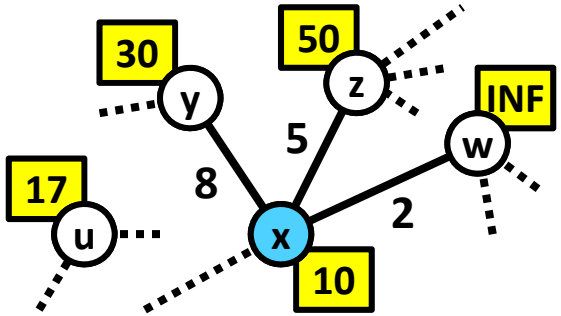
Notes on Minimum Spanning Tree problem

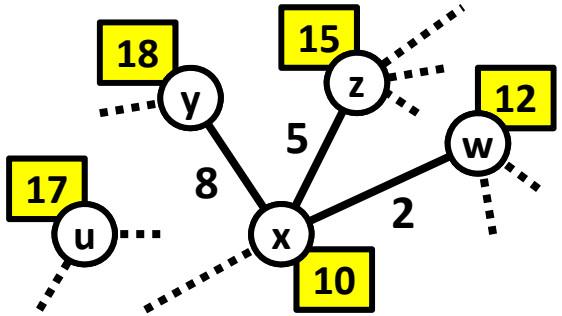
A Priority queue trouble

Usual theoretical and conceptual descriptions of Prim's (Dijkstra's, etc.) algorithm say:
"... store nodes in a priority queue".
Technically, this is nearly impossible to do.
The graph and the nodes are defined separately and stored elsewhere in the memory.
The node has no reference to its position in the queue.
The programmer does not know where is the node in the queue.
So, how to move a node inside the queue according to the algorithm demands?
Standard solution:
Do not move a node, enqueue a "copy of a node", possibly more times.

When a copy of the node with the smallest value (=highest priority) among all its copies appears at the top of the queue it does its job exactly according to the algorithm prescription.
From that moment on, all other copies of the node which are still in the queue become useless and must be ignored. The easiest way to ignore a copy is to check it when it later appears at the top of the queue: If the node is already closed, ignore the copy, pop it and process the next top of the queue. If the node is still open, process it according to the algorithm.



Easy



x	..	...	u	...	y	...	z	...	w	...
10	..	...	17	...	30	...	50	...	INF	...

Difficult

...	..	w	...	z	...	u	...	y	...
...	..	12	...	15	...	17	...	18	...

Easy

q.insert(y);
q.insert(z);
q.insert(w);

..	w	...	z	...	u	...	y	...	y	...	z	...	w	...
..	12	...	15	...	17	...	18	...	30	...	50	...	INF	...

// push the nodes once more to the queue.

The older copies of nodes will get to the top of the queue later than the new copies (which have higher priority) . The older copy gets to the top when the node had been processed and closed earlier. Thus:
If the node at the top of the queue is closed just pop it and do not process it any more.

Example of Prim algorithm implementation using standart library priority queue

```
void MST_Prim(Graph g, int start, final int [] dist, int [] pred ) {
  // allocate structures
  int currnode = start, currdist, neigh;
  int INF = Integer.MAX_VALUE;
  boolean [] closed = new boolean[g.N];

  PriorityQueue <Integer> pq
    = new PriorityQueue<Integer>( g.N,
          new Comparator<Integer>() {
            @Override
            public int compare(Integer n1, Integer n2) {
              if( dist[n1] < dist[n2] ) return -1;
              if( dist[n1] > dist[n2] ) return 1;
              return 0;
          } }  );

  // initialize structures
  pq.add( start );
  for( int i = 0; i < g.N; i++ ) pred[i] = i;
  Arrays.fill( dist, INF );
  Arrays.fill( closed, false );
  dist[start] = 0;
```

Example of Prim algorithm implementation using standart library priority queue

```
  for( int i = 0; i < g.N; i++ ) {
    // take the closest node and skip the closed ones
    while( closed[currnode = pq.poll()] == true );
    // and expand the closest node
    for( int j = 0; j < g.dg[currnode]; j++ ){
      neigh = g.edge[currnode][j];
      if( !closed[neigh] &&
         ( dist[neigh] > g.w[currnode][j]) ) {
           dist[neigh] = g.w[currnode][j];
           pred[neigh] = currnode;
           pq.add(neigh);
      }
     closed[currnode] = true;
    }
  }
}
```

A very small change produces Dijkstra's algorithm:

```
      if( !closed[neigh] &&
         ( dist[neigh] > g.w[currnode][j] + dist[currnode]) ) {
           dist[neigh] = g.w[currnode][j] + dist[currnode];
```

Prim and Dijkstra Algorithms compared

```
void MST_Prim( Graph G, function weight, Node startnode )
  for each u in G.V:  u.dist = INFINITY; u.parent = NIL
  startnode.dist = 0; PriorityQueue Q = G.V

  while !Q.isEmpty()
    u = Extract-Min(Q)
    for each v in G.Adj[u]
      if (v in Q) and v.dist > weight(u,v)
        v.parent = u
        v.dist = weight(u,v)
```

```
void Dijkstra( Graph G, function weight, Node startnode )
  for each u in G.V:  u.dist = INFINITY; u.parent = NIL
  startnode.dist = 0; PriorityQueue Q = G.V

  while !Q.isEmpty()
    u = Extract-Min(Q)
    for each v in G.Adj[u]
      if (v in Q) and v.dist > weight(u,v) + u.dist
        v.parent = u
        v.dist = weight(u,v) + u.dist
```

Disjoint-set data structure alias Union-Find structure

Only 3 operations are needed:

Initialize()

Union(representativeA, representativeB) // merges the two sets represented
// by the given two representatives

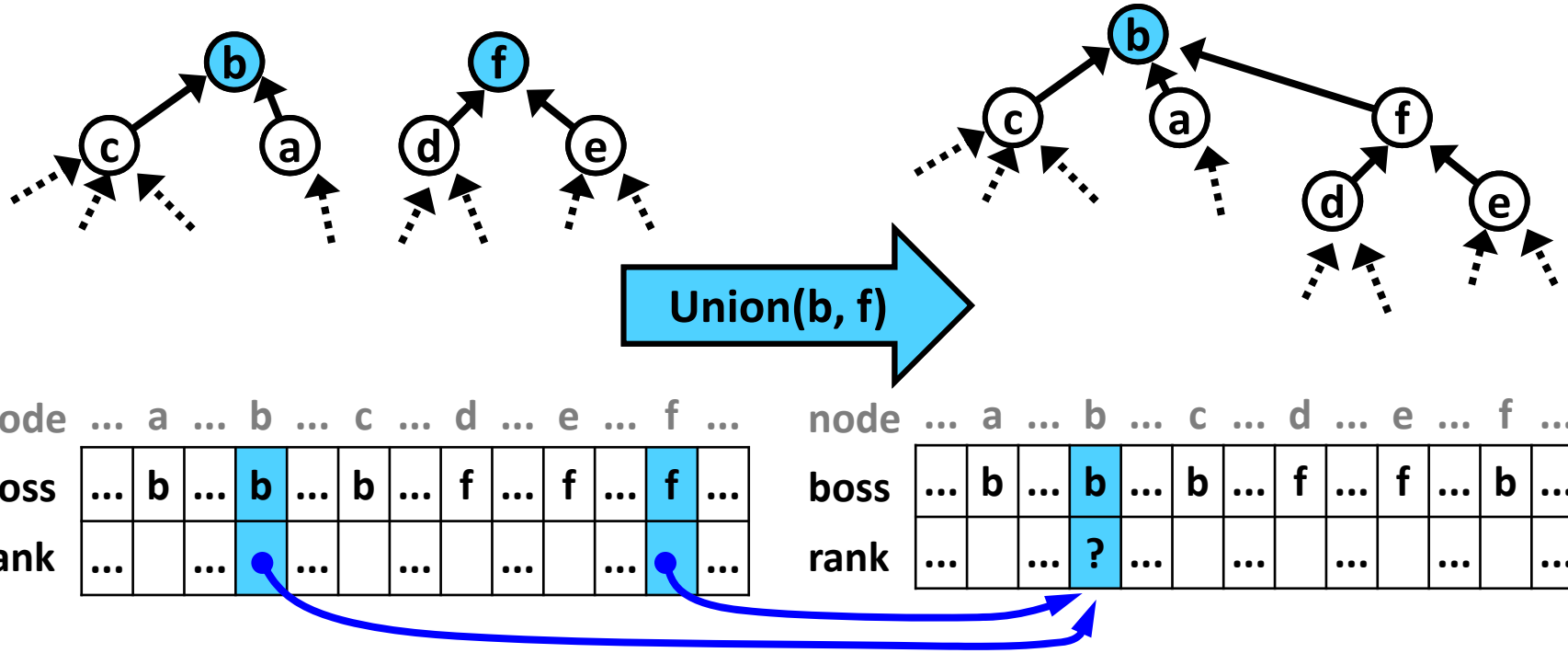
Find(nodeX) // returns a representative of the set to which X belongs

```
int [] boss;
int [] rank;

void UF_init( int n ) {
  boss = new int [n];
  rank = new int [n];
  for( int i = 0; i < n; i++ ) {
    boss[i] = i;   // everybody's their own boss
    rank[i] = 0;   // necessary?
} }
```

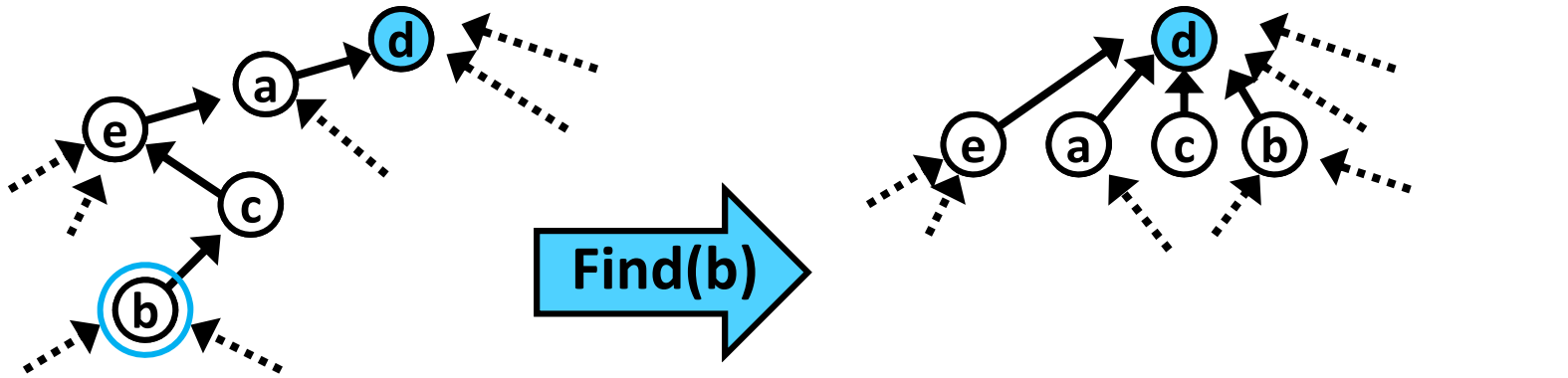
Easy experiment, try it at home:
When the end nodes of the inspected edges are chosen uniformly randomly
then the average depth of a queried node in the Union-Find forest is less than 2.

Union with rank comparison




```
void UF_union( int rootA, int rootB ) {
  if( rank[rootB] > rank[rootA] )
      boss[rootA] = rootB;
  else {
    boss[rootB] = rootA;
    if( rank[rootB] == rank[ rootA ] )   // change rank?
      rank[rootA]++;
} }
```

Find with path compression



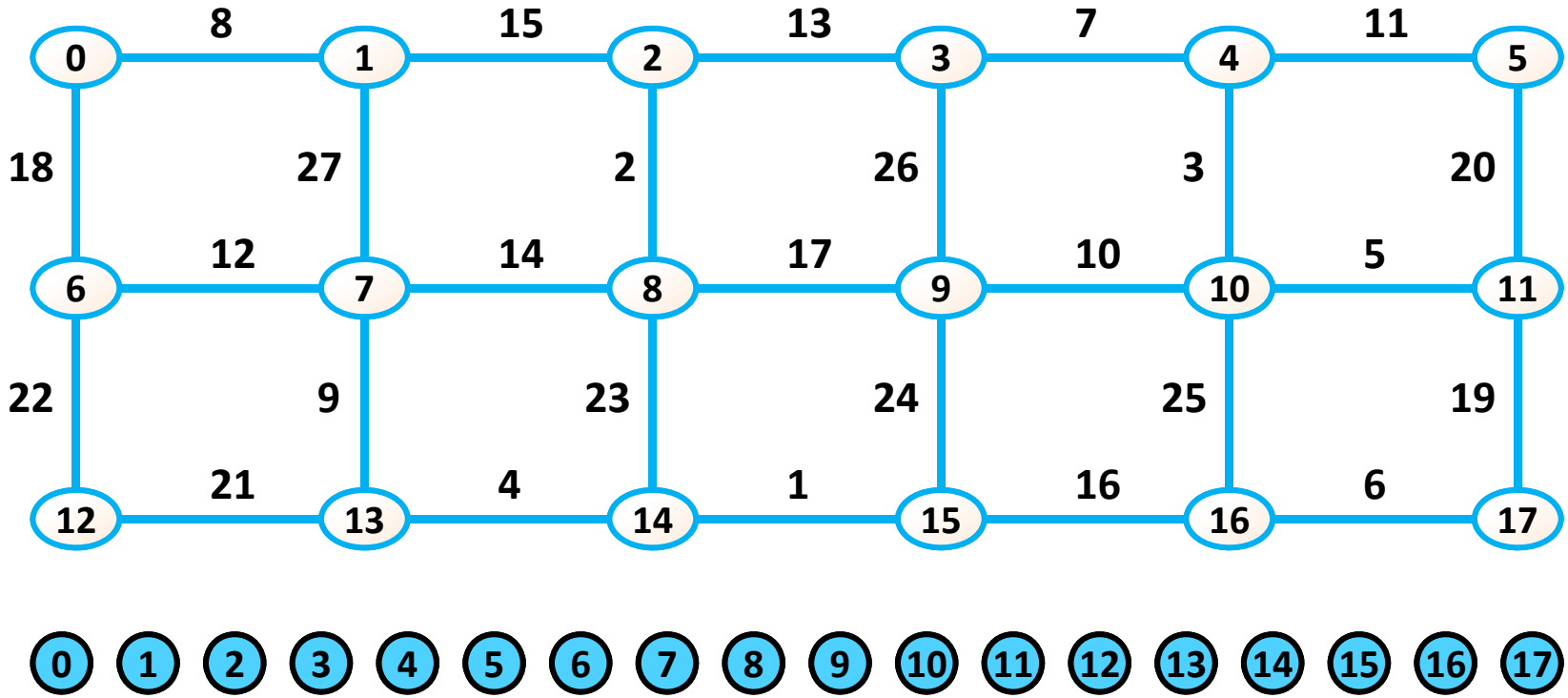
node	…	a	…	b	…	c	…	d	…	e	…
boss	…	d	…	c	…	e	…	d	…	a	…

node	…	a	…	b	…	c	…	d	…	e	…
boss	…	d	…	d	…	d	…	d	…	d	…

```
int UF_find( int a ) {
  int parent = boss[a];
  if( parent != a )
    boss[a] = UF_find( parent ); // path compression
  return boss[a];
}
```

```
int find( int a )  // for C experts
 { return( boss[a] == a ? a : (boss[a] = find(boss[a])) );}
```

