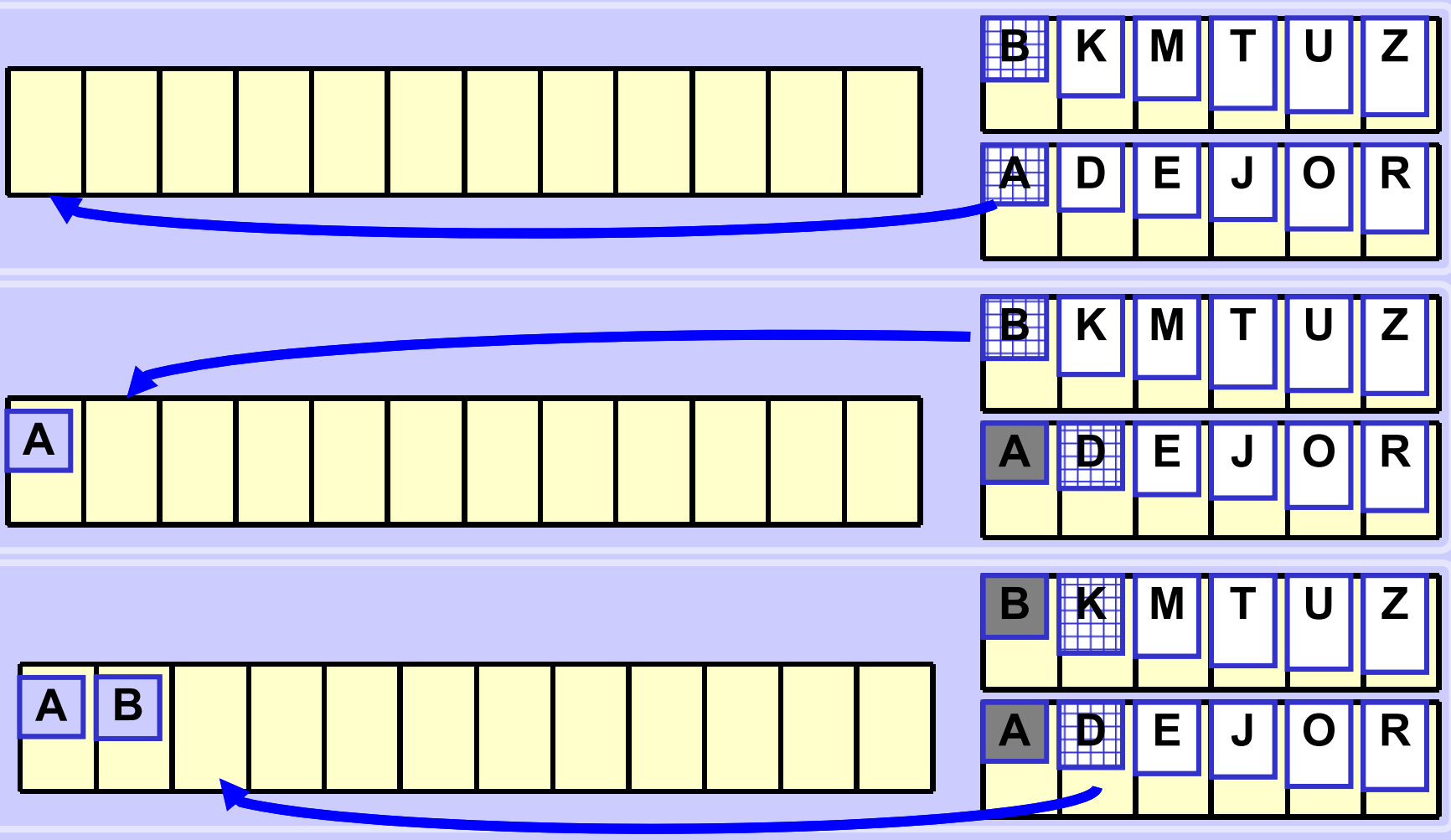
# Merge sort

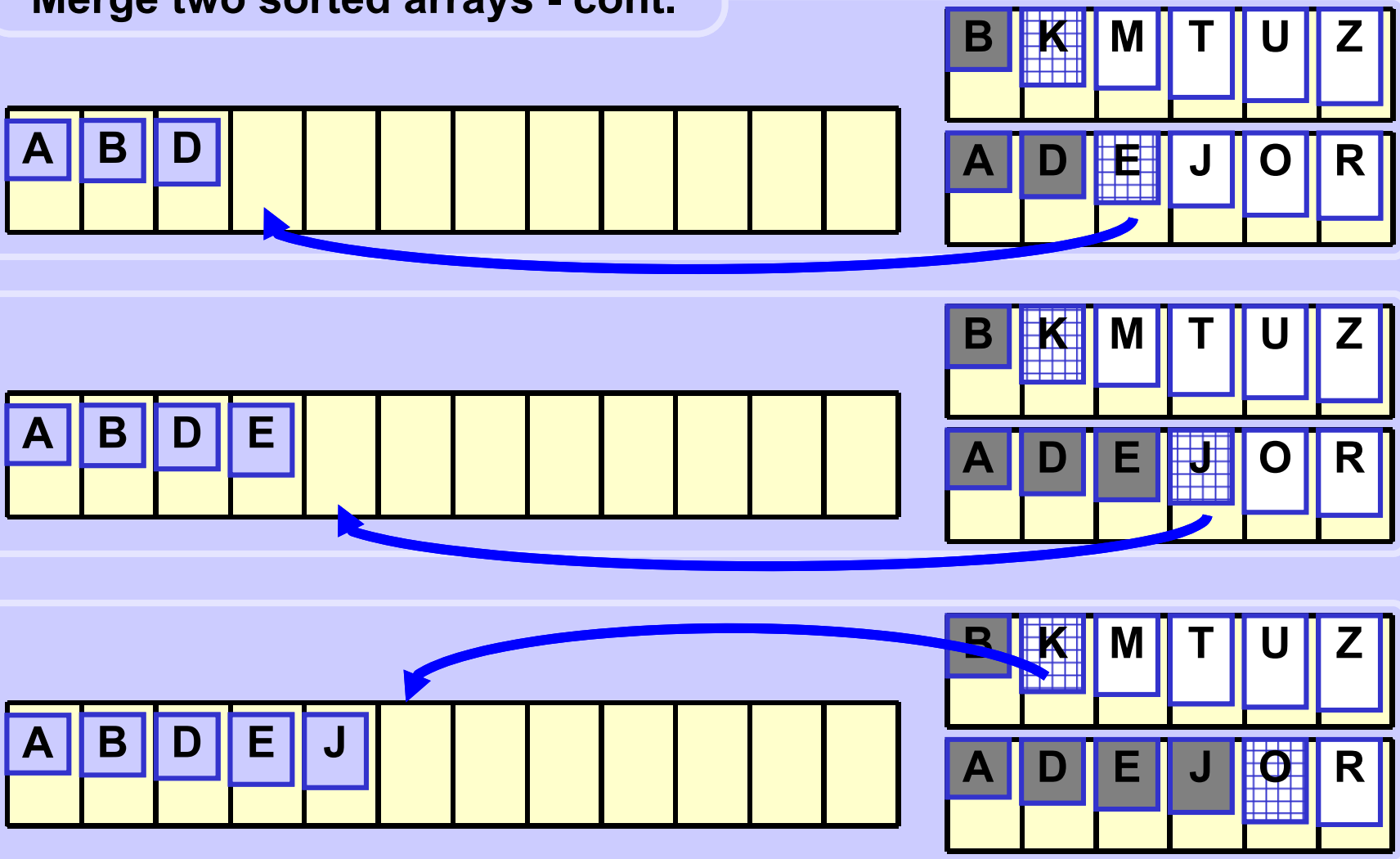
