Computed-Torque Control

Dynamics:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) + \tau_d = \tau$$

Properties:

- The inertia matrix $M(q)$ is *symmetric, positive definite*, and *bounded* so that $\mu_1 I \le M(q) \le \mu_2 I \quad \forall q(t)$.
- Coriolis/centripetal vector $V_m(q,\dot{q})\dot{q}$ is *quadratic* in $\dot{q}$. V_m is *bounded* so that $\| V_m \dot{q} \| \le v_B \| \dot{q} \|^2$.
- The Coriolis/centripetal matrix can always be selected so that the matrix $S(q,\dot{q}) \equiv \dot{M}(q) - 2V_m(q,\dot{q})$ is *skew symmetric*. Therefore, $x^T S x = 0$ for all vectors x.

Properties:

- The friction term is bounded so that $\| F(\dot{q}) \| \leq f_B \| \dot{q} \| + k_B$.
- The gravity vector is *bounded* so that $\| G(q) \| \leq g_B$.
- The disturbances are *bounded* so that $\| \tau_d(t) \| \leq d_B$.

It is often convenient to write the robot dynamics as

$$M(q)\ddot{q} + N(q, \dot{q}) + \tau_d = \tau$$

where $N(q, \dot{q}) \equiv V_m(q, \dot{q})\dot{q} + F(\dot{q}) + G(q)$ represents a vector of nonlinear terms. It is assumed that $\tau_d = 0$.

Reconsider the robot dynamics

$$M(q)\ddot{q} + N(q, \dot{q}) = \tau$$

Position/velocity state-space form: Defining state vector as $x \equiv [q^T \quad \dot{q}^T]^T$, we can rewrite robot dynamics as follows:

$$\dot{x} = \begin{bmatrix} \dot{q} \\ -M^{-1}(q)N(q, \dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(q) \end{bmatrix} \tau$$

$$\ddot{q} = -M^{-1}N + M^{-1}\tau = u$$

By defining a state vector $x \equiv [q^T \quad \dot{q}^T]^T$, we get following state space model:

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

where $u = -M^{-1}(q)[N(q, \dot{q}) - \tau]$

It uses feedback-linearization technique. Defi ne the tracking error as

$$e(t) = q_d(t) - q(t)$$

Differentiating twice we get

$$\ddot{e} = \ddot{q}_d - \ddot{q} = \ddot{q}_d - M^{-1}\tau + M^{-1}N = \ddot{q}_d + M^{-1}[N - \tau]$$

By choosing a state vector $x = [e^T \quad \dot{e}^T]^T$, we obtain the Brunovsky canonical form as

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

with $u(t) \equiv \ddot{q}_d + M^{-1}(q)[N(q, \dot{q}) - \tau]$.

Now choose u to stabilize the error dynamics and then compute the required arm torques as

$$\tau = M(q)(\ddot{q}_d - u) + N(q, \dot{q})$$

Two forms of Computed-torque control are given below:

PD CT control

$$\tau = M(q)(\ddot{q}_d + K_v\dot{e} + K_p e) + V_m(q, \dot{q})\dot{q} + F(\dot{q}) + G(q)$$

PID CT control

$$\begin{aligned} \dot{\varepsilon} &= e \\ \tau &= M(q)(\ddot{q}_d + K_v\dot{e} + K_p e + K_i\varepsilon) + V_m(q, \dot{q})\dot{q} + F(\dot{q}) + G(q) \end{aligned}$$

Consider single-link manipulator dynamics given by

$$a\ddot{q} + bsin\ q = \tau$$

where $a = ml^2$ and $b = mgl$. Differentiating tracking error $e = q_d - q$ twice, we get

$$\begin{aligned} a\ddot{e} &= a\ddot{q}_d - a\ddot{q} \\ &= a\ddot{q}_d + bsin\ q - \tau = f(q) - \tau \end{aligned}$$

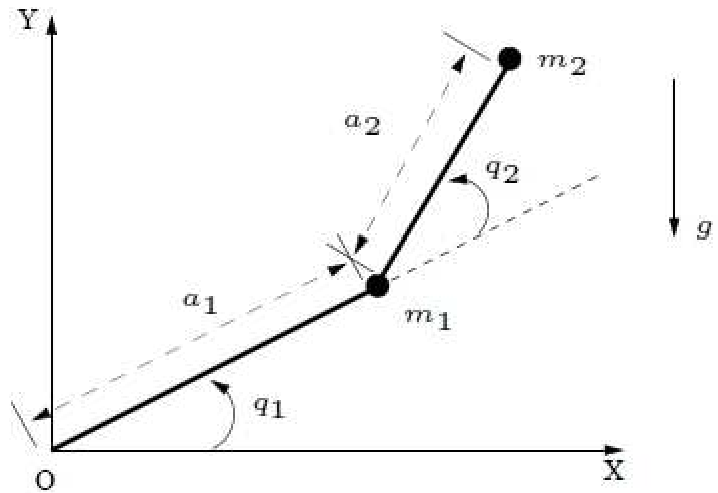
The computed torque control $\tau = f(q) + K_d\dot{e} + K_p e$ yields following closed loop error dynamics

$$a\ddot{e} + K_d\dot{e} + K_p e = 0$$

which is stable and achieves trajectory tracking.