Kruskal algorithm and Union-Find scheme example



node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
rank	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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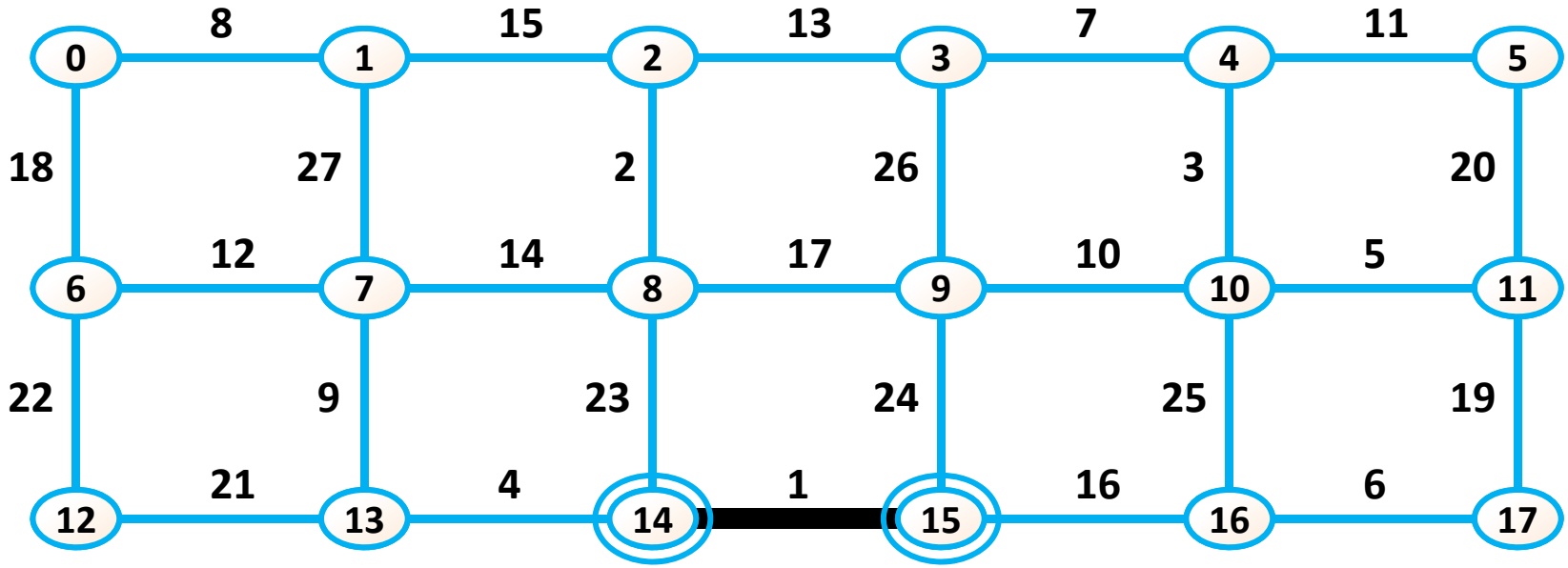

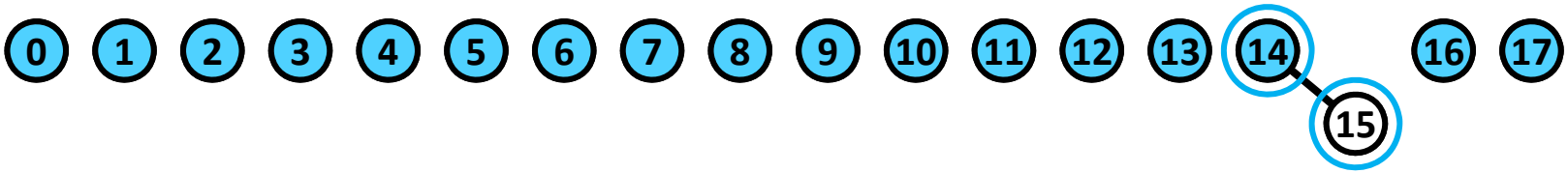

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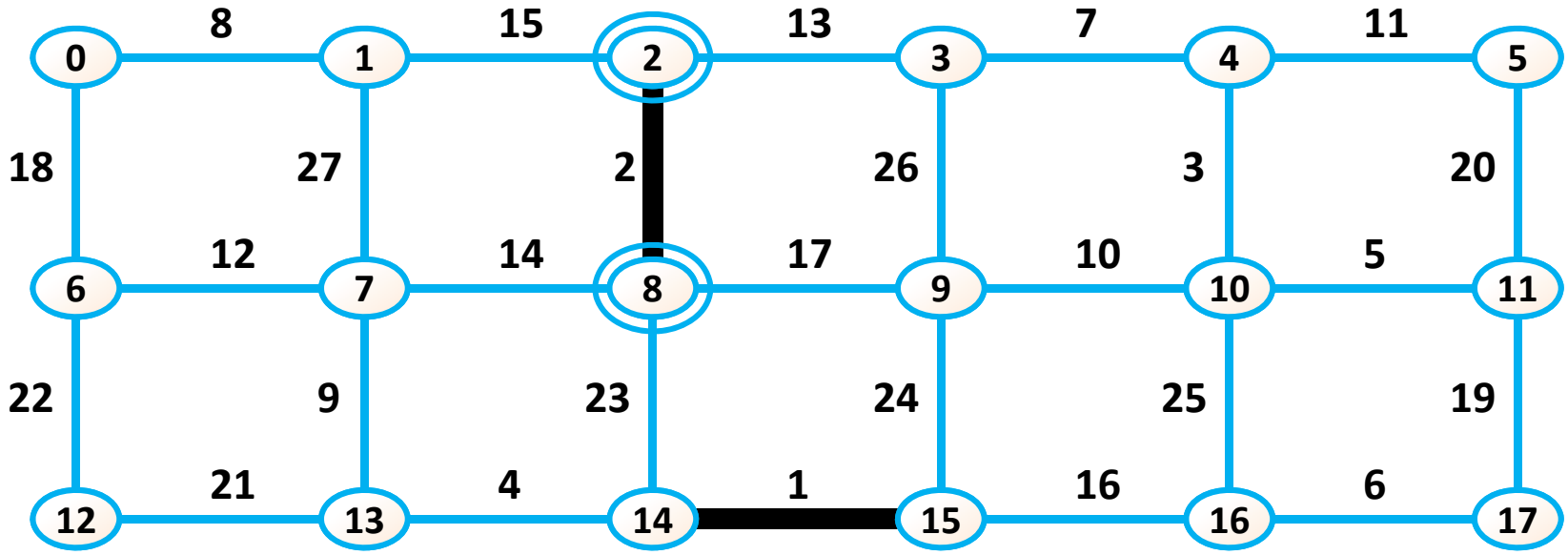

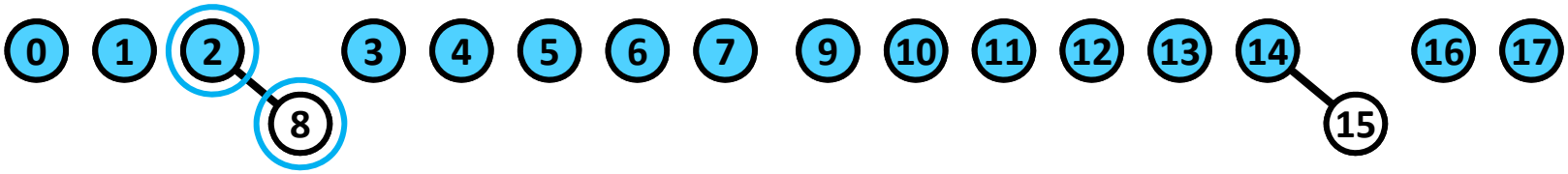

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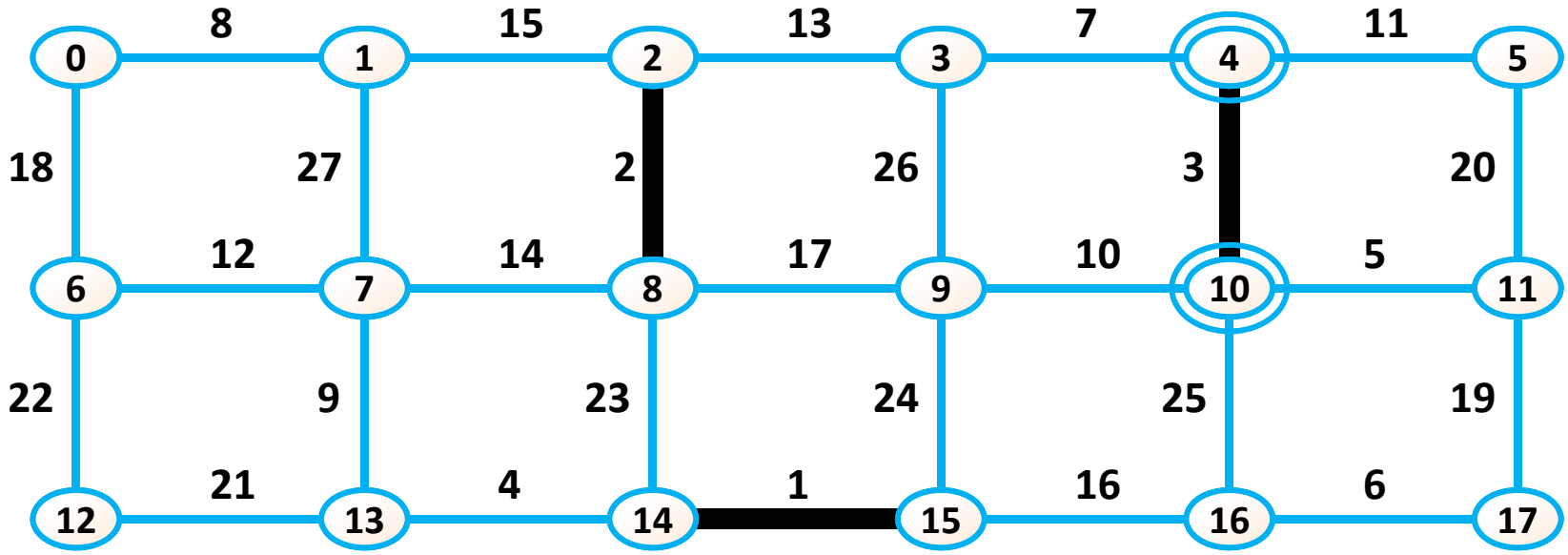

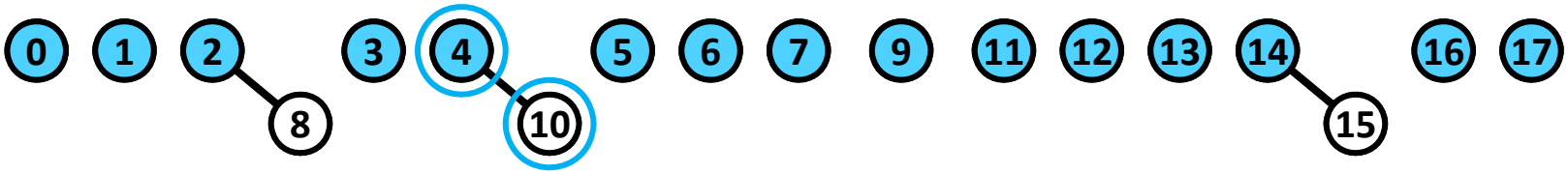

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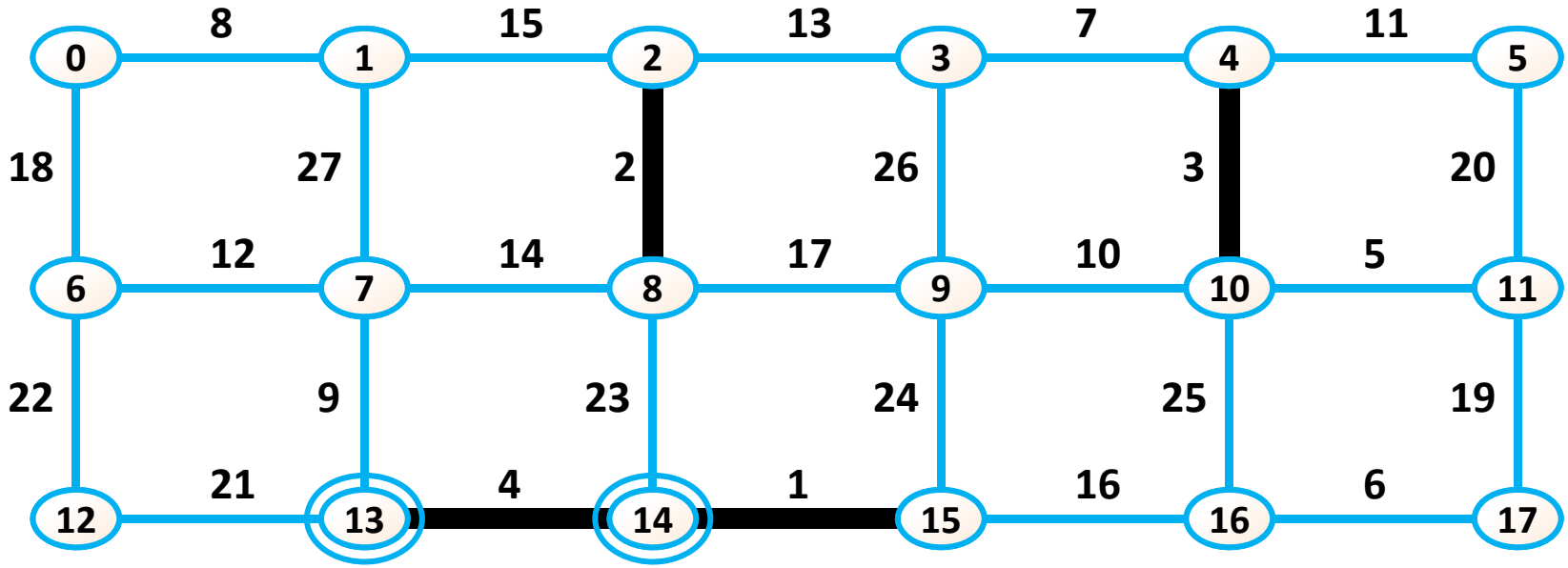