## Merge two sorted arrays

Compared elements



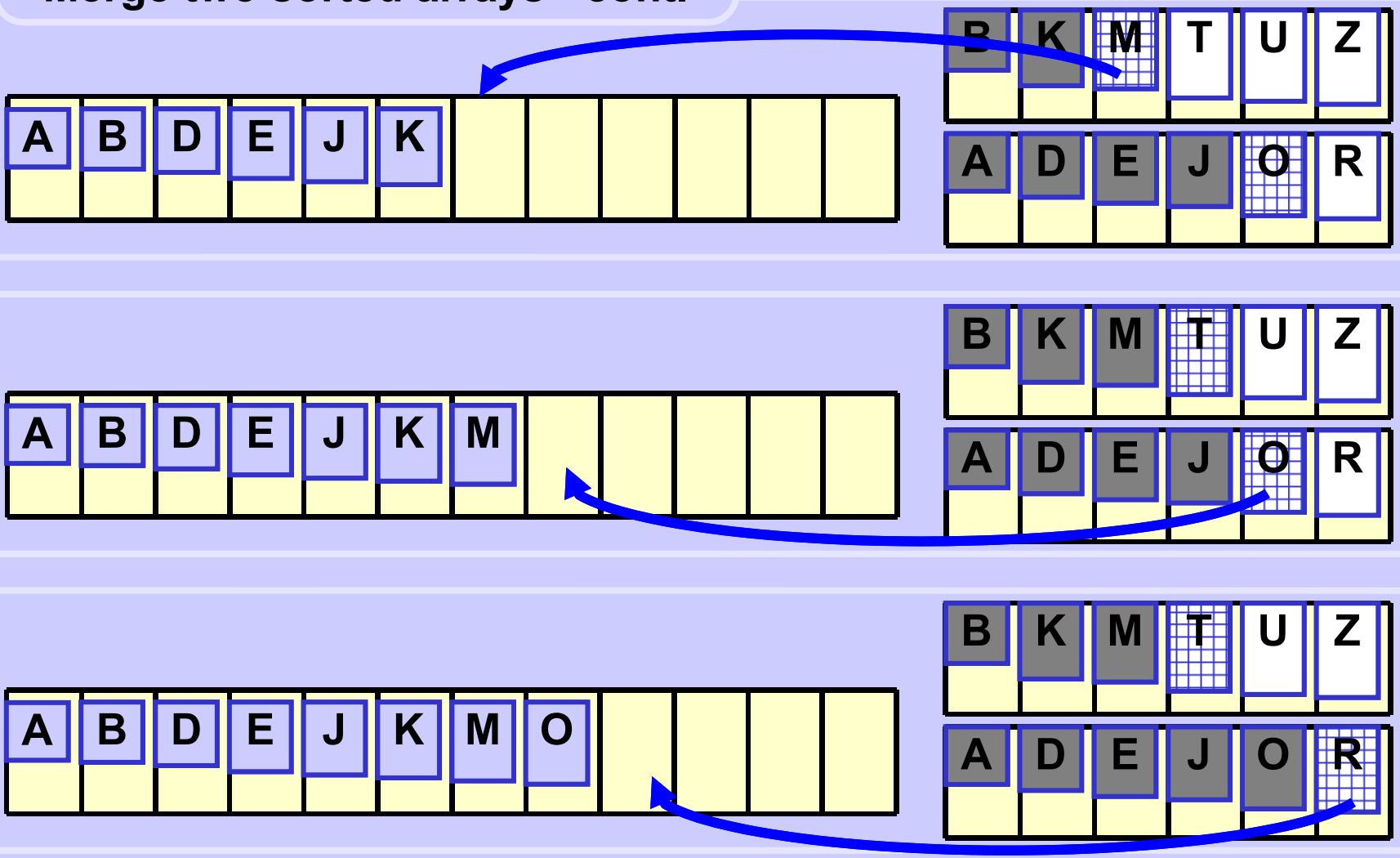
# Merge sort

## Merge two sorted