Chapter II Graphical models on general graphs

12. Markov random fields, Gibbs random fields

A. Model definitions

- (V, E) - undirected graph (or hypergraph)
  outer boundary ∂M of a vertex set $M \subset V$
  $\partial M = \{ i \in V \setminus M \mid \exists\, j \in M \text{ s.t. } \{i,j\} \in E \}$, $\mathcal{N}_i = \partial\{i\}$
- A field (collection) of K-valued random variables s_i indexed by graph vertices $i \in V$.
  $s_i \in K$, $s = \{ s_i \mid i \in V \}$, $s_M = \{ s_i \mid i \in M \subset V \}$ denotes the subset of variables indexed by vertices of M
- A joint p.d. $p(s)$ defined on K^V

Definition 1 A joint p.d. defined on K^V is a Markov random field w.r.t. the graph structure (V, E) if

$$p(s_M, s_{\tilde{M}} \mid s_{\partial M}) = p(s_M \mid s_{\partial M})\, p(s_{\tilde{M}} \mid s_{\partial M})$$

holds for each $M \subset V$ and $\tilde{M} = V \setminus (M \cup \partial M)$. ■

It follows that an MRF has the property

$$p(s_M \mid s_{\partial M}, s_{\tilde{M}}) = p(s_M \mid s_{\partial M}).$$

The converse is true as well.

Definition 2 Let $(V, \mathcal{C})$ be a hypergraph. A joint p.d. defined on K^V is a Gibbs random field w.r.t. the hypergraph structure $(V, \mathcal{C})$ if it factorises into a product of functions depending on s_c, $c \in \mathcal{C}$, i.e.

$$p(s) = \prod_{c \in \mathcal{C}} f_c(s_c).$$

If $p(s)$ is strictly positive, it can be written as

$$p(s) = \frac{1}{Z} \exp \sum_{c \in \mathcal{C}}{}' u_c(s_c),$$

where $u_c : K^c \to \mathbb{R}$ are arbitrary functions (a.k.a. Gibbs potentials) and Z is a normalising constant (a.k.a. partition sum). ∎

Theorem 1 (Hammersley, Clifford, 1971)
Let (V, E) be a graph and let C denote the system of its cliques. Every strictly positive MRF w.r.t. (V, E) is also a GRF w.r.t. $\mathcal{C}$ and vice versa. ∎

Remark 1 Def 2 does not require $\mathcal{C}$ to be the system of cliques of some graph. Consider e.g. a complete graph (V, E). Any p.d. on K^V is an MRF w.r.t. (V, E)! However, the class of GRFs w.r.t. $\mathcal{C} = V \cup E$ is a proper subclass.

Remark 2 Definitions 1 and 2 generalise to random fields with random variables having different co-domains, i.e. $s_i \in K_i$, $i \in V$.

B. Examples (computer vision)

Example 1 (segmentation)

- $x : V \to F$ an (colour) image defined on $V \subset \mathbb{Z}^2$
- $s : V \to K$ a segmentation (K - set of segment labels)

A model for a joint p.d. $p(x, s)$ e.g. $p(x, s) = p(x \mid s)\, p(s)$ where

(1) $p(s)$ is a GRF on the lattice (V, E)

$$p(s) = \frac{1}{Z} \exp \sum_{ij \in E}' u(s_i, s_j)$$

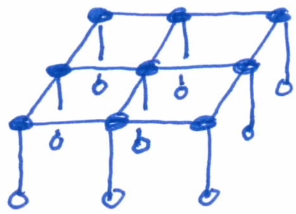
Simple variant: Potts model $u(k, k') = \alpha\, \delta_{kk'}$, $\alpha > 0$

(2) $p(x|s)$ is a conditionally independent appearance model

$$p(x|s) = \prod_{i \in V} p(x_i|s_i),$$

where $p(x_i|s_i)$ are e.g. (mixtures of) Gaussians.