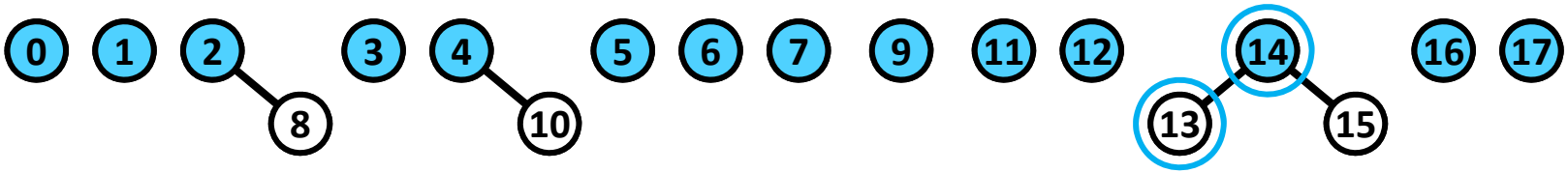

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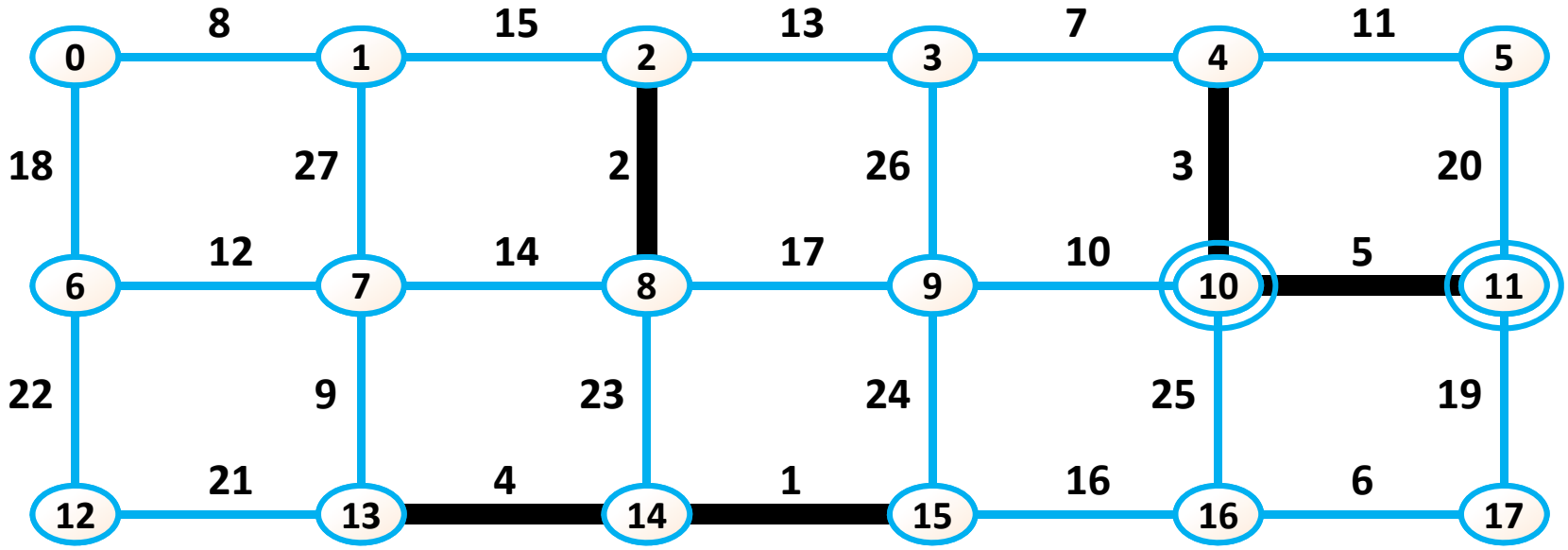

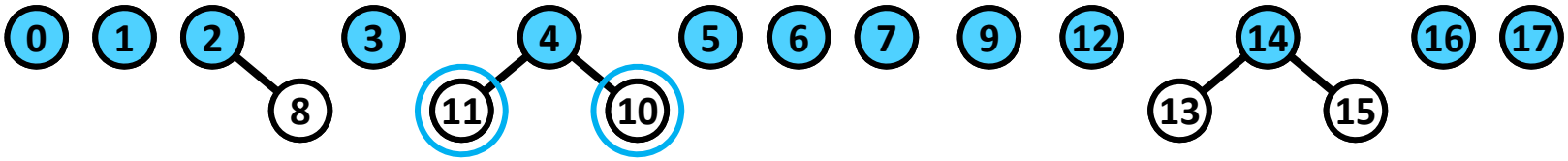

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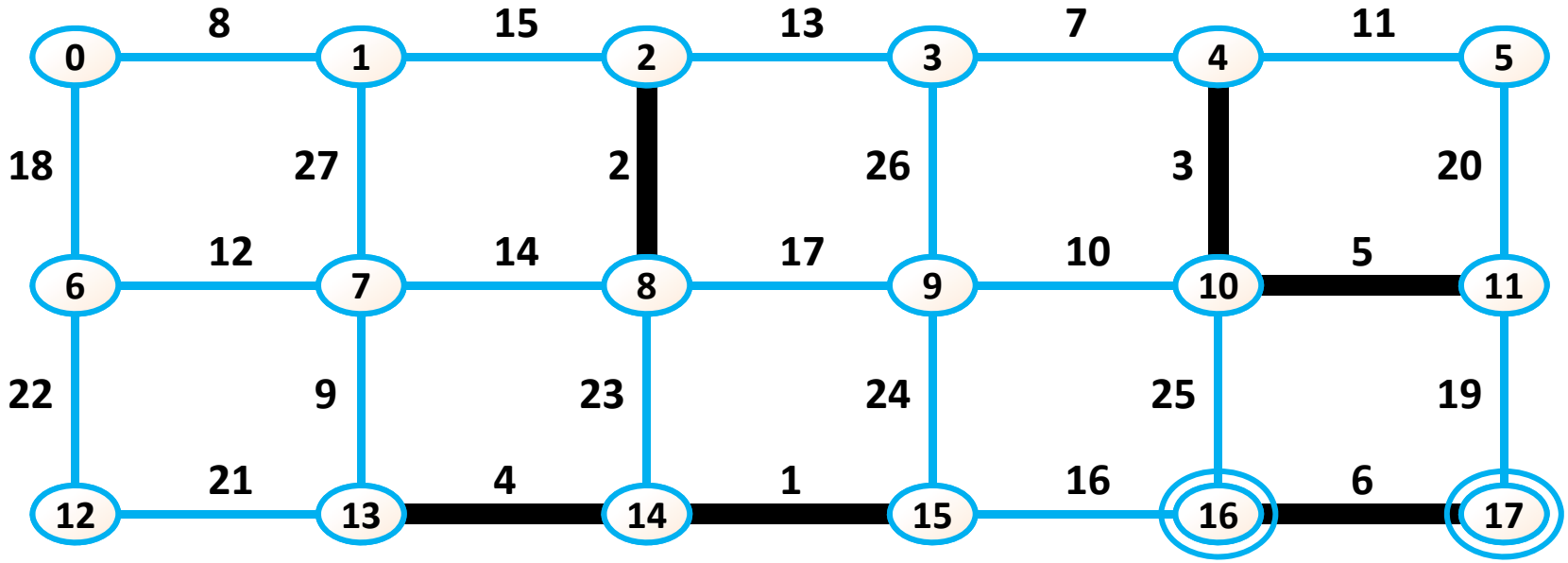

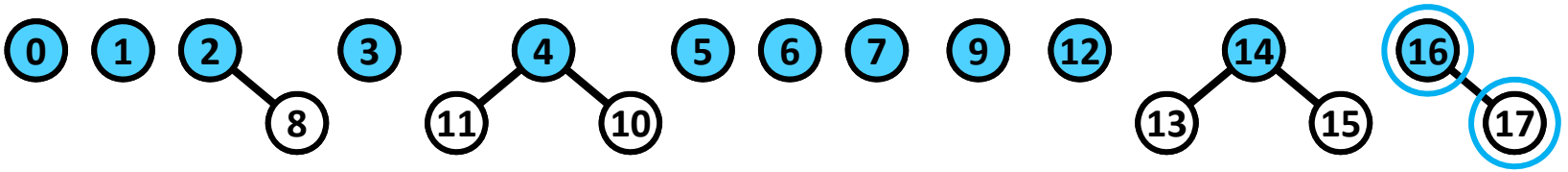

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rank	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

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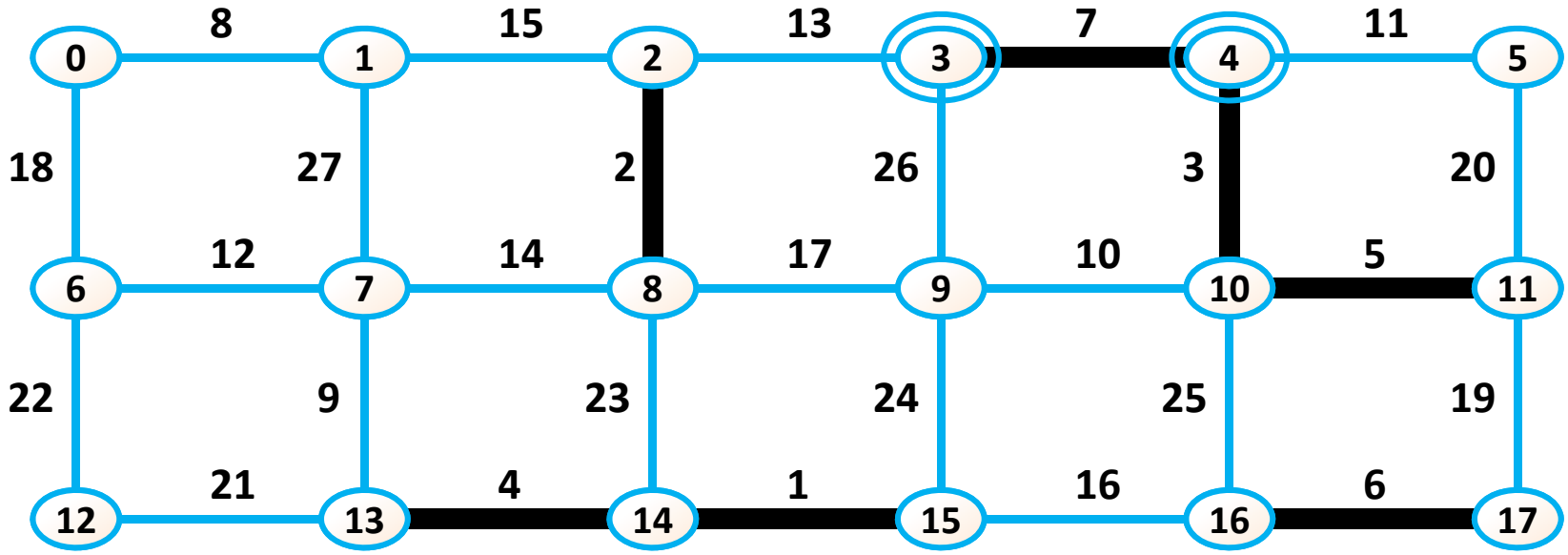

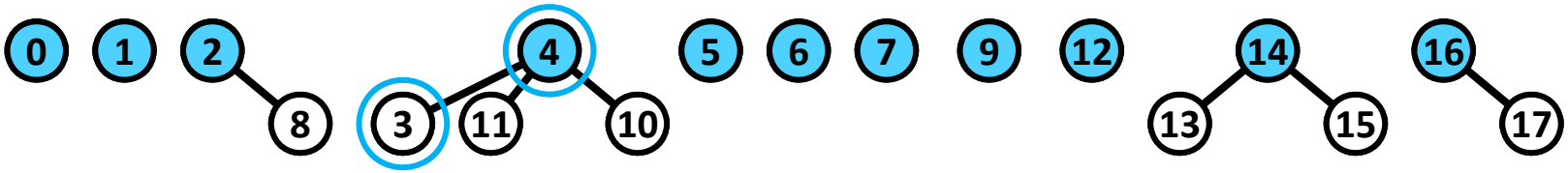

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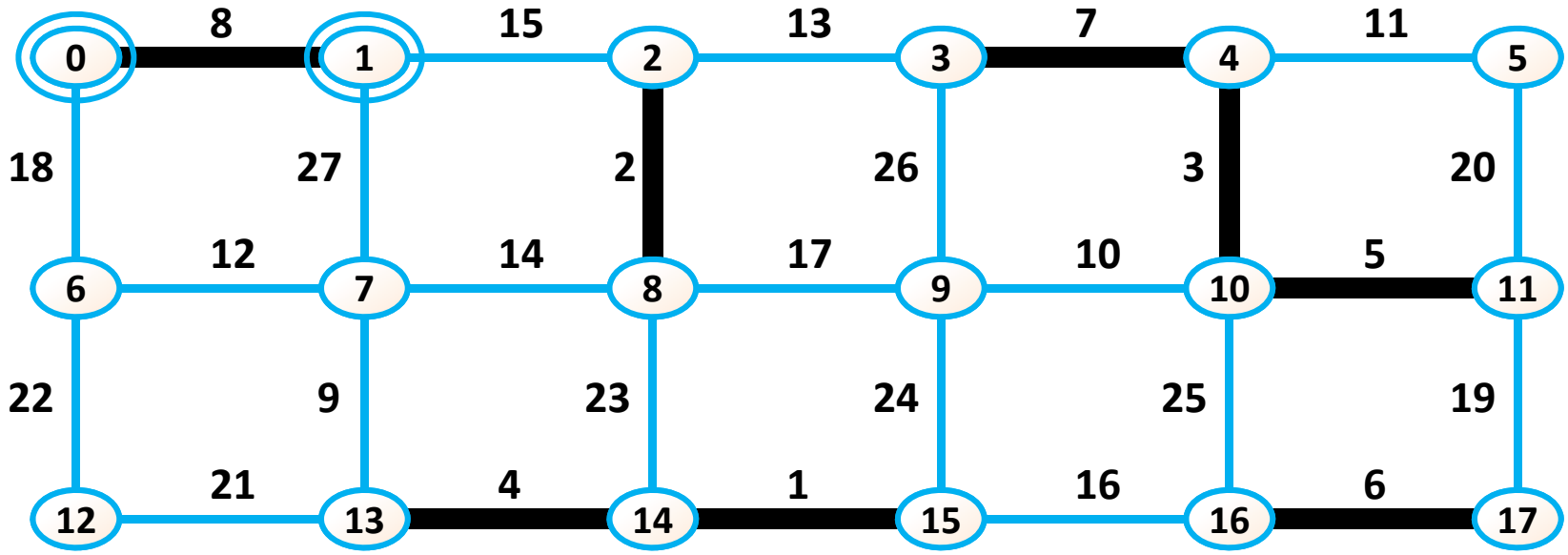