arrays - cont.



# Merge sort

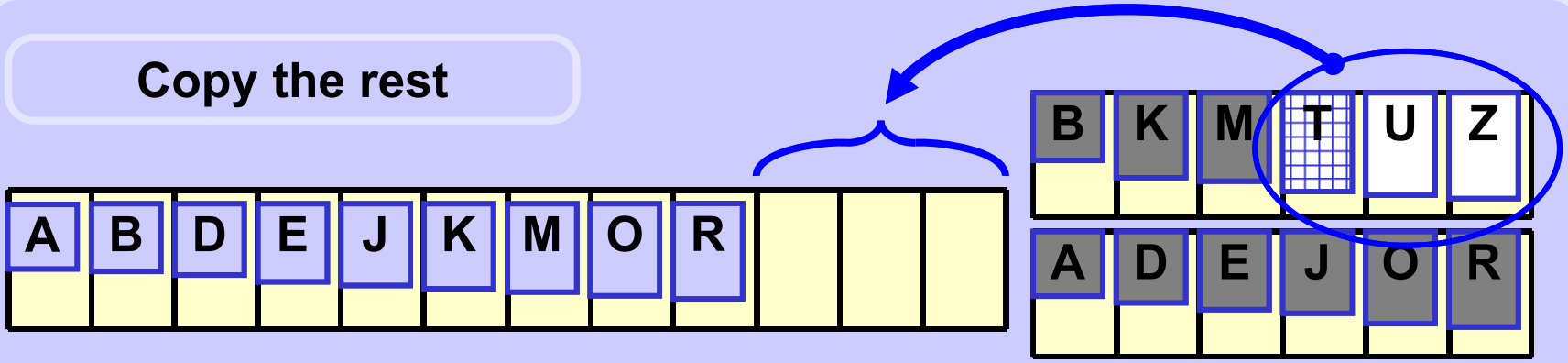
## Merge two sorted arrays - cont.



