Gaussian derivatives

UCU Winter School 2017

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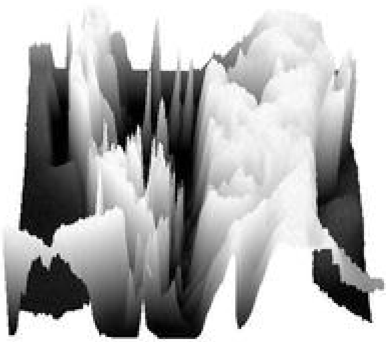
[1] Images taken from Noah Snavely's and Robert Collins's course notes

Definition

An *image* (grayscale) is a function, I, from $\mathbb{R}^2$ to $\mathbb{R}$ such that $I(x, y)$ gives the intensity at position (x, y).



Definition

A *digital image* (grayscale) is a sampled and quantized version of I. The discrete values are indexed as $I[x, y]$.

Why do we need image derivatives?

Image derivatives will be used to construct discrete operators that detect salient differential geometry in the scene.

Some desiderata

- sparse representation of the image
- repeatability
- salient features
- invariance to photometric and geometric transforms of the image

Examples include

- Harris corners
- Hessian Affine (blobs)
- Maximally Stable Extremal Regions

Finite forward difference

- Taylor series expansion

$$I(x+h) = I(x)+hI_x(x)+\frac{1}{2}h^2 I_x(xx)+\frac{1}{3!}h^3 I_{xxx}(x)+\mathcal{O}(h^4) \implies$$

$$\frac{I(x+h)-I(x)}{h} = I_x(x)+\mathcal{O}(h) \implies$$

$$I_x[x] \approx \frac{I[x+h]-I[x]}{h}$$

- Template

-1	1

Finite backward difference

- Taylor series expansion

$$I(x-h) = I(x) - hI_x(x) + \frac{1}{2}h^2 I_{xx}(x) - \frac{1}{3!}h^3 I_{xxx}(x) + \mathcal{O}(h^4) \implies$$

$$\frac{I(x) - I(x-h)}{h} = I_x(x) + \mathcal{O}(h) \implies$$

$$I_x[x] \approx \frac{I[x] - I[x-h]}{h}$$

- Template

-1	1

Central difference

- Taylor series expansion

$$I(x+h) = I(x) + hI_x(x) + \frac{1}{2}h^2 I_{xx}(x) + \frac{1}{3!}h^3 I_{xxx}(x) + \mathcal{O}(h^4)$$
$$I(x-h) = I(x) - hI_x(x) + \frac{1}{2}h^2 I_{xx}(x) - \frac{1}{3!}h^3 I_{xxx}(x) + \mathcal{O}(h^4) \implies$$

$$I(x+h) - I(x-h) = I(x) + 2hI_x(x) + \frac{2}{3!}h^3 I_{xxx}(x) \implies$$

$$\frac{I(x+h) - I(x-h)}{2h} = I_x(x) + \mathcal{O}(h^2) \implies$$

$$I_x[x] \approx \frac{I[x+h] - I[x-h]}{2h}$$

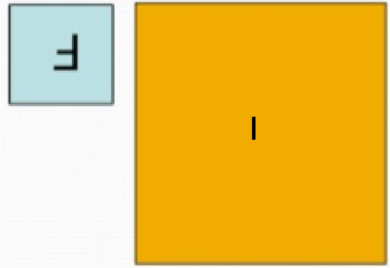
- Template

$-\frac{1}{2}$	0	$\frac{1}{2}$

Convolution in spatial domain

$$(I * f)[x, y] = \sum_{i=-k}^{k} \sum_{j=-k}^{k} I[i, j] * f[x - i][y - j]$$

- convolution is equivalent to flipping the filter in both dimensions and correlating
- same result for symmetric kernels
- many libraries conflate convolution and correlation
- so why NOT just use cross-correlation?



Applying derivative to an image

- row i forward difference

```
for (j = jstart; j <= jend; j++)
   h[j]= I[j+1]-I[j];
```

let $f[0] = 1$ and $f[1] = -1$, then

- row i Forward difference with mask $f[0] = 1$ and $f[1] = -1$.

```
for (j = jstart; j <= jend; j++)
   h[j]=f[0]*I[j+1]+f[1]*I[j];
```

- row i with arbitrary derivative defined by mask f, e.g, $f[-1] = \frac{1}{2}, f[0] = 0, f[1] = -\frac{1}{2}$

```
for (j = jstart; j <= jend; j++)
{
   h[j] = 0;
   for (b = bstart; b <= bend; b++)
      h[j] += I[b]*f[j-b];
}
```

Image derivatives are convolutions

- row i with arbitrary derivative defined by mask f.

