

II Graphical models on general graphs

11. Markov random fields, Gibbs random fields

A. Model definitions

- (V, E) - undirected graph (or hypergraph)
 outer boundary ∂M of a set $M \subset V$
 $\partial M = \{i \in V \setminus M \mid \exists j \in M \text{ s.t. } \{i, j\} \in E\}$, $N_i = \partial\{i\}$
- A field (collection) of K -valued random variables s_i
 indexed by graph nodes $i \in V$, $S = \{s_i \mid i \in V\}$, $S_M = \{s_i \mid i \in M\}$
- A joint p.d. defined on K^V

Definition 1 A joint p.d. defined on K^V is a Markov random field w.r.t. the graph structure (V, E) if

$$P(S_M, S_{\tilde{M}} \mid S_{\partial M}) = P(S_M \mid S_{\partial M}) P(S_{\tilde{M}} \mid S_{\partial M})$$

hold for each $M \subset V$ and $\tilde{M} = V \setminus (M \cup \partial M)$. ■

It follows that an MRF has the property

$$P(S_M \mid S_{\partial M}, S_{\tilde{M}}) = P(S_M \mid S_{\partial M}).$$

Definition 2 Let (V, C) be a hypergraph. A joint p.d. defined on K^V is a Gibbs random field w.r.t. the hypergraph structure (V, C) if it factorises into a product of functions depending on s_c , $c \in C$, i.e.

$$P(S) = \prod_{c \in C} f_c(s_c).$$

If $p(S)$ is strictly positive, it can be written as

$$p(s) = \frac{1}{Z} \exp \sum_{c \in \mathcal{C}} u_c(s_c)$$

where $u_c: K^c \rightarrow \mathbb{R}$ are arbitrary functions (aka Gibbs potentials) and Z is a normalising constant. \blacksquare

Theorem 1 (Hammersley Clifford, 1971)

Let (V, E) be a graph and let \mathcal{C} denote the system of its cliques. Every strictly positive MRF w.r.t. (V, E) is also a GRF w.r.t. (V, \mathcal{C}) and vice versa. \blacksquare

Remark 1

Def. 2 does not require that \mathcal{C} has to be the system of cliques for some graph. Consider e.g. a complete graph (V, E) . Any p.d. on K^V is an MRF w.r.t. (V, E) ! However, the class of GRFs w.r.t. $\mathcal{C} = V \cup E$ is a proper subclass of p.d.s on K^V .

Example 1 (segmentation of images)

- $x: V \rightarrow F$ an (colour) image defined on $V \subset \mathbb{Z}^2$
- $s: V \rightarrow K$ a segmentation with labels from K

A model for a joint p.d. $p(x, s) = p(x|s)p(s)$:

(1) $p(s)$ is a GRF on the lattice (V, E)

$$p(s) = \frac{1}{Z} \exp \sum_{ij \in E} u(s_i, s_j)$$

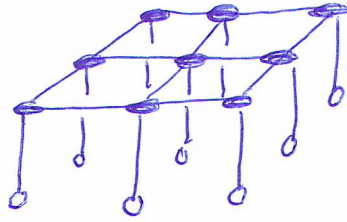
a simple variant is $u(k, k') = \alpha \delta_{kk'}$, $\alpha > 0$
(Potts model)

(2) $p(x|S)$ is a conditionally independent appearance model

$$p(x|S) = \prod_{i \in V} p(x_i | S_i)$$

where $p(x_i | S_i)$ are e.g. (mixtures of) Gaussians

The model is a GRF on the graph



B. Equivalent transformations for GRFs

Consider a GRF w.r.t. $\mathcal{C} = V \cup E$ defined on K^V

$$p(S) = \frac{1}{Z(u)} \exp \left[\sum_{i \in V} u_i(S_i) + \sum_{ij \in E} u_{ij}(S_i, S_j) \right]$$

Q: Are the functions u_i, u_{ij} uniquely defined by $p(S)$?

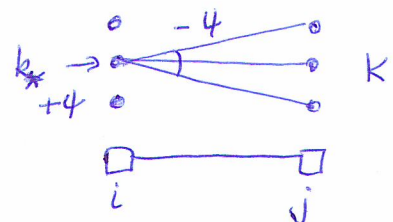
(1) Clearly, adding a constant to any of them will not change p

$$\tilde{u}_i(S_i) = u_i(S_i) + c$$

(2) Consider a node $i \in V$, an edge $\{i, j\} \in E$ and fix a state $k_* \in K$. Change

$$u_i(k_*) \rightarrow u_i(k_*) + \psi$$

$$u_{ij}(k_*, k) \rightarrow u_{ij}(k_*, k) - \psi \quad \forall k \in K$$



This will not change $p(S)$!

(3) "Elementary" transformations as in (2) can be applied for any triple $i \in V$, $\{i, j\} \in E$, $k \in K$

$$\tilde{u}_i(s_i) = u_i(s_i) - \sum_{j \in N_i} \psi_{ij}(s_i)$$

$$\tilde{u}_{ij}(s_i, s_j) = \psi_{ij}(s_i) + u_{ij}(s_i, s_j) + \psi_{ji}(s_j)$$

Theorem 2 (w/o proof)

The equivalent transformations (aka reparametrisations) given above describe all possible equivalent transformations. □