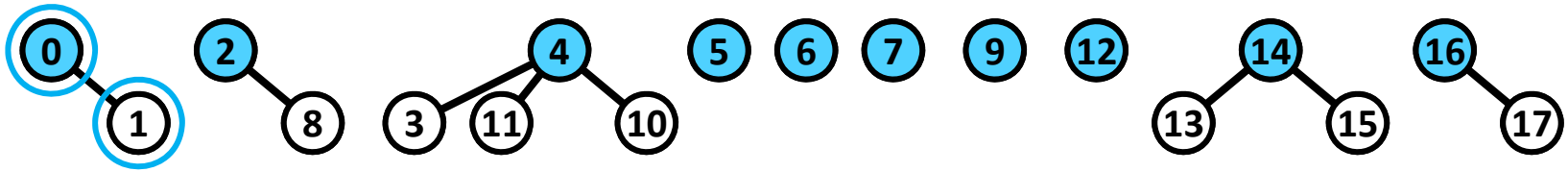

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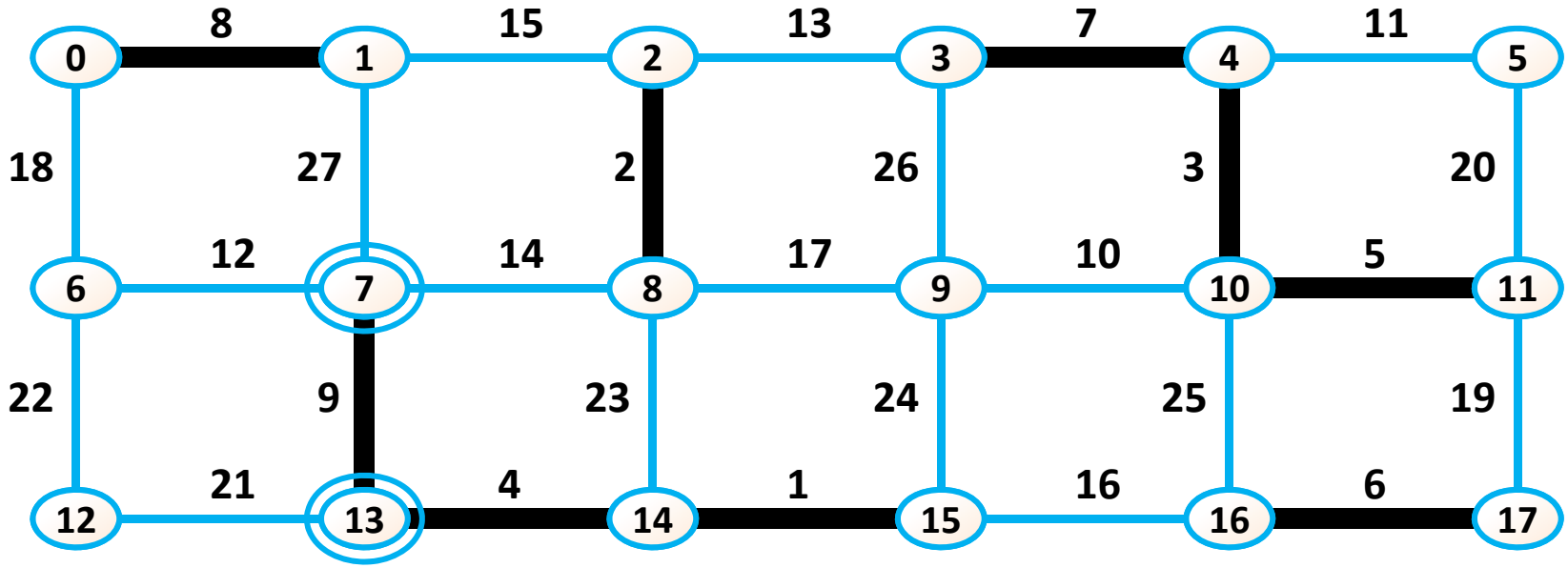
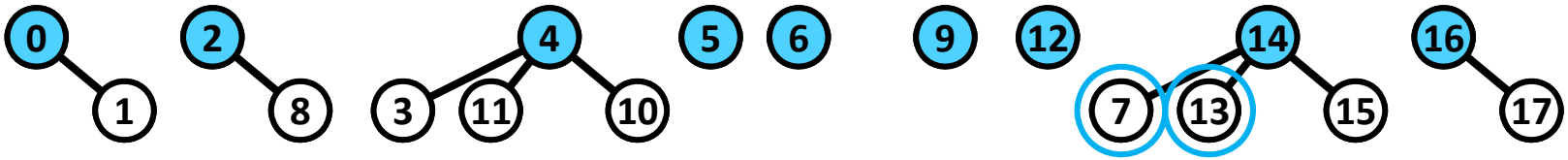
Kruskal algorithm and Union-Find scheme example





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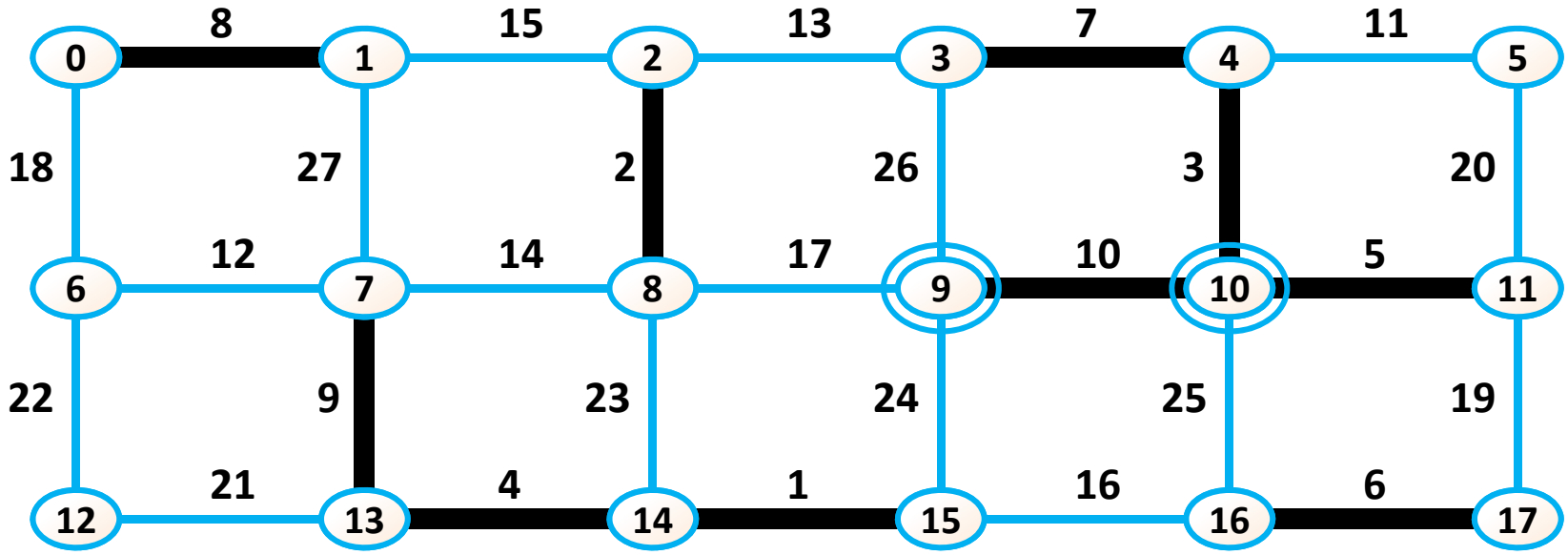

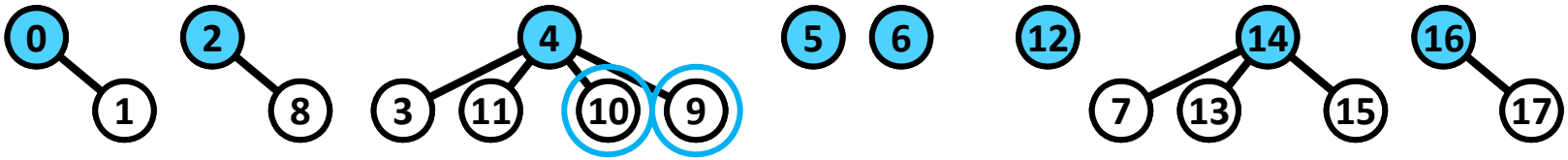

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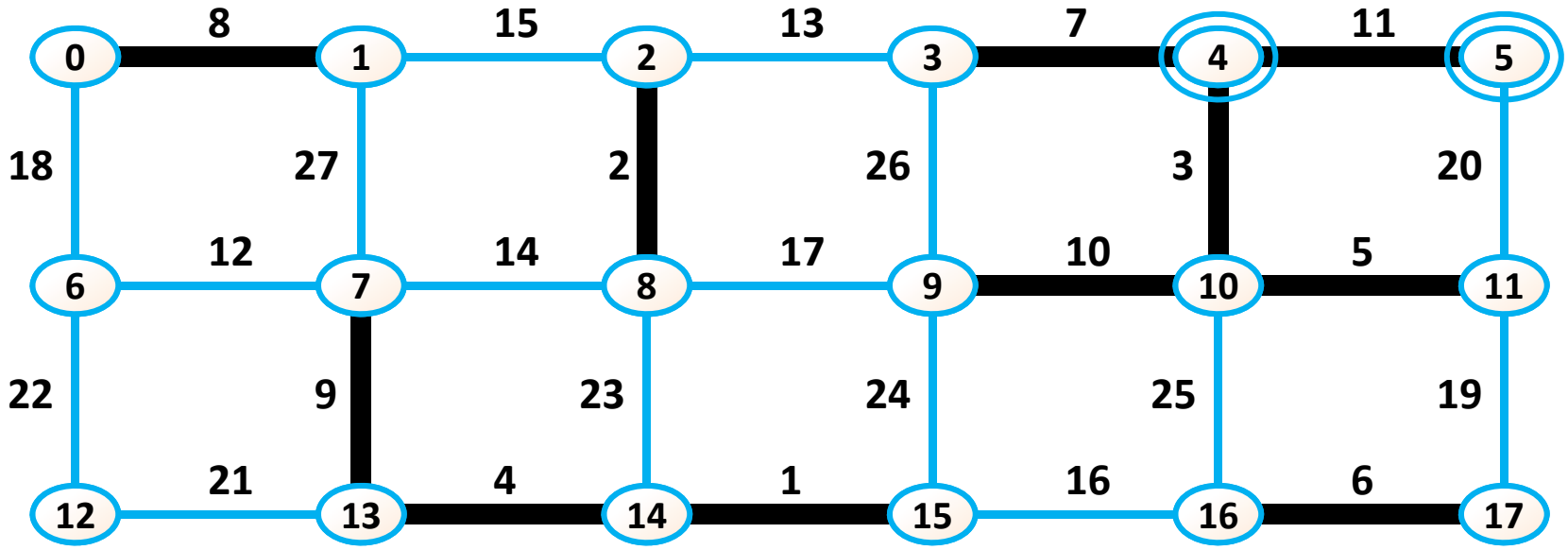

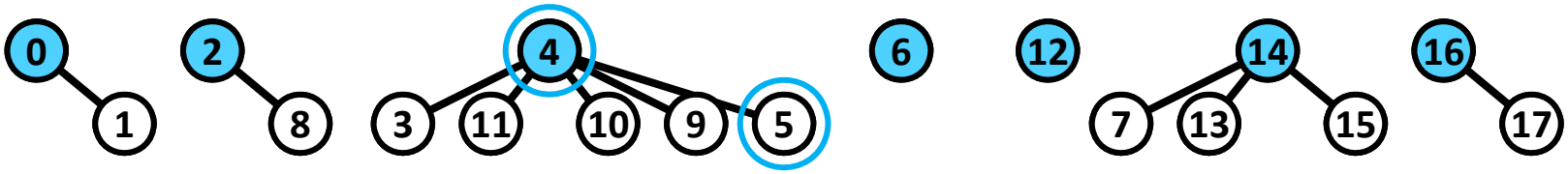

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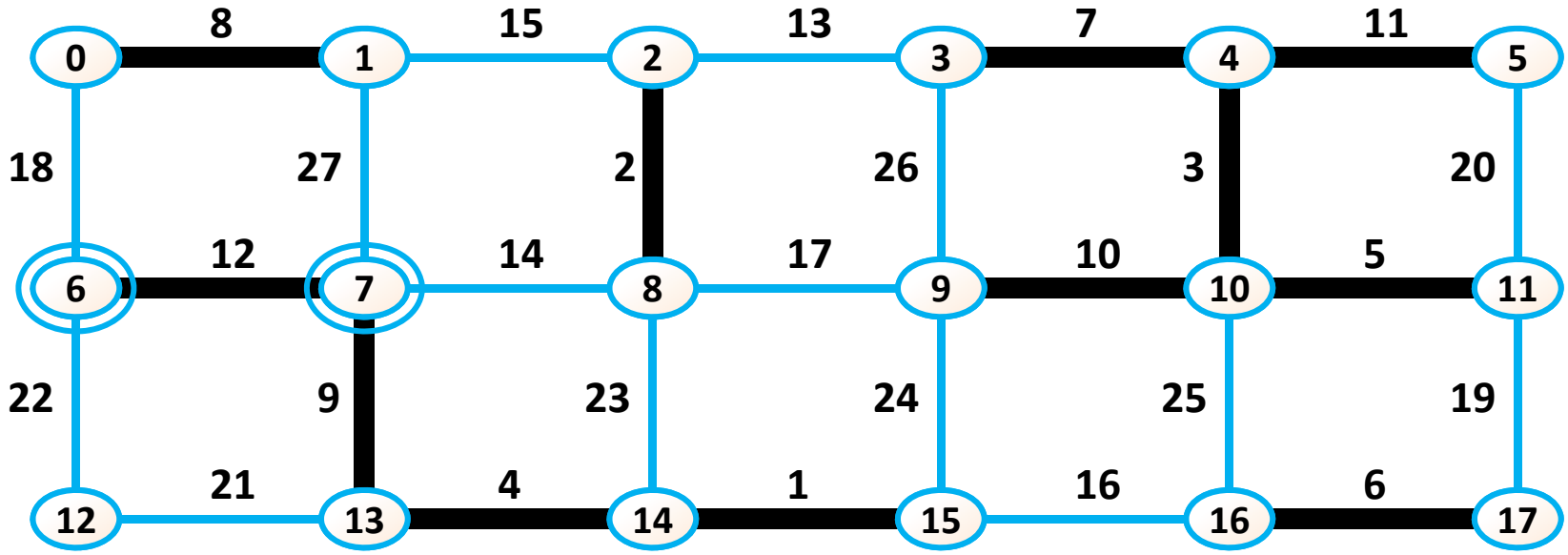
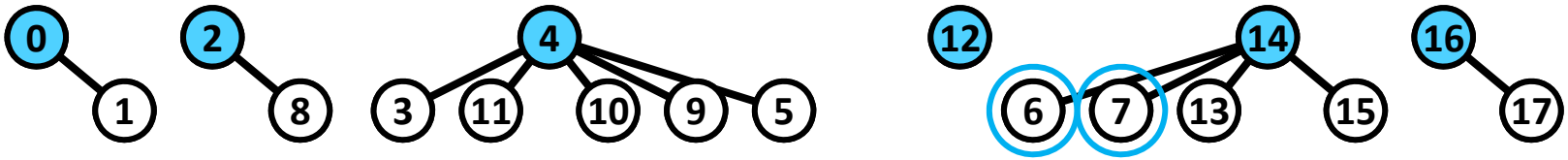
Kruskal algorithm and Union-Find scheme example





