14. Approximation algorithms for (Min,+)-problems

We still consider the task

$$s^* \in \underset{s \in K^V}{\operatorname{argmin}} \, U(s) = \underset{s \in K^V}{\operatorname{argmin}} \left[\sum_{i \in V} u_i(s) + \sum_{ij \in E} u_{ij}(s_i, s_j) \right]$$

A Iterated descent

- define a family of neighbourhoods $N_m(s)$, $m = 1, \dots, M$
- repeatedly solve the restricted problem

$$s^{(t+1)} \in \underset{s \in N_m(s^{(t)})}{\operatorname{argmin}} \, U(s)$$

until no further improvement is possible, i.e.

$$s^{(t)} \in \underset{s \in N_m(s^{(t)})}{\operatorname{argmin}} \, U(s) \quad \forall\, m = 1, \dots, M$$

Iterated conditional modes (ICM)

Choose a very simple system of neighbourhoods $N_i(s)$, $i \in V$

$$N_i(s) = \{ s' \in K^V \,/\, s'_j = s_j \ \ \forall j \neq i \}$$

The task $\underset{s' \in N_i(s)}{\operatorname{argmin}} \, U(s')$ is easy to solve. However, the neighbourhoods $N_i(s)$ are very small and their are no approximation bound guaranties.

α-Expansions (Boykov et al. 2001)

For each label $\alpha \in K$ define

$$N_\alpha(s) = \{ s' \in K^V \,/\, s'_i = \alpha \text{ if } s'_i \neq s_i \ \ \forall i \in V \}.$$

Is the task $s^* \in \underset{s' \in N_\alpha(s)}{\operatorname{argmin}} \, U(s')$ solvable in polynomial time?

Yes, if

$$u_{ij}(k,k') + u_{ij}(\alpha,\alpha) \leq u_{ij}(k,\alpha) + u_{ij}(\alpha,k')$$

holds for $\forall \{ij\} \in E$ and $\forall k,k' \in K \setminus \alpha$.
This can be seen by constructing a binary (min,+)-problem, which is equivalent to the considered reduced optimisation task:

$V' = \{ i \in V \mid s_i \neq \alpha \}$, $y_i = 0,1$ encode the labelling s', i.e.

$$s_i' = s_i \Leftrightarrow y_i = 0$$
$$s_i' = \alpha \Leftrightarrow y_i = 1$$

The pairwise functions of this equivalent task are submodular if the condition given above holds.

Example 1 Consider the Potts model $u_{ij}(k,k') = a_{ij}(1-\delta_{kk'})$, $a_{ij} > 0$. It is not submodular if $|K| > 2$. However, it fulfills the above conditions.

Theorem 1 (w/o proof)
Let $\bar{s}$ be a fixpoint of α-expansions $\forall \alpha \in K$. Then
$U(\bar{s}) \leq 2c \min_{s \in K^V} U(s)$, where c is defined as

$$c = \max_{ij \in E} \left[\frac{\max_{k \neq k'} u_{ij}(k,k')}{\min_{k \neq k'} u_{ij}(k,k')} \right].$$

$\alpha\beta$-Swaps (Boykov et al. 2001)
Define $N_{\alpha\beta}$ for each pair of labels $\alpha,\beta \in K$ as follows

$$N_{\alpha\beta}(s) = \left\{ s' \in K^V \,\middle|\, s_i' = \begin{cases} s_i & \text{if } s_i \neq \alpha,\beta \\ \alpha,\beta & \text{otherwise} \end{cases} \quad \forall i \in V \right\}$$

Is the task $s^* \in \operatorname{argmin}_{s' \in N_{\alpha\beta}(s)} U(s)$ solvable in polynomial time?

Yes, if the restriction of every $u_{ij}: K^2 \to \mathbb{R}$ to $\{\alpha,\beta\}^2 \subset K$ is submodular for every pair $\alpha,\beta \in K$ of labels.

Example 2 Consider the truncated metric for $K \subset \mathbb{Z}$

$$u_{ij}(k,k') = a_{ij} \min(C, |k-k'|), \quad a_{ij} > 0.$$

H is not submodular. H allows α-β swaps but does not allow α-expansions.

B. Algorithms based on equivalent transformations and LP relaxations

- Loopy belief propagation: Apply equivalent transformations, which resemble dynamic programming on trees, iteratively until convergence. Popular, but not well grounded. See next section.

More principled approaches start from LP-relaxations of the discrete optimisation problem

$$U(s) = \sum_{i \in V}' u_i(s_i) + \sum_{ij \in E}' u_{ij}(s_i, s_j) \rightarrow \min_{s \in K^V}$$

A lower bound is given by

$$\sum_i' \min_{k \in K} u_i(k) + \sum_{ij \in E}' \min_{k,k' \in K} u_{ij}(k,k') \leq \min_{s \in K^V} U(s)$$

Combine it with equivalent transformations and maximise the lower bound w.r.t. them

$$B(\varphi) = \sum_i' \min_{k \in K} \Big[u_i(k) - \sum_{j \in N_i}' \varphi_{ij}(k) \Big] +$$

$$+ \sum_{ij \in E}' \min_{kk'} \Big[\varphi_{ij}(k) + u_{ij}(k,k') + \varphi_{ji}(k') \Big] \rightarrow \max_{\varphi}$$

This can be expressed as a linear optimisation task by introducing additional variables

$$\sum_{i\in V} c_i + \sum_{ij\in E} c_{ij} \to \max_{c,\psi}$$

s.t. $$c_i + \sum_{j\in N_i} \psi_{ij}(k) \leq u_i(k) \quad \forall i\in V,\ \forall k\in K$$

$$c_{ij} - \psi_{ij}(k) - \psi_{ji}(k') \leq u_{ij}(k,k') \quad \forall ij\in E,\ \forall k,k'\in K$$

It is important to notice, that this LP-task is <u>dual</u> to the following direct relaxation of the discrete optimisation task

$$\sum_{i\in V}\sum_{k\in K} \lambda_i(k) u_i(k) + \sum_{ij\in E}\sum_{k,k'\in K} \lambda_{ij}(k,k') u_{ij}(k,k') \to \min_{\lambda\geq 0}$$

s.t. $$\lambda_i(k) = \sum_{k'} \lambda_{ij}(k,k') \quad \forall ij\in E,\ \forall k\in K$$

$$\sum_{k\in K} \lambda_i(k) = 1 \quad \forall i\in V$$

$$\sum_{k,k'\in K} \lambda_{ij}(k,k') = 1 \quad \forall ij\in E$$

There are several algorithms, which try to solve the primal or dual LP-task (or both simultaneously):

- tree reweighted message passing (Kolmogorov 2006)
- (min,+) - diffusion
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