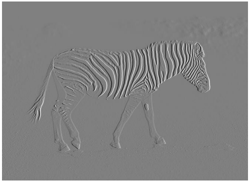
```
for (i = istart; i <= iend; i++)
  for (j = jstart; j <= jend; j++)
  {
    h[i][j] = 0;
    for (a = astart; a <= aend; a++)
      for (b = bstart; b <= bend; b++)
        h[i][j] += I[a][b]*f[i-a][j-b];
  }
```

- Loops over a, b is discrete convolution at i, j.

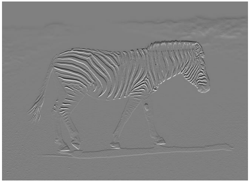
$$h[i,j] := (I * f)[i,j] \equiv \sum_a \sum_b I[a,b] f[i-a, j-b]$$

Image Gradient

I



$\frac{\partial I}{\partial x}$



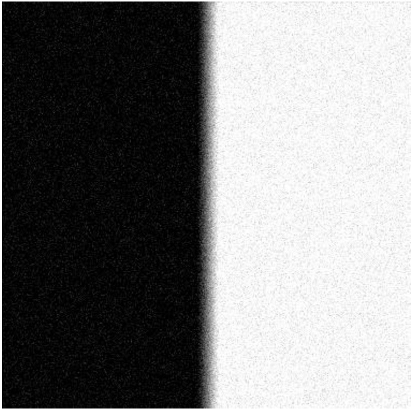
$\frac{\partial I}{\partial y}$

$$\|\nabla I\| = \sqrt{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}$$

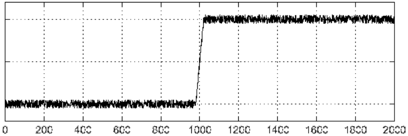
Artificially added white noise

$\sim \mathcal{N}(\mu = 0, \sigma = 16)$

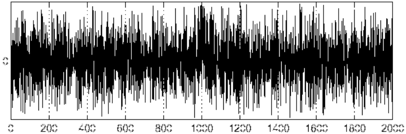
Effect of white noise image derivatives



Noisy image



$$I[x, y = j]$$



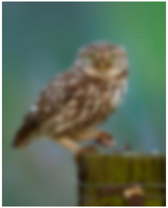
$$\frac{d}{dx} I[x, y = j]$$

Why do we use the Gaussian as a low-pass filter?

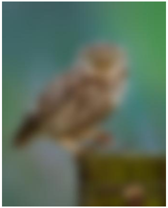
- white noise exists at all frequencies
- corners and edges are represented by high frequencies
- need to remove noise but maintain details
- Fourier transform of Gaussian is Gaussian: Gaussian does not have a sharp cutoff at some pass band frequency and does not oscillate.
- weighted averaging is spatial blurring is low-pass filtering.
- we will blur in the spatial domain with Gaussian

$\sigma = 0$

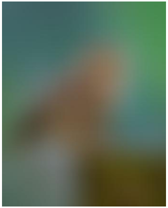
$\sigma = 1$



$\sigma = 5$



$\sigma = 10$



$\sigma = 30$

Box-filter vs Gaussian

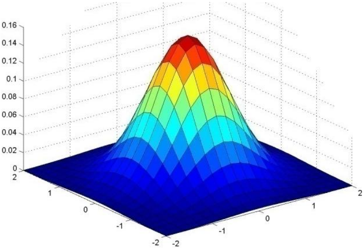
original

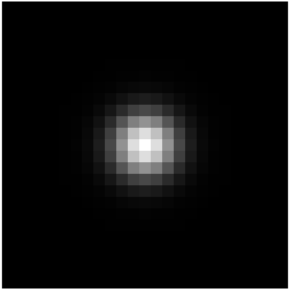
box

Gaussian

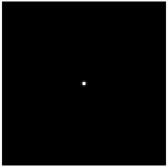
- Box-filter garbles high-frequency signal while removing noise. (not doing *good* things in the frequency domain)

