4: Approximation algorithms for (Min,+)-problems

We still consider the task

$$\min_{s \in K^{|V|}} G(s) = \min_{s \in K^{|V|}} \Big[\sum_i{}' g_i(s_i) + \sum_{ij \in E}{}' g_{ij}(s_i, s_j) \Big]$$

A: Iterated descent

Iteratively improve a current estimate $s^{(t)}$ of the optimal labelling:

- define a family of „neighbourhoods"
  $\mathcal{U}_m(\tilde{s}) \subset K^{|V|}, \quad m = 1, \dots, M$
- repeatedly solve

$$s^{(t+1)} = \operatorname*{argmin}_{s \in \mathcal{U}_m(s^{(t)})} G(s)$$

until

$$\operatorname*{argmin}_{s \in \mathcal{U}_m(s^{(t)})} G(s) = s^{(t)} \;\; \forall\, m = 1, 2, \dots, M$$

<u>Iterated Conditional Modes (ICM)</u>

- choose simple neighbourhoods $\mathcal{U}_i(\tilde{s}),\ i \in V$

$$s \in \mathcal{U}_i(\tilde{s}) \Leftrightarrow s_j = \tilde{s}_j \;\; \forall\, j \in V \setminus i$$

- The task

$$\operatorname*{argmin}_{s \in \mathcal{U}_i(\tilde{s})} G(s)$$

  is easy to solve.

However, the neighbourhoods $\mathcal{U}_i$ are very small and there are no approximation bound guarantees.

α-β Swaps

For each pair $\alpha, \beta \in K$ of labels define $\mathcal{U}_{\alpha\beta}(\tilde{s})$ by

$$s \in \mathcal{U}_{\alpha\beta}(\tilde{s}) \iff s_i = \begin{cases} \tilde{s}_i & \text{if } \tilde{s}_i \neq \alpha, \beta \\ \alpha, \beta & \text{else} \end{cases}$$

Is the task

$$s^* = \operatorname*{argmax}_{s \in \mathcal{U}_{\alpha\beta}(\tilde{s})} G(s)$$

solvable in polynomial time? Yes, if the restriction of g_{ij} to $\{\alpha, \beta\}^2 \subset K^2$ is submodular $\forall$ $ij \in E$ and $\forall \alpha, \beta \in K$

<u>Example 1</u> Consider the Potts model

$$g_{ij}(k, k') = a_{ij} [1 - \delta_{kk'}]$$

where $a_{ij} > 0$. It is not submodular if $|K| > 2$. However, its restriction to $\{\alpha, \beta\}^2 \subset K^2$ is submodular for any pair of labels $\alpha, \beta \in K$

α - Expansions

For each label $\alpha \in K$ define $\mathcal{U}_\alpha(\tilde{s})$ by

$$s \in \mathcal{U}_\alpha(\tilde{s}) \Leftrightarrow s_i = \alpha \text{ if } s_i \neq \tilde{s}_i$$

Is the task

$$s^* = \operatorname*{argmax}_{s \in \mathcal{U}_\alpha(\tilde{s})} G(s)$$

solvable in polynomial time? Yes, if

$$g_{ij}(k,k') + g_{ij}(\alpha,\alpha) \leq g_{ij}(k,\alpha) + g_{ij}(\alpha,k')$$

$\forall\, ij \in E$ and $k, k' \in K \setminus \alpha$

B. Algorithms based on equivalent transformations

see Remark in next section.