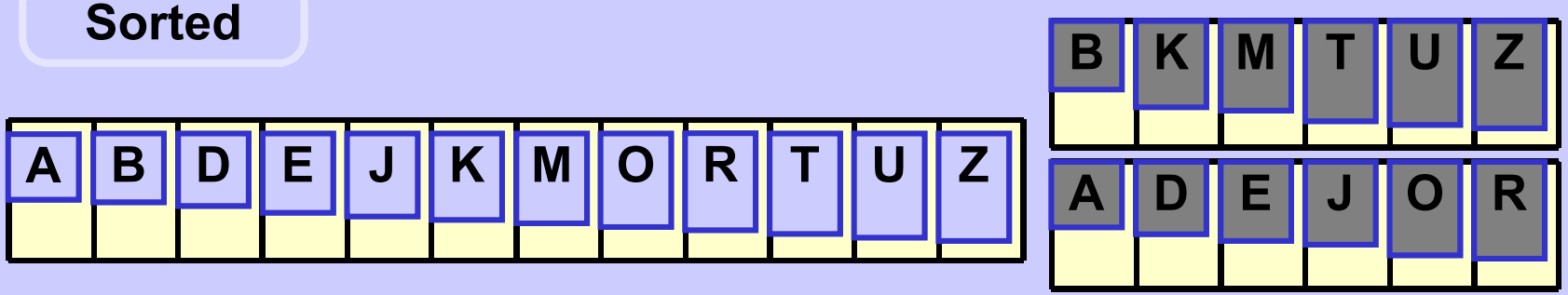
# Merge sort

## Merge two sorted arrays - cont.