Gaussian kernel

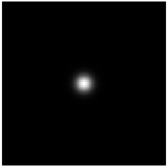




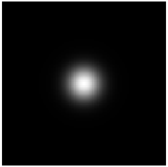
$$g_\sigma[x,y] = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}}$$



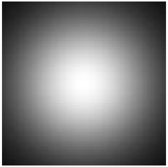
$\sigma = 1$



$\sigma = 5$

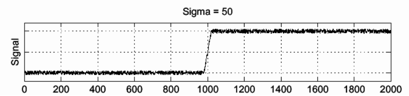


$\sigma = 10$

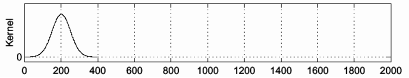


$\sigma = 30$

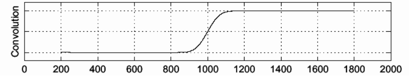
Gaussian smoothing



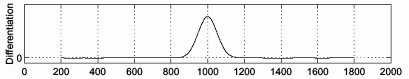
f



g



$I * g$



$\frac{d}{dx}(I * g)$

Separability of Gaussian kernels

Definition

A 2D kernel g is called separable if it can be broken down into the convolution of two kernels: $g = g^{(1)} * g^{(2)}$.

$$
\begin{aligned}
g_\sigma[x, y] &= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \\
&= g_\sigma^{(1)}[x] \cdot g_\sigma^{(2)}[y]
\end{aligned}
$$

and

$$
(I * g_\sigma)[x, y] = \sum_i \sum_j g_\sigma[x - i, x - j] I[i, j] = \dots
$$

$$
\sum_i \sum_j g_\sigma^{(1)}[x-i] g_\sigma^{(2)}[x-j] I[i, j] = \sum_i g_\sigma^{(1)}[x-i] \sum_j g_\sigma^{(2)}[x-j] I[i, j] = \dots
$$

$$
(g_\sigma^{(1)} * (g_\sigma^{(2)} * I))[x, y]
$$

Complexity of convolution in spatial domain

What are the number of operations and complexity of *kxk*-dimension kernel on and *mxn*-dimension image for a

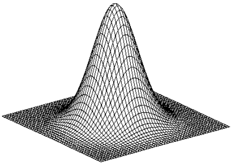
- non-separable kernel: $k^2 \cdot m \cdot n$ operations, complexity $\mathcal{O}(k^2)$
- separable kernel: $2 \cdot k \cdot m \cdot n$ operations, complexity $\mathcal{O}(k)$

It pays to take advantage of separability!

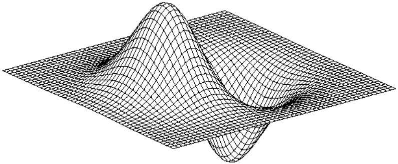
Important Gaussian derivative properties

- Image differentiation $\frac{d}{dx}$ is a convolution on image I.
- Smoothing by Gaussian kernel g is a convolution on image I.
- 2D Gaussian kernel is separable $g = g^{(1)} * g^{(2)}$.
- Convolution is
  - commutative $I * g = g * f$
  - associative $(I * g) * h = I * (g * h)$

So $\frac{d}{dx}(I * g) = I * \frac{d}{dx}g = (I * (\frac{d}{dx}g^{(1)})) * g^{(2)}$

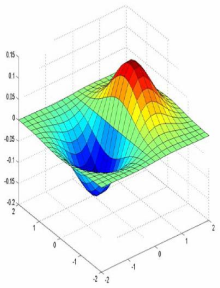


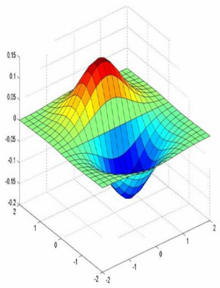
g

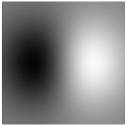


$\frac{d}{dx}g$

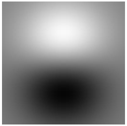
First Derivatives of a Gaussian







$\frac{d}{dx}g$



$\frac{d}{dy}g$

Gaussian derivatives like a boss.

If you want to level up, then you can exploit a recurrence relation of Hermite polynomials to algorithimically construct Gaussian derivatives of any order without convolution or symbolic differentiation.

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1,$$