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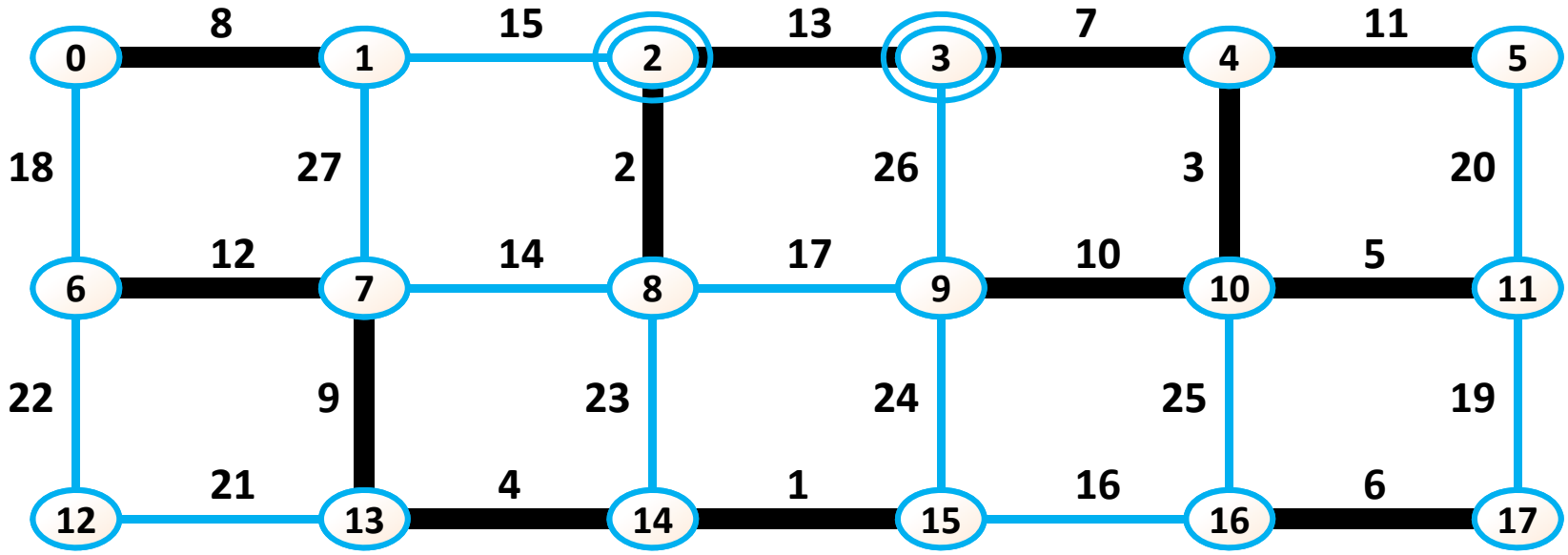

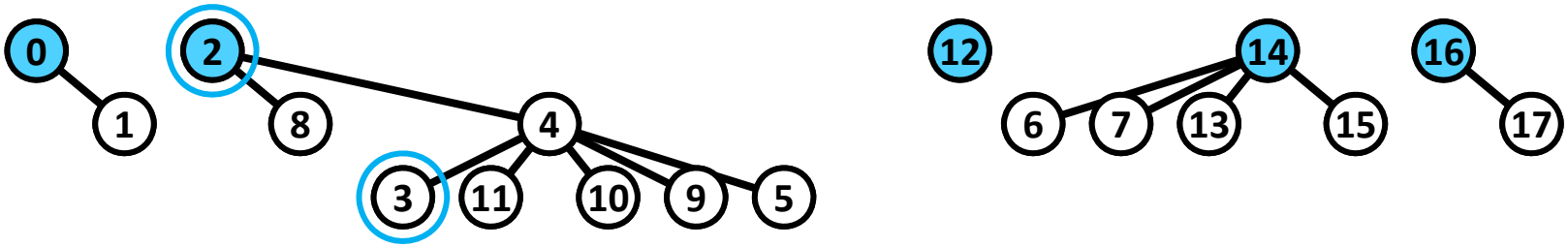

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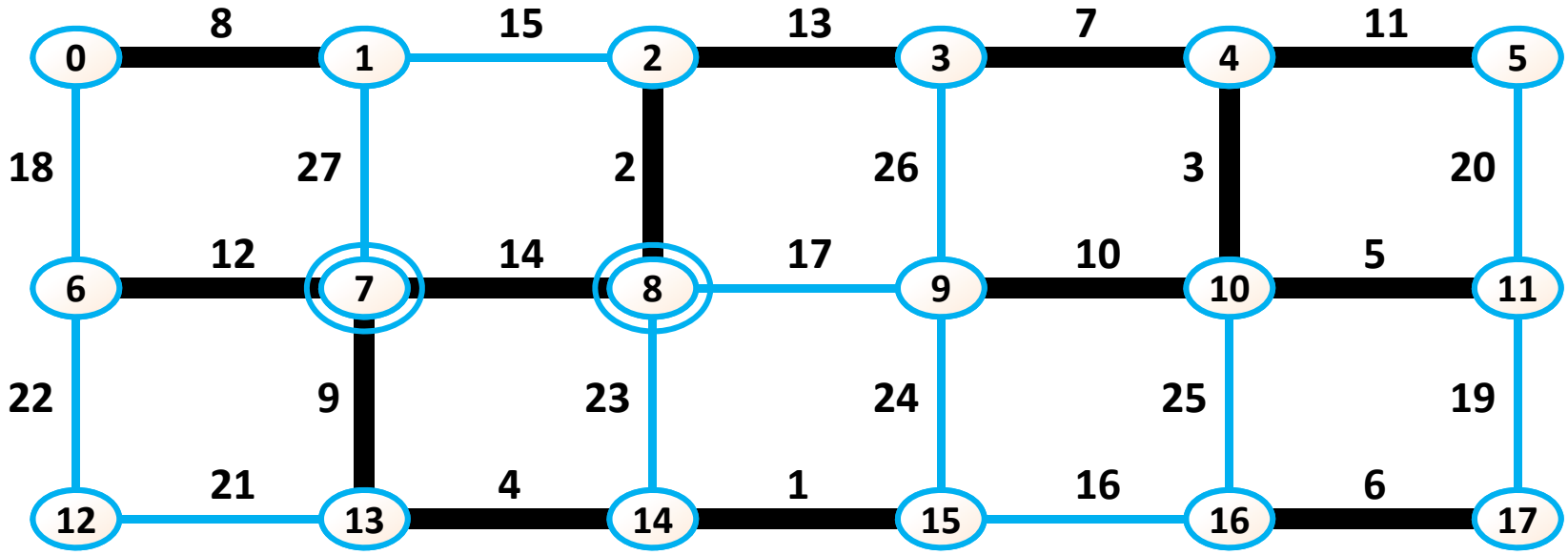
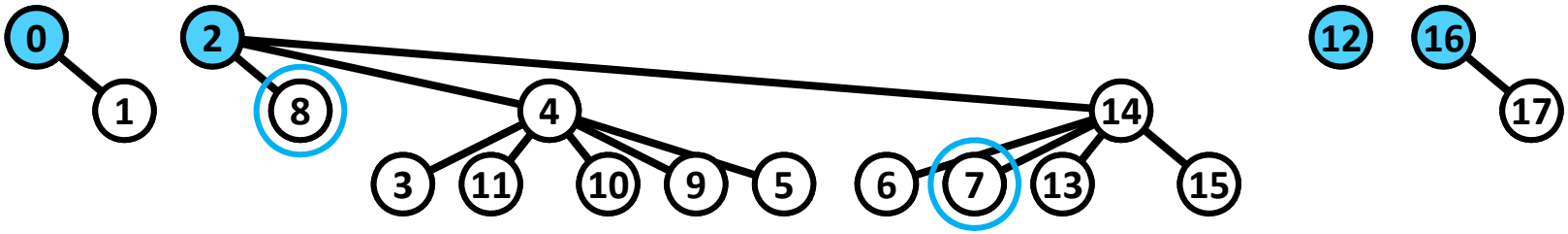
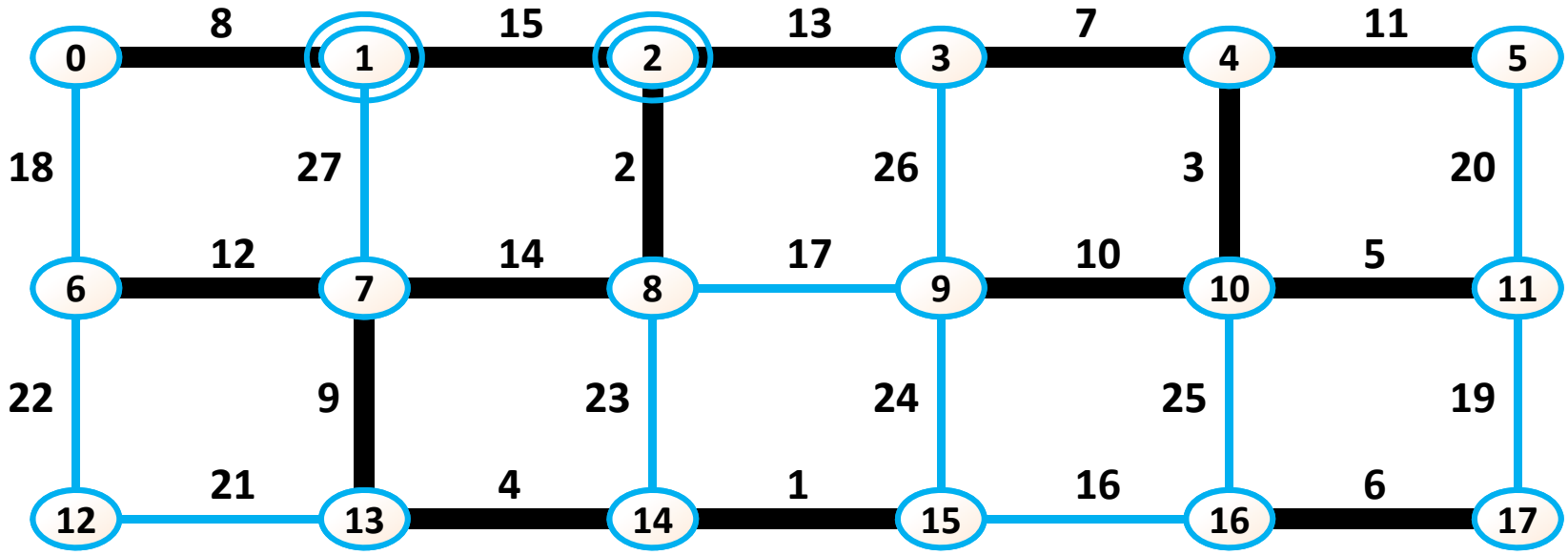
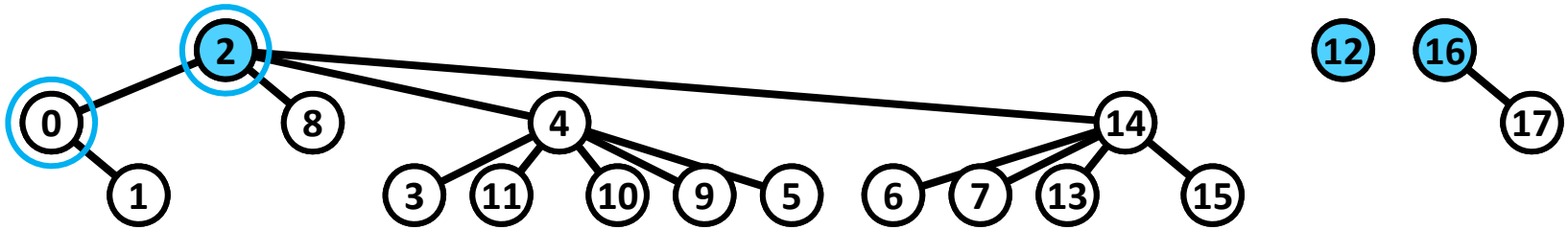




node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	0	0	2	4	2	4	14	14	2	4	4	4	12	14	2	14	16	16
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Kruskal algorithm and Union-Find scheme example