This model is a GRF on the graph

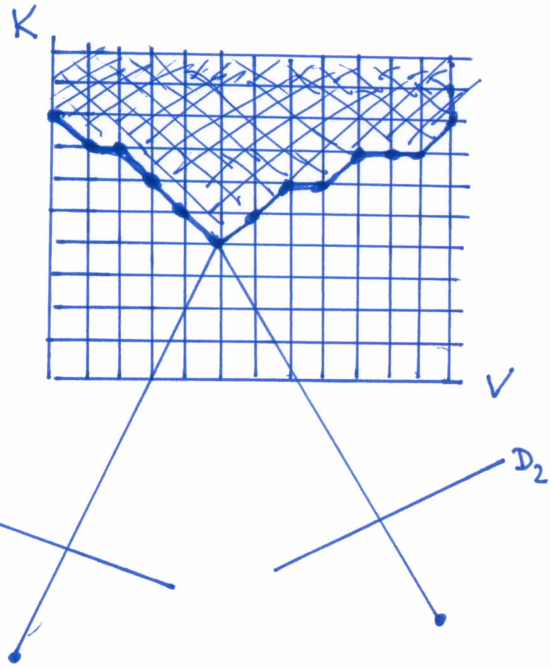


■

Example 2 (stereo reconstruction)

Given: pair of calibrated images of an almost everywhere smooth 3D surface

Task: reconstruct the surface



- $x_{1,2} : D_{1,2} \to F$
- $s : V \to K$ depth map, where $V \subset \mathbb{Z}^2$, $K \subset \mathbb{N}$
- calibration:

$$(V, K) \ni (i, k) \xleftrightarrow{T} (i_1, i_2) \in (D_1, D_2)$$

We don't want to model colours in the images $\Rightarrow$ model only $p(s|x_1, x_2)$ as a conditional random field

$p(s|x_1, x_2) \sim$

$$\exp\left[-\alpha \sum_{ij \in E} (s_i - s_j)^2 - \beta \sum_{i \in V} \| x_1(T_1(i, s_i)) - x_2(T_2(i, s_i)) \|^2 \right]$$

■

1C. Equivalent transformations for GRFs

Consider a GRF w.r.t. $C = V \cup E$ defined on K^V

$$p(s) = \frac{1}{Z(u)} \exp\Big[\sum_{i \in V} u_i(s_i) + \sum_{ij \in E} u_{ij}(s_i, s_j)\Big]$$

<u>Question</u>: Are the functions u_i, u_{ij} uniquely defined by $p(s)$?

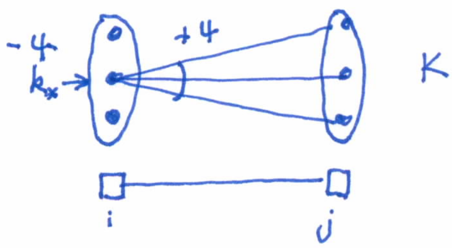
(1) Clearly, adding a constant to any of them will not change p

$$\tilde{u}_i(s_i) = u_i(s_i) + c_i$$

(2) Consider a vertex $i \in V$, an edge $\{i,j\} \in E$ and fix a state $k_* \in K$

$$u_i(k_*) \to u_i(k_*) - \varphi$$

$$u_{ij}(k_*, k) \to u_{ij}(k_*, k) + \varphi \quad \forall k \in K$$



This will not change $p(s)$!

(3) We can apply such „elementary" transformations for each triple $i \in V$, $\{i,j\} \in E$, $k \in K$:

$$\tilde{u}_i(s_i) = u_i(s_i) - \sum_{j \in N_i} \varphi_{ij}(s_i)$$

$$\tilde{u}_{ij}(s_i, s_j) = \varphi_{ij}(s_i) + u_{ij}(s_i, s_j) + \varphi_{ji}(s_j)$$

<u>Theorem 2</u> (w/o proof)

The equivalent transformations (a.k.a. reparametrisations) given above describe <u>all</u> possible equivalent transformations. ∎