4. Approximation algorithms for (Min,+) problems

We still consider the task

$$\min_{s\in K^{|V|}} U(s) = \min_{s\in K^{|V|}} \Big[\sum_{i\in V}' u_i(s_i) + \sum_{ij\in E}' u_{ij}(s_i, s_j) \Big]$$

4A Iterated descent

- define a family of „neighbourhoods"

$$N_m(s) \subset K^{|V|}, \qquad m = 1,..,M$$

- repeatedly solve the restricted problem

$$s^{(t+1)} = \operatorname*{argmin}_{s\in N_m(s^{(t)})} U(s)$$

until

$$\operatorname*{argmin}_{s\in N_m(s^{(t)})} U(s) = s^{(t)} \quad \forall\, m = 1,..,M$$

Iterated Conditional Modes (ICM)

- choose simple system of neighbourhoods $N_i(s)$, $i\in V$

$$N_i(s) = \{ s' \in K^{|V|} \mid s'_j = s_j \ \forall j\in V, j\neq i \}$$

- The task $\operatorname*{argmin}_{s'\in N_i(s)} U(s')$ is easy to solve

However, the neighbourhoods $N_i(s)$ are very small and there are no approximation bound guaranties

α-Expansions (Boykov et al., 2001)

For each label $\alpha \in K$ define

$$N_\alpha(s) = \{ s' \in K^{|V|} \mid s'_i = \alpha \text{ if } s'_i \neq s_i,\ \forall i \in V \}$$

Is the task

$$s_* = \underset{s' \in N_\alpha(s)}{\operatorname{argmin}} \ \mathcal{U}(s')$$

solvable in polynomial time? Yes, if

$$u_{ij}(k,k') + u_{ij}(\alpha,\alpha) \leq u_{ij}(k,\alpha) + u_{ij}(\alpha,k') \qquad (*)$$

holds for $\forall\ ij \in E$, $\forall\ k,k' \in K \backslash \alpha$.

Example 1 Consider the Potts model

$$u_{ij}(k,k') = a_{ij}(1 - \delta_{kk'}), \quad a_{ij} > 0$$

It is not submodular if $|K| > 2$. However, it fulfills $(*)$. ∎

Theorem 1 (w/o proof)

Let $\tilde{s}$ be a fixpoint of α-expansions ($\forall \alpha \in K$).
Then $\mathcal{U}(\tilde{s}) \leq 2c \min_s \mathcal{U}(s)$, where c is defined as

$$c = \max_{ij \in E} \left[\frac{\max_{k \neq k'} u_{ij}(k,k')}{\min_{k \neq k'} u_{ij}(k,k')} \right].$$

∎

$\alpha\beta$ - Swaps (Boykov et al. 2001)

Define $N_{\alpha\beta}(s)$ for each pair of labels $\alpha, \beta \in K$

$$N_{\alpha\beta}(s) = \left\{ s' \in K^{|V|} \;\middle|\; s_i' = \begin{cases} s_i & \text{if } s_i \neq \alpha, \beta \\ \alpha, \beta & \text{otherwise} \end{cases} \quad \forall i \in V \right\}$$

Is the task

$$s_* = \operatorname*{argmin}_{s' \in N_{\alpha\beta}(s)} \mathcal{U}(s')$$

solvable in polynomial time? Yes, if the restriction of each $u_{ij} : K^2 \to \mathbb{R}$ to $\{\alpha, \beta\}^2 \subset K$ is submodular for every pair $\alpha, \beta \in K$.

Example 2 Consider the truncated metric for $K \subset \mathbb{Z}$

$$u_{ij}(k, k') = a_{ij} \min(M, |k - k'|), \quad a_{ij} > 0$$

It is not submodular. It allows $\alpha\beta$-swaps (but not allows α-expansion). ∎

B. Algorithms based on equivalent transformations

- loopy belief propagation: popular but not well grounded. See next section

- there are many better grounded algorithms based on LP relaxations and reparametrisations