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boss	2	0	2	4	2	4	14	14	2	4	4	4	12	14	2	14	16	16
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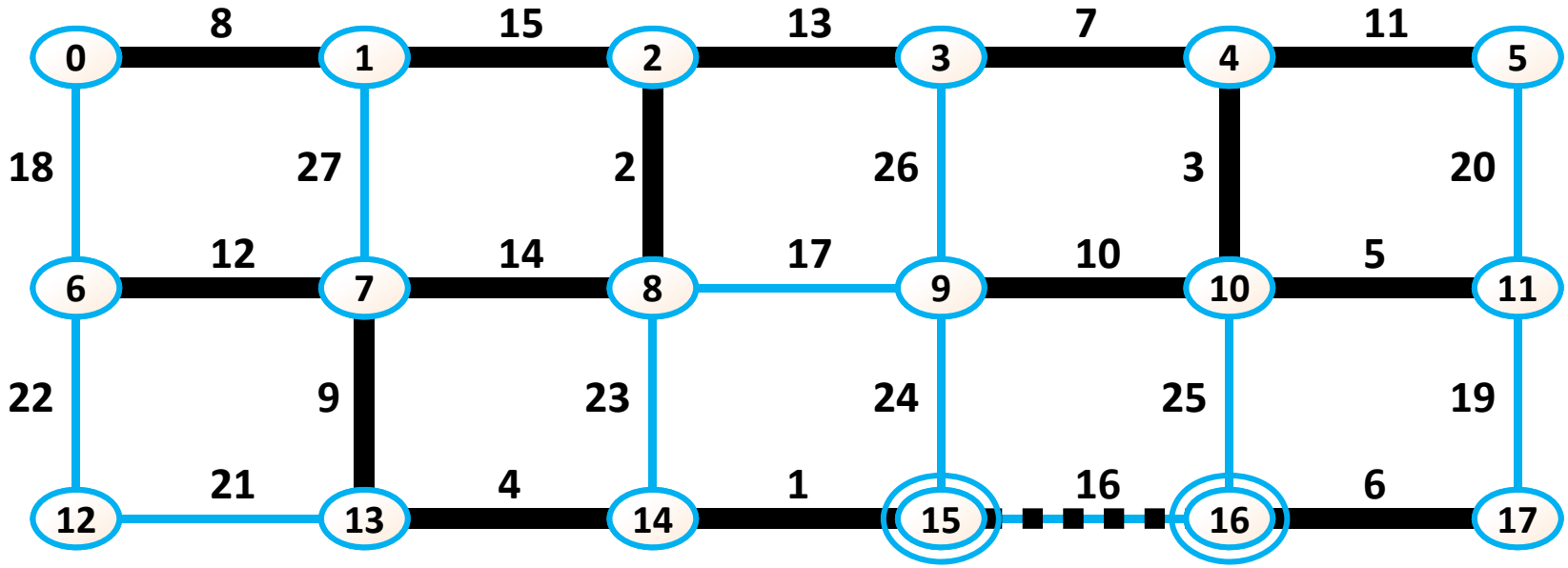
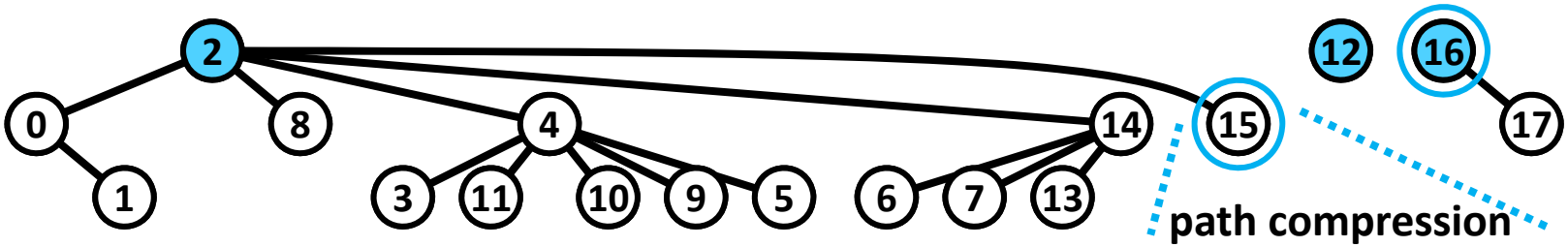
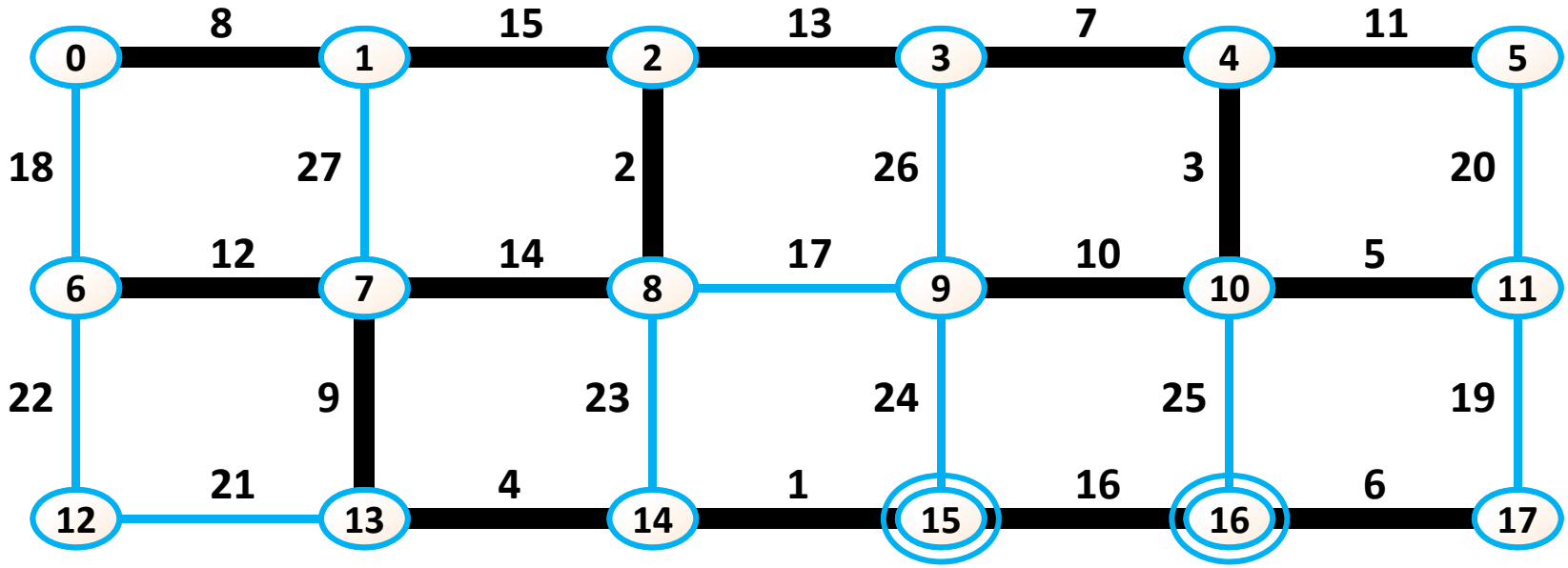
Kruskal algorithm and Union-Find scheme example

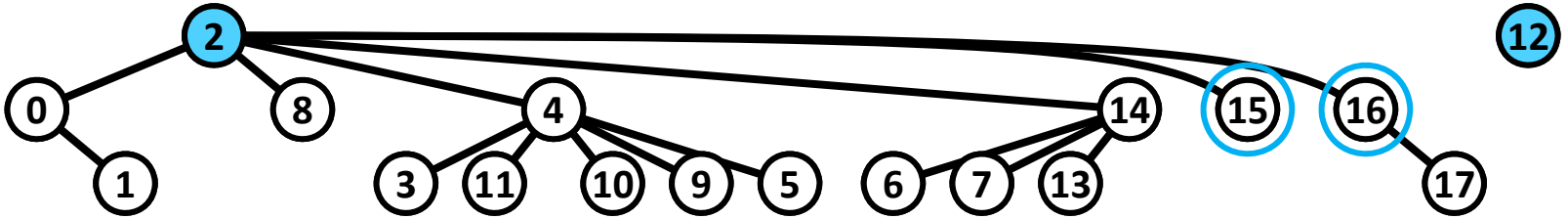




node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	2	0	2	4	2	4	14	14	2	4	4	4	12	14	2	2	16	16
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

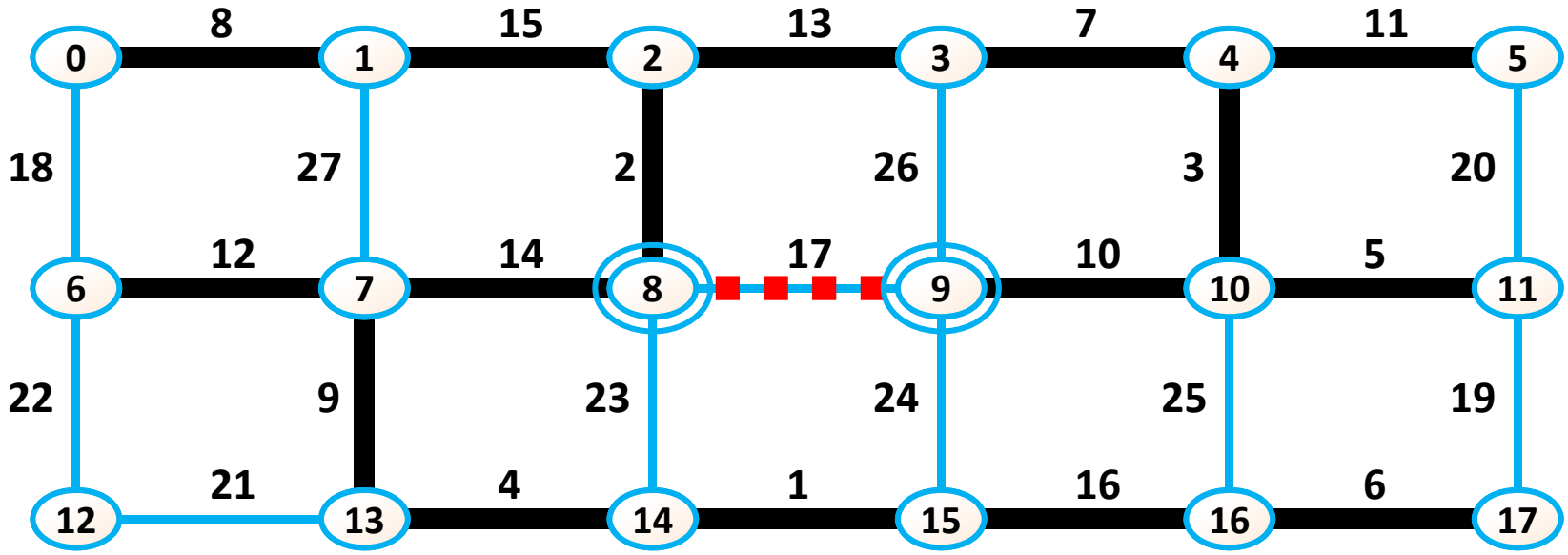
Kruskal algorithm and Union-Find scheme example

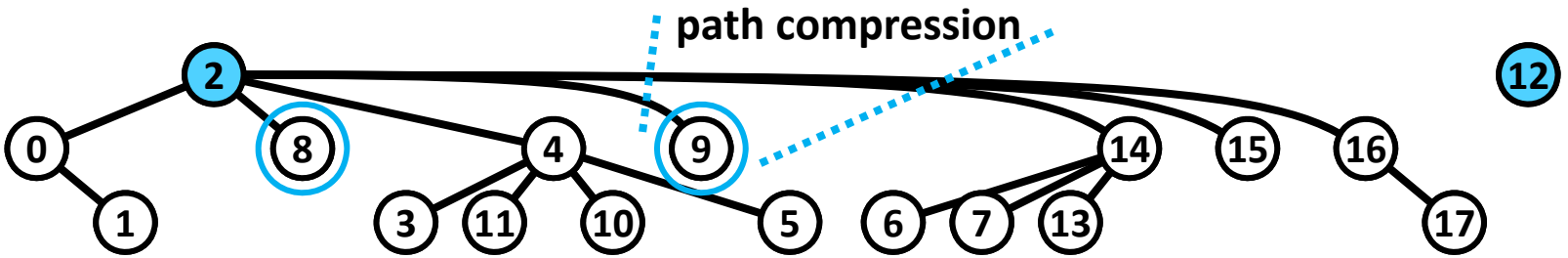




node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	2	0	2	4	2	4	14	14	2	4	4	4	12	14	2	2	2	16
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Kruskal algorithm and Union-Find scheme example





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boss	2	0	2	4	2	4	14	14	2	2	4	4	12	14	2	2	2	16
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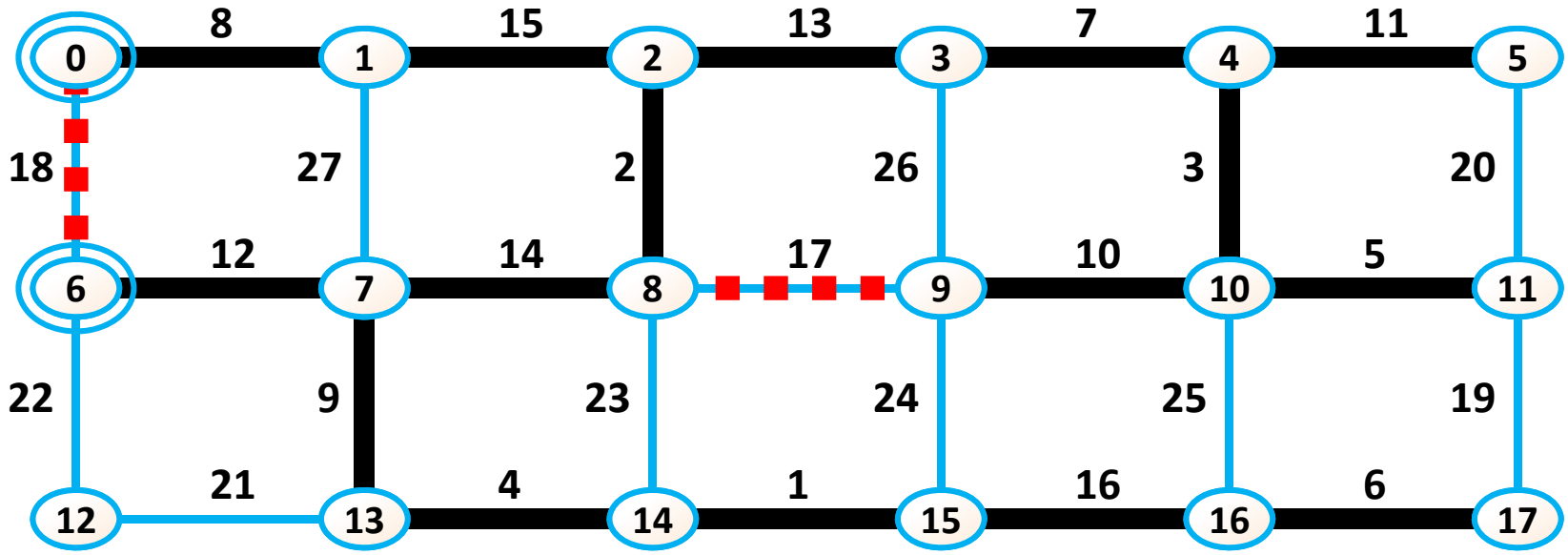
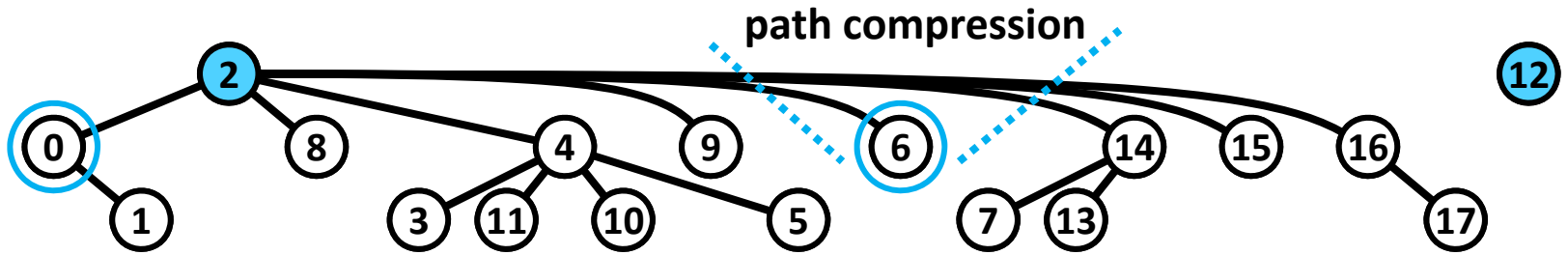
Kruskal algorithm and Union-Find scheme example