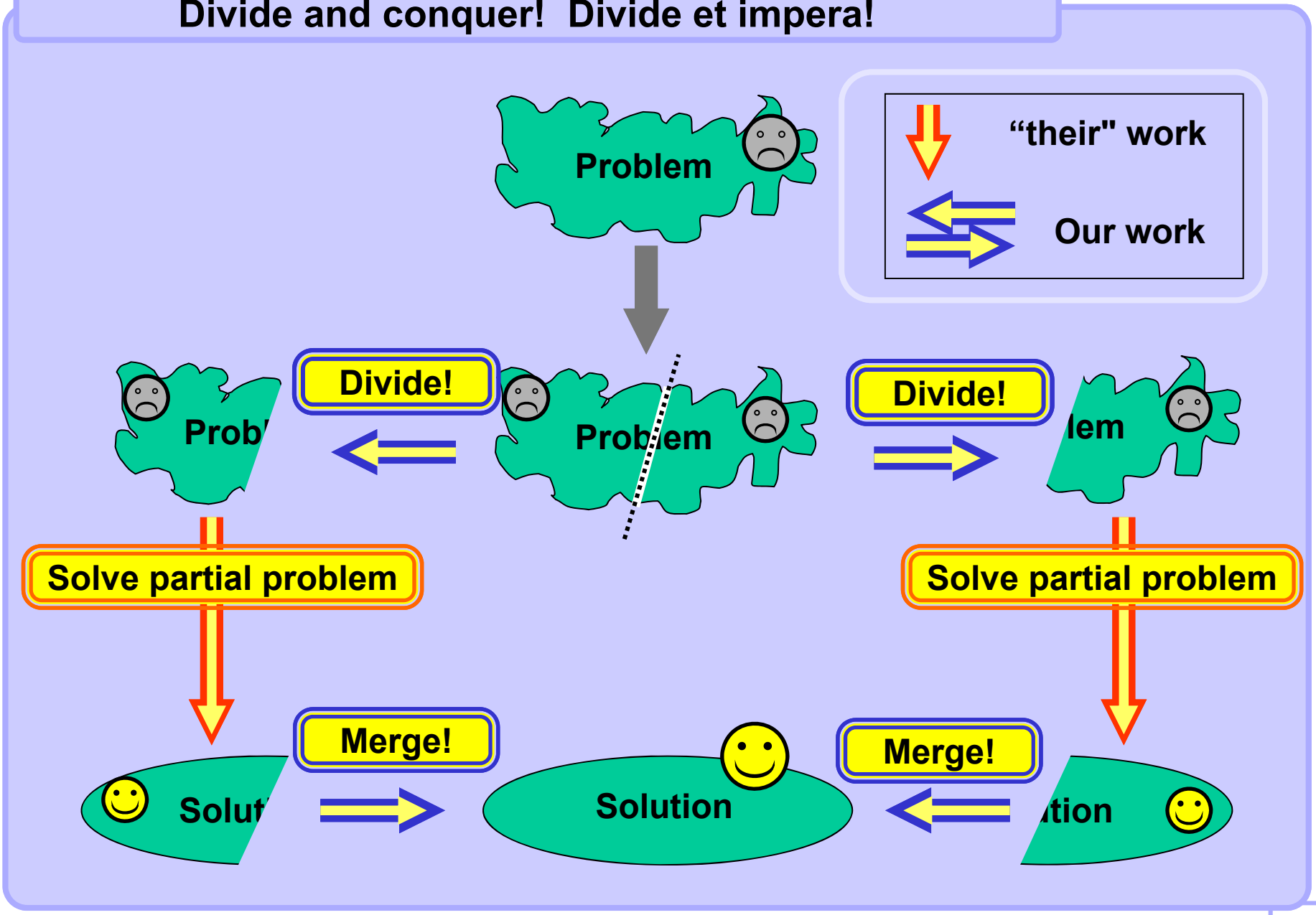
### Copy the rest



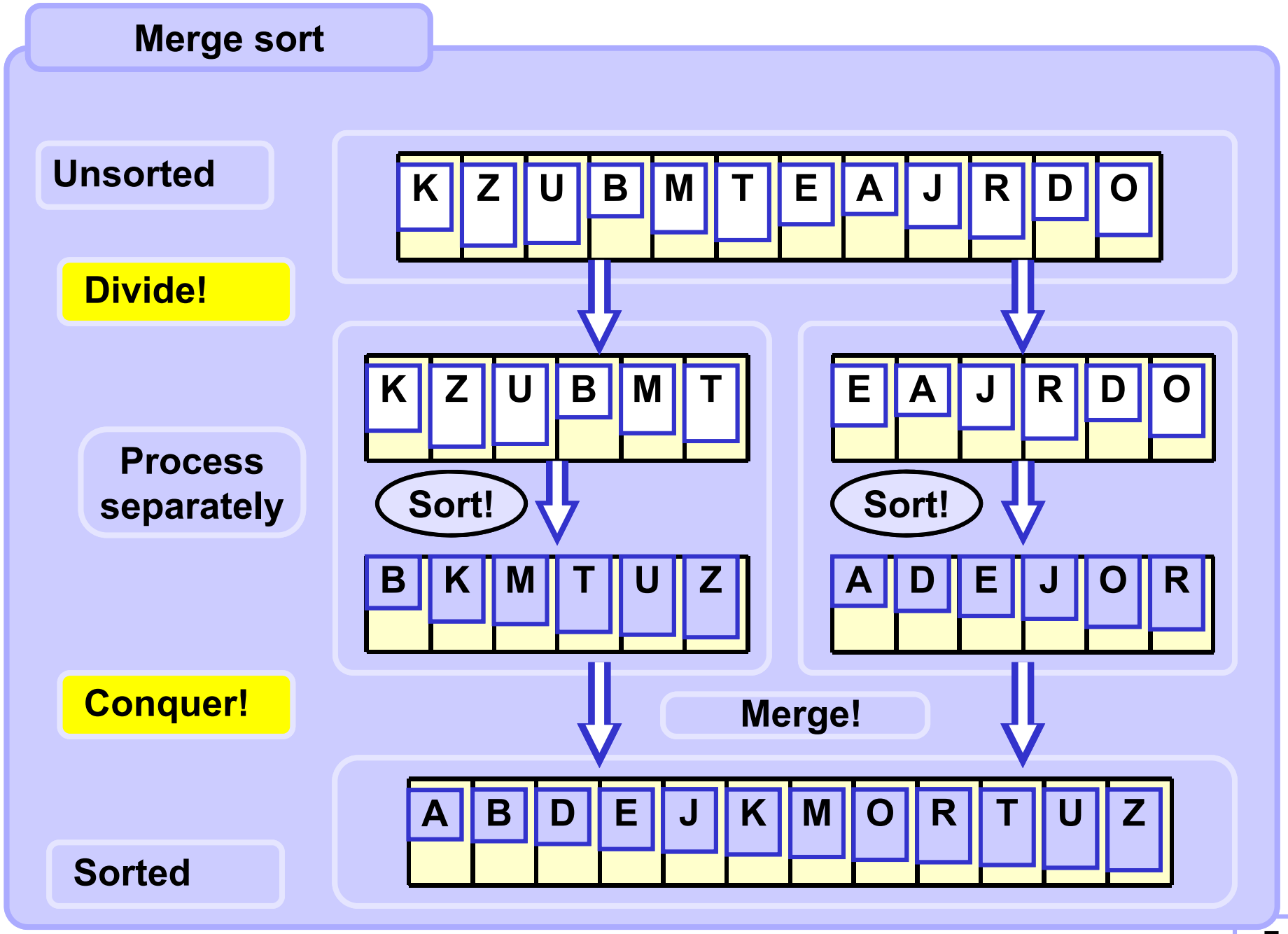