Consider a planar 2-link manipulator in standard form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

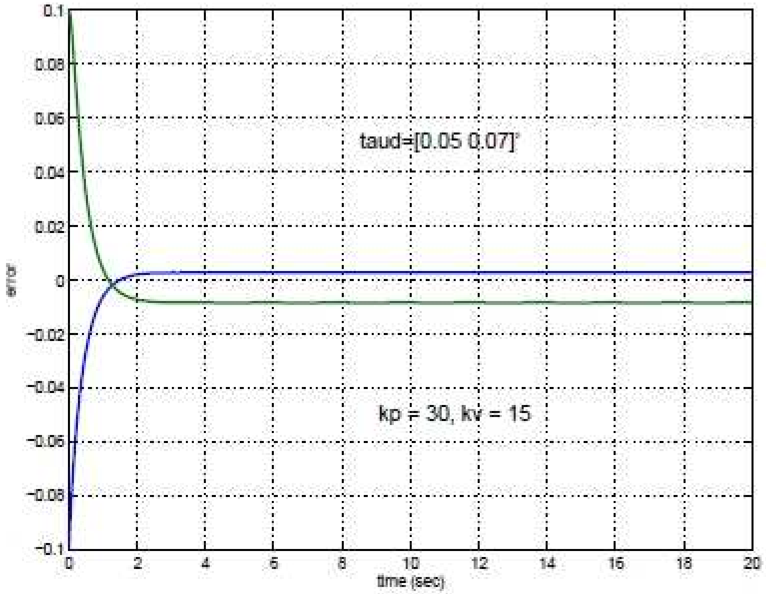


Defi ne

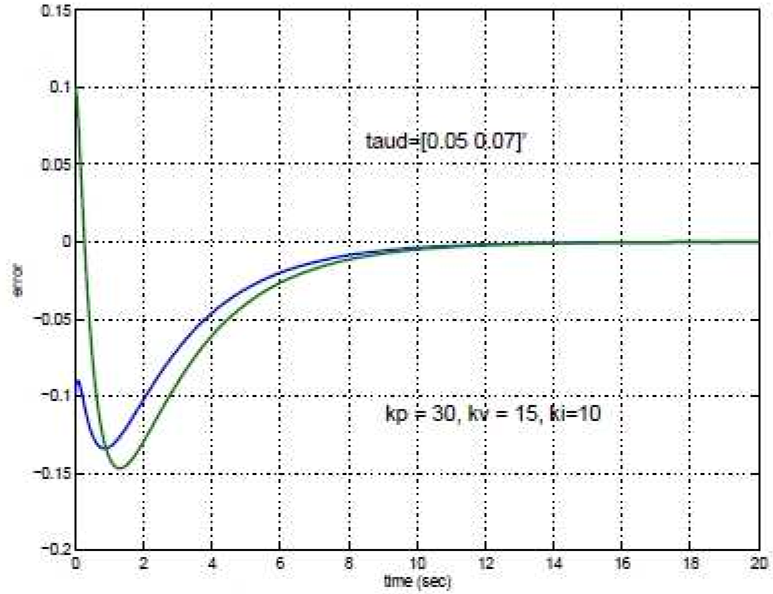
$\alpha = (m_1 + m_2)a_1^2$, $\beta = m_2 a_2^2$, $\eta = m_2 a_1 a_2$ and $e_1 = g/a_1$.

The dynamics may be re-written as

$$\begin{bmatrix} \alpha + \beta + 2\eta cos\ q_2 & \beta + \eta cos\ q_2 \\ \beta + \eta cos\ q_2 & \beta \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -\eta(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) sin\ q_2 \\ \eta\dot{q}_1^2 sin\ q_2 \end{bmatrix} + \begin{bmatrix} \alpha e_1 cos\ q_1 + \eta e_1 cos(q_1 + q_2) \\ \eta e_1 cos(q_1 + q_2) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$



(a) CT-PD control



(b) CT-PID control

Computed-Torque control of a 2-link manipulator under constant torque disturbance