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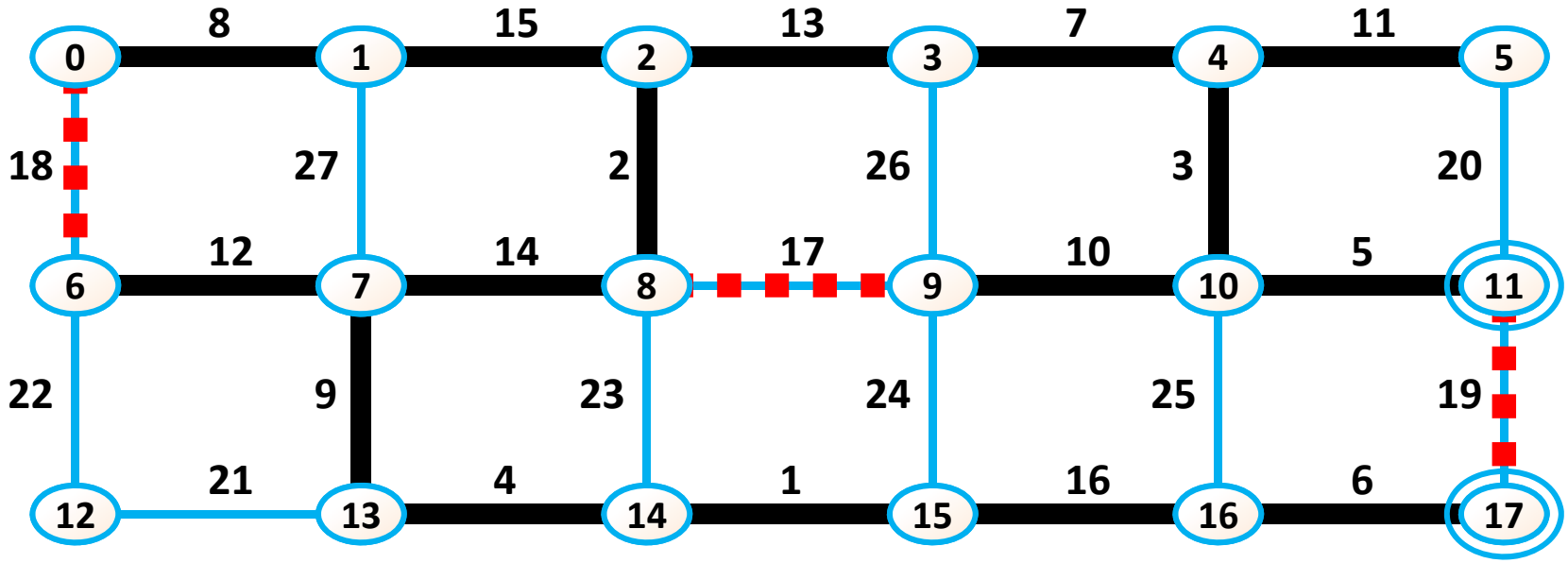
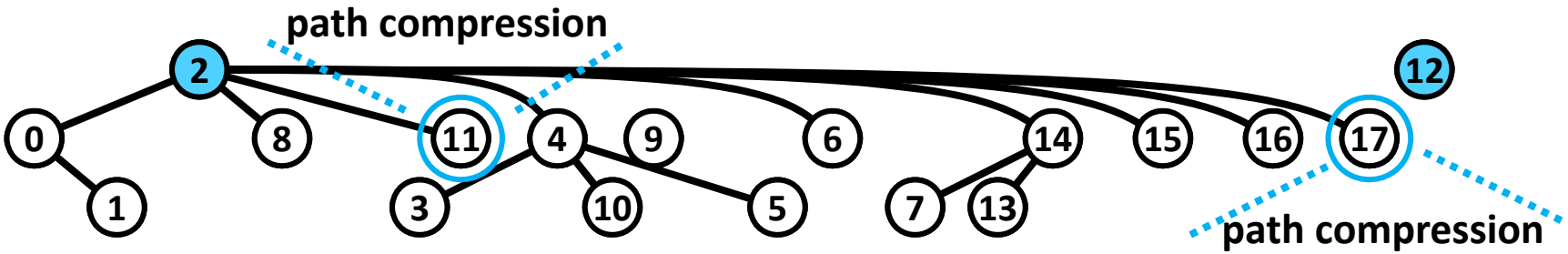
boss	2	0	2	4	2	4	2	14	2	2	4	4	12	14	2	2	2	16
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

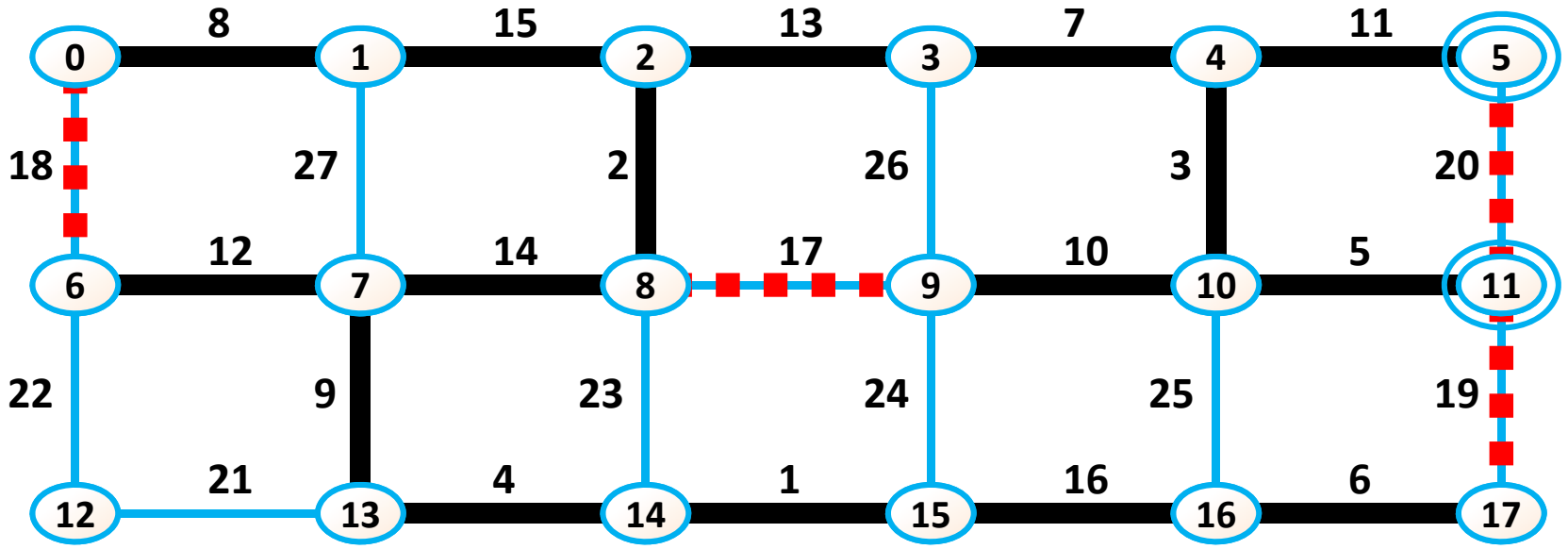
Kruskal algorithm and Union-Find scheme example

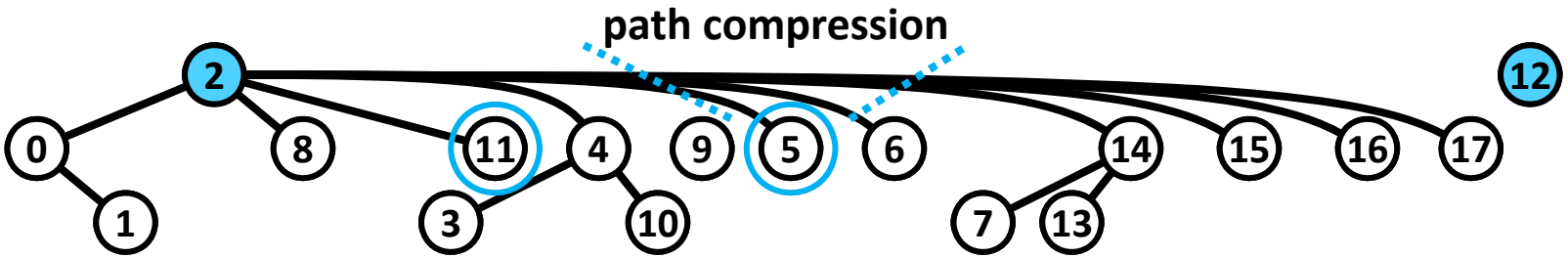




node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	2	0	2	4	2	4	2	14	2	2	4	2	12	14	2	2	2	2
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Kruskal algorithm and Union-Find scheme example





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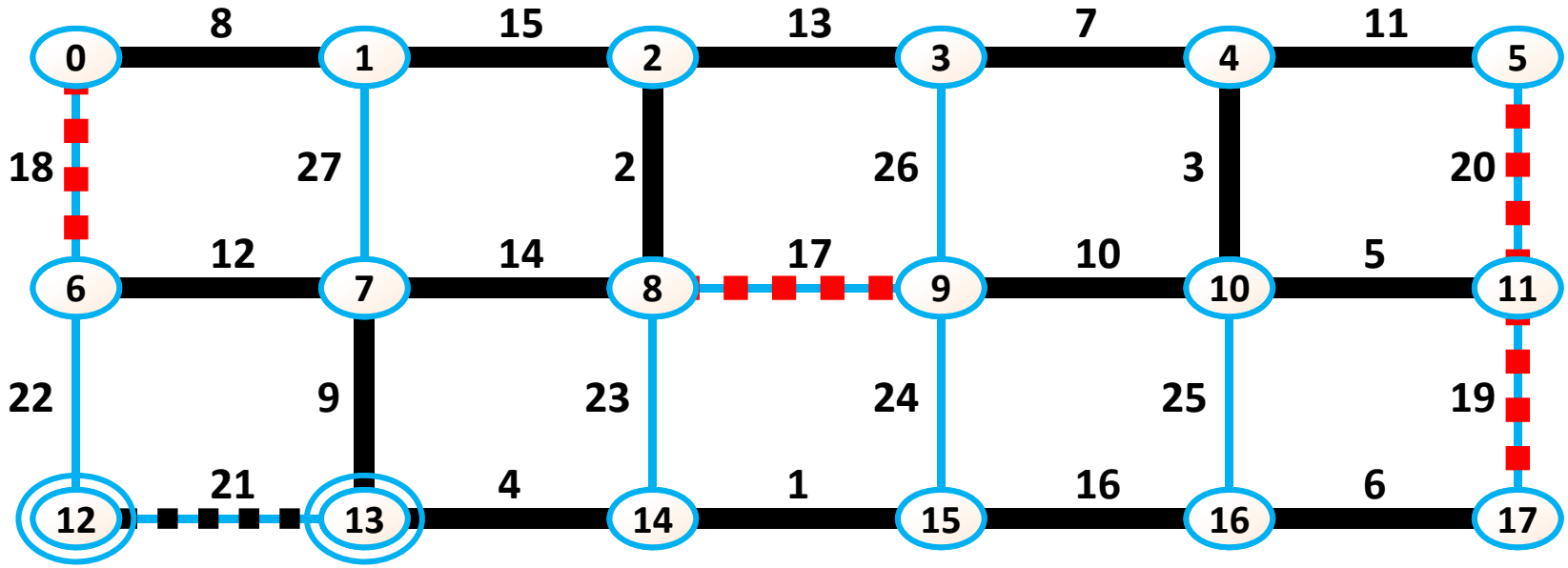
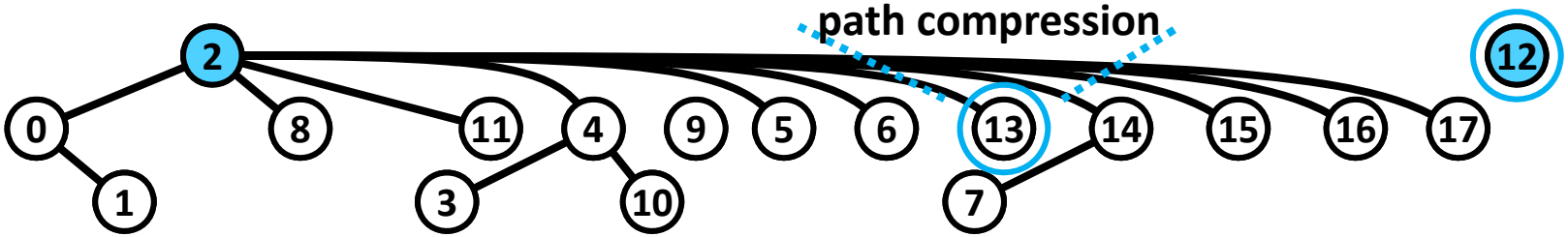
boss	2	0	2	4	2	2	2	14	2	2	4	2	12	14	2	2	2	2
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