### Sorted



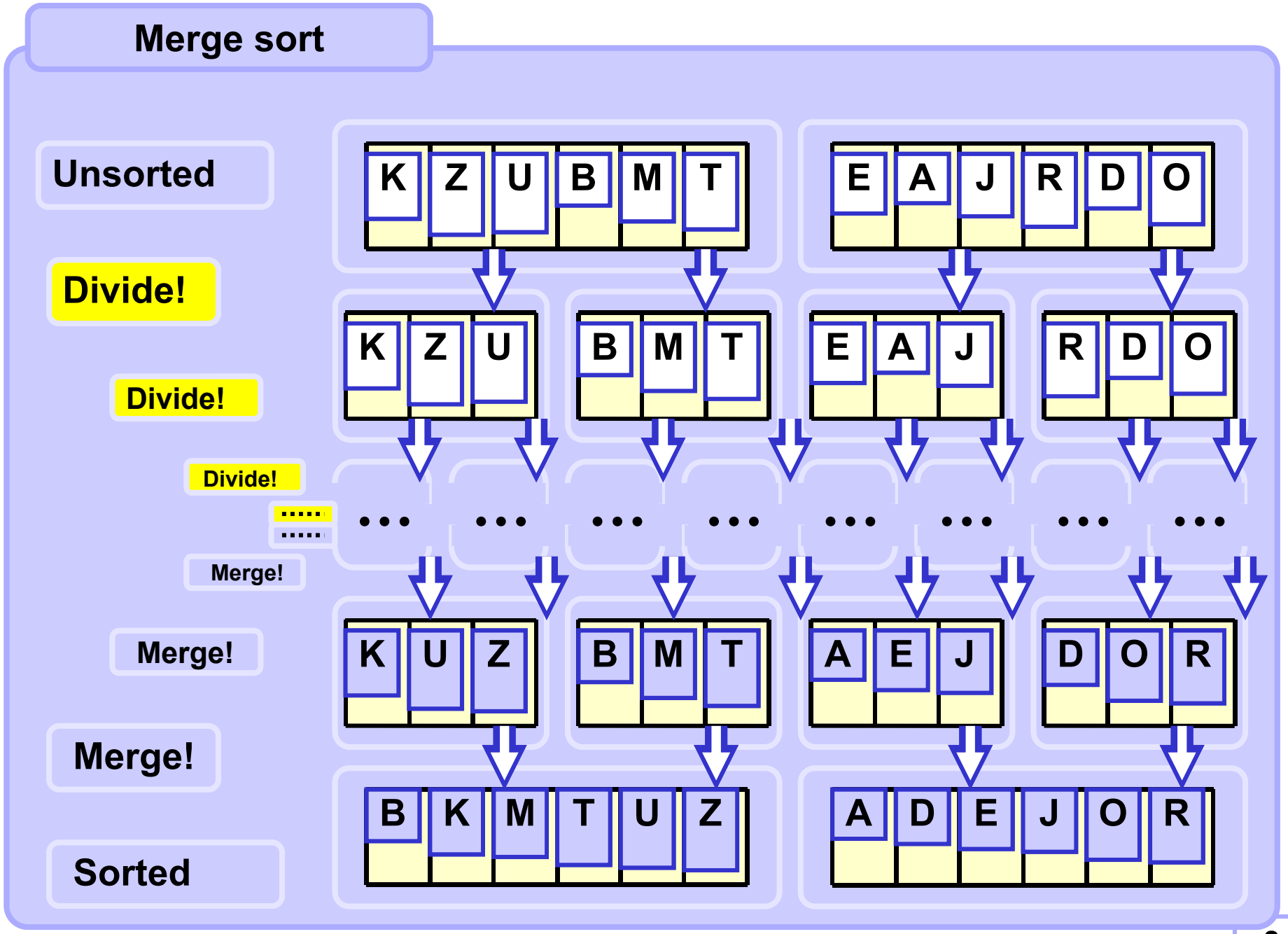
# Divide and conquer! Divide et impera!



# Merge sort


Unsorted
K Z U B M T E A J R D O
Divide!
Process separately
K Z U B M T
Sort!
B K M T U Z
E A J R D O
Sort!
A D E J O R
Conquer!
Merge!
Sorted
A B D E J K M O R T U Z

# Merge sort


Unsorted
K Z U B M T
E A J R D O
Divide!
K Z U
B M T
E A J
R D O
Divide!
Divide!
Merge!
K U Z
B M T
A E J
D O R
Merge!
Merge!
B K M T U Z
A D E J O R
Sorted

# Merge sort

```
def merge (inArr, outArr, low, high):
    half = (low+high) // 2
    i1 = low
    i2 = half + 1
    j = low;
    # compare and merge
    while i1 <= half and i2 <= high:
        if inArr[i1] <= inArr[i2]:
            outArr[j] = inArr[i1]; i1 += 1
        else:
            outArr[j] = inArr[i2]; i2 += 1
        j += 1
    # copy the rest
    while i1 <= half:
        outArr[j] = inArr[i1]; i1 += 1; j += 1
    while i2 <= high:
        outArr[j] = inArr[i2]; i2 += 1; j += 1
```

## Merge sort

```
def _mergeSort (arr, auxArr, low, high):

  if low >= high: return   # too small!
  half = (low+high) // 2

  # sort to auxArr
  _mergeSort(arr, auxArr, low, half)    # left half
  _mergeSort(arr, auxArr, half+1, high) # right half
  merge(arr, auxArr, low, high)

  # copy back from auxArr
  for i in range(low, high+1):
    arr[i] = auxArr[i]
```

# Merge sort - improved use of auxArr

```
def _mergeSortX (arr, auxArr, low, high, depth):

  if low >= high: return   # too small!
  half = (low+high) // 2

  _mergeSortX(arr, auxArr, low,    half, depth+1)
  _mergeSortX(arr, auxArr, half+1, high, depth+1)

  # note the swaping of arr and auxArr
  if depth%2 == 0: merge(auxArr, arr, low, high)
  else:            merge(arr, auxArr, low, high)


def mergeSortX (arr):
  auxArr = arr[:]  # auxArr = copy(arr)
  _mergeSortX(arr, auxArr, 0, len(arr)-1, 0)
```

## Merge sort

### Asymptotic complexity

Divide! ..... $\log_2(n)$ times $\Rightarrow$

$\Rightarrow$ Merge! ..... $\log_2(n)$ times

Divide! ..... $\Theta(1)$ operations

Merge! ..... $\Theta(n)$ operations

---

Total ..... $\Theta(n) \cdot \Theta(\log_2(n)) = \Theta(n \cdot \log_2(n))$ operations

**Asymptotic complexity of Merge sort is $\Theta(n \cdot \log_2(n))$**

# Merge sort

## Stability

Divide! ..... Does not move the elements.

