is assumed to be conditional independent

$$p(\vec{x}_j \mid s_j) = \prod_{i \leqslant s_j} p_1(x_{ij}) \cdot \prod_{i > s_j} p_2(x_{ij}),$$

where $p_1()$ and $p_2()$ are two pd-s for gray values.

a) Deduce an efficient algorithm for determining the most probable boundary.

b**) Suppose that the loss function $c(s, s')$ for incorrectly recognised boundaries is defined by

$$c(s, s') = \sum_{j=1}^{m} (s_j - s'_j)^2.$$

Formulate the segmentation task for this case. Deduce an efficient inference algorithm.