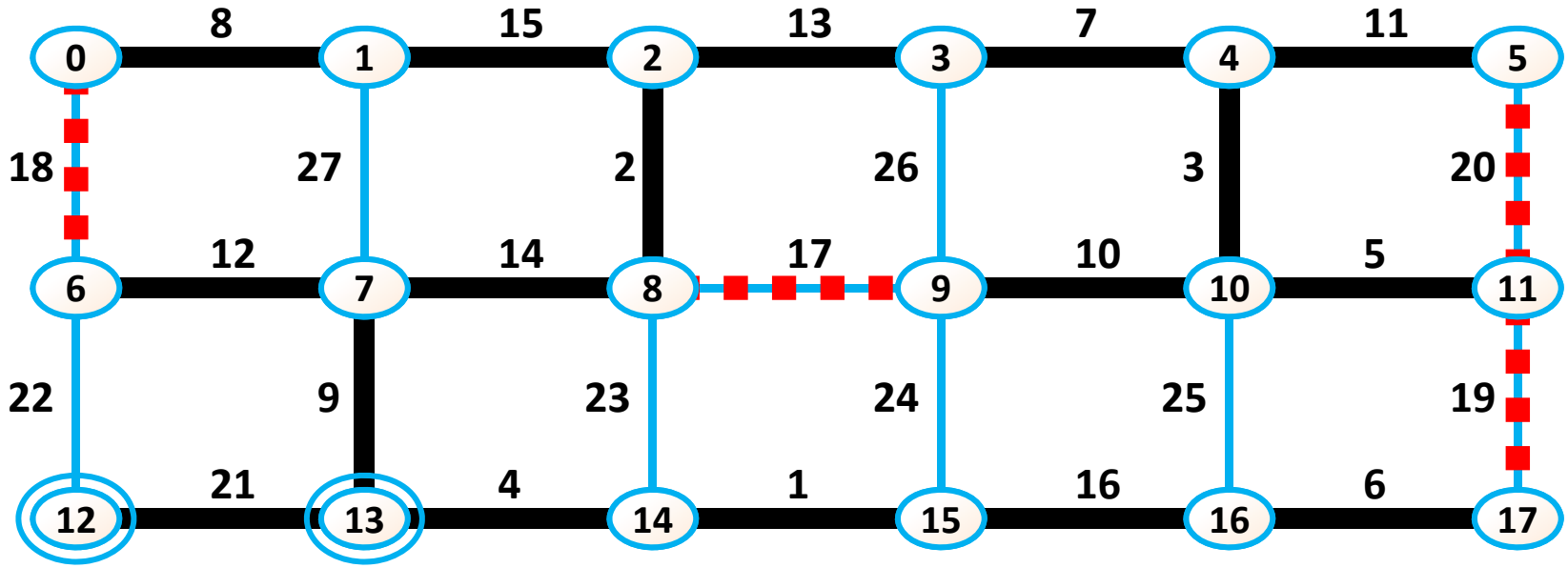
Kruskal algorithm and Union-Find scheme example

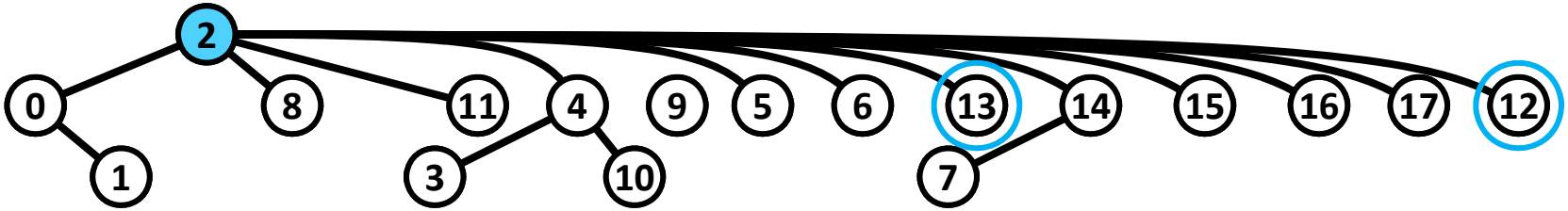




node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	2	0	2	4	2	2	2	14	2	2	4	2	12	2	2	2	2	2
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Kruskal algorithm and Union-Find scheme example





node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
boss	2	0	2	4	2	2	2	14	2	2	4	2	2	2	2	2	2	2
rank	1	0	2	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Kruskal algorithm running time improvement

When both union by rank and path compression are used, the running time spent on Union-Find operations in a graph with N nodes and M edges is $O(M \cdot \alpha(N))$, where $\alpha(N)$ is the inverse function of $f(x) = A(x, x)$, where $A(x, y)$, is the Ackermann function, known to grow quite fast. In fact, for *any* graph representable in *any* conceivable machine $\alpha(N) < 4$.

Thanks to the inverse Ackermann function, all Union-Find operations run in amortized constant time in all practical situations.
It means that the speed bottleneck is the initial edge sorting.
Sorting can be done in linear time when:

- edge values are strings (does not happen too often), apply **Radix sort**,
- edge values are integers in some (relatively) moderate range (e.g 0..10 000, etc.), apply **Counting sort**,
- edge values are floats (more or less) uniformly distributed over some interval, apply **Bucket sort**.

Conclusion
In many practical situations, a careful implementation of Kruskal algorithm runs in $\Theta(M)$ time.

Ackermann function

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ A(m-1,1) & \text{if } m > 0, n = 0, \\ A(m-1, A(m,n-1)) & \text{if } m > 0, n > 0. \end{cases}$$

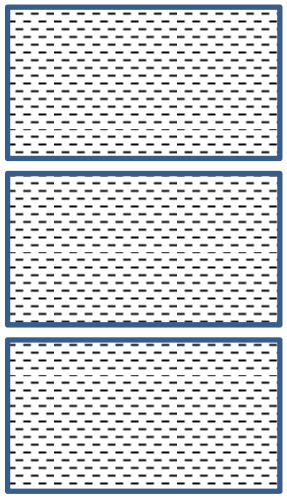
$m \backslash n$	0	1	2	3	4	5	n
0	1	2	3	4	5	6	$n+1$
1	2	3	4	5	6	7	$n+2$
2	3	5	7	9	11	13	$2n-3$
3	5	$13 = 2^4 - 3$	$= 2^5 - 3 = 29$	$= 2^6 - 3 = 61$	$= 2^7 - 3 = 125$	$= 2^8 - 3 = 253$	$2^{n+3} - 3$
4	$13 = 2^{2^2} - 3$	$65533 = 2^{2^{2^2}} - 3$	$(\#) = 2^{2^{2^{2^2}}} - 3$	$(\#\#) = 2^{2^{2^{2^{2^2}}}} - 3$	$(\#\#\#) = 2^{2^{2^{2^{2^{2^2}}}}} - 3$	$(\#\#\#\#) = 2^{2^{2^{2^{2^{2^{2^2}}}}}} - 3$	$2^{2^{2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}}} - 3$ } $n+3$
5	$65533 = 2^{2^{2^2}} - 3$	$2^{2^{2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}}} - 3$ } 65536					*too big to fit here …*
6	$A(6,0) = A(5,1)$						*too big to fit here …*
7							

$2^{2^{2^{2^{2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}}}}} - 3$ } $2^{2^{2^{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}}}$ } 65536

1. 0-th value in each line = 1st value in the previous line.
2. Value X in the j-th column (j > 0) on the current line is equal to the value on the previous line which column index is equal to the value written to left of the value X on the current line (= in the (j–1)-th column).

$$A(4,2) = (\#) = 2^{2^{2^{2}}} - 3 = 2^{2^{2^{4}}} - 3 = 2^{2^{16}} - 3 = 2^{65536} - 3 =$$

= (next 3 slides with 19729 decimal digits) =



$$A(4,3) = (\#\#) = 2^{(\#)} - 3 = ?$$

$$A(4,4) = (\#\#\#) = 2^{2^{(\#)}} - 3 = ??$$

Informal discussions concerning big integers:

http://waitbutwhy.com/2014/11/1000000-grahams-number.html
http://www.scottaaronson.com/writings/bignumbers.html

```
20035299304068464649790723515602557504478254755697514192650169737108940595563114530895061308809333481010382343429072631818229493821188126688695063647615470291650418719163515879663472194429309279820483091048559905701593189596395248633723672030029169695921561087649488892540908059114570376752085002066715637023661263597471448071117748158809141357427209671901518362825606180914588526998261414250301233911082736038437678764490432059603791244909057075603140350761625624760318637931264847037437829549756137709816046144133086921181024859591523801953310302921628001605686701056516467505680387415294638422448452925373614425336143737290883037946012747249584148649159306472520151556939226281806916507963810641322753072671439981585088112926289011342377827055674210800700652839633221550778312142885516755540733451072131124273995629827197691500548839052238043570458481979563931578535100189920000241419637068135598404640394721940160695176901561197269823378900176415171900511334663068981402193834814354263873065395529696913880241581618595611006403621197961018595348027871672001226046424923851113934004643516238675670787452594646709038865477434832178970127644555294090920219595857516229733335761595523948852975799540284719435299135437637059869289137571537400019863943324648900525431066296691652434191746913896324765602894151997754777031380647813423095961909606545913008901888875880847336259560654448885014473357060588570801621084997145295683440619796905654698136311620537936979140323632849623304642106613620022017578785185740916205048971178182040018728293994344618622432800983732376493181478984811945271300744022076568091037620399920349202390662626449190916798546151577883906039772075927937885224129430101745808686226336928472585140303961555856433038545068865221311481363840838477826379045960718687672850976347127198889068047824323039471865052566097815072986114143030581692792497140916105941718535227588750447759221830115878070197553572224140001954810200566177358978149953232520858975346354700778669040642901676380816174055040511767009367320280454933902799249186730653993164072049223847481528061916690093380573212081635070763435166986962502096902316285935007187419057916124153689751480826190484794657173660100589247665544584083833479054414481768425532720731558634934760513741977952519036503219802010876473836868253102518337753390886142618480037400808223810407646887847164755294532694766170042446106331123802113458869453220011656407632702307429242605158281107038701834532456763562595143003203743274078087905628366340696503084422585596703927186946115851379338647569974856867007982396060439347885086164926030494506174341236582835214480672667684180708375486221140823657980296120002744132443843240233125740354501935242877643088023285085588608996277445816468085787511580701474376386797695504999164399828435729041537814343884730348426190338884149403136613985425763557710533558020662218557706008255128889333222643628198483861323957067619140963853383237434375883085923372228464428799624560547693242899843265267737837317328806321075321123868060467470842805116648870908477029120816110491255559832236624486850664140268464120969498259056551921618810434122683899628307165486852553691485029953967550395493837185340590009618748947399288043249637316575380367358671017578399481847179849824694806053208199606618343401247609663951977802144119975254670408060849934417825628509272652370989865153946219300460736450792621297591769829389236701517099209153156781443979124847570623780460000991829332130688057004659145838720808801688744583555792625846512476308714856631352893416611749061752667149267217612833084527393646924458289257138887839056300482483799839692029222215486145902373478222682521639957440801727144146179559226175083889020074169926238300282286249284182671243405751424188569994272331606998712986882771820617214453142574944015066139463169197629181506579745526236191224848063890033669074365989226349564114665503062965960199720636202603521917767406687774635493753188995878662821254697971020657472327213729181446666594218720034745089428309115351892711142871083761592223802766053278233516615551493693757784666701457179719012271178127804502400263847587883393968179629506907988171216906869295382485298300234760684541141781391106485602365497542274972310076151318700240539105109138178437217914225285874320985249578780346837033378184214440171386881242499844186181292711985333153825673218704215306311977485352146709553346263366108646673322924098798492566911095161436186015489097402419135096230436121961281659505186660220307156136847323646608689050142639139065150639081993788523183650598972991254044794434251667742996598118492331515552728832740283526884424087528112832899806259126736995462473415433335001472314306127503903073971352520693381738433229507010490618675394331307847980156551303847581556852362180104196502555961819349863159132330360964619059902361126811960234418433633345949276319461017166529138237171823942992162725384617760656945422978770713831988170369645886898118632109769003557358846244648357062914530527571012788720279653644797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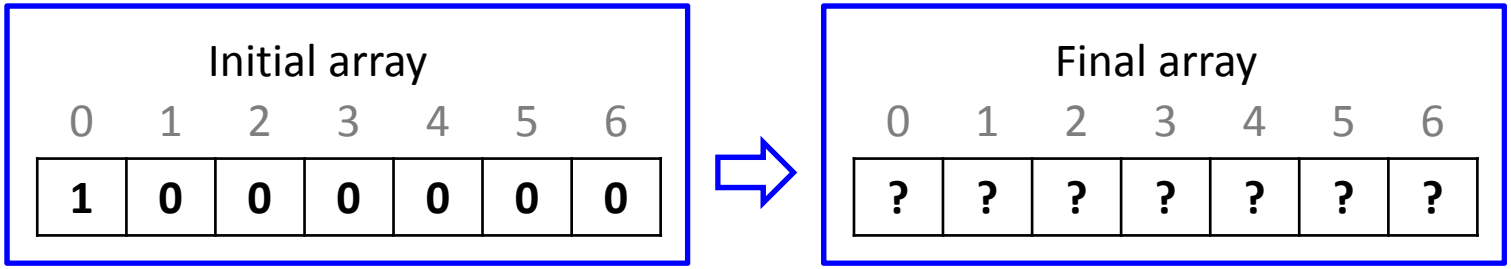
A game with some relation to the Ackermann function



On a table:

Init: Array with 7 fields, leftmost field contains 1 token, other fields are empty.

In each step:
-- Remove a token from any field F
-- Choose one of the following:
- If there is a field to the right of F
  add two tokens to that field
- If there are two fields to the right of F
  swap the contents of those fields

The objective:
Maximize the number of tokens in the array.

In a computer

Init: A[0] = 1; A[k] = 0, k = 1..6

In each step:
-- choose index k
-- if (A[k] > 0)
  A[k] –= 1;
  choose one of the following:
  either A[k+1] += 2; (if k < 6)
  or swap (A[k+1], A[k+2]) (if k < 5)

The objective:
Maximize sum(A[k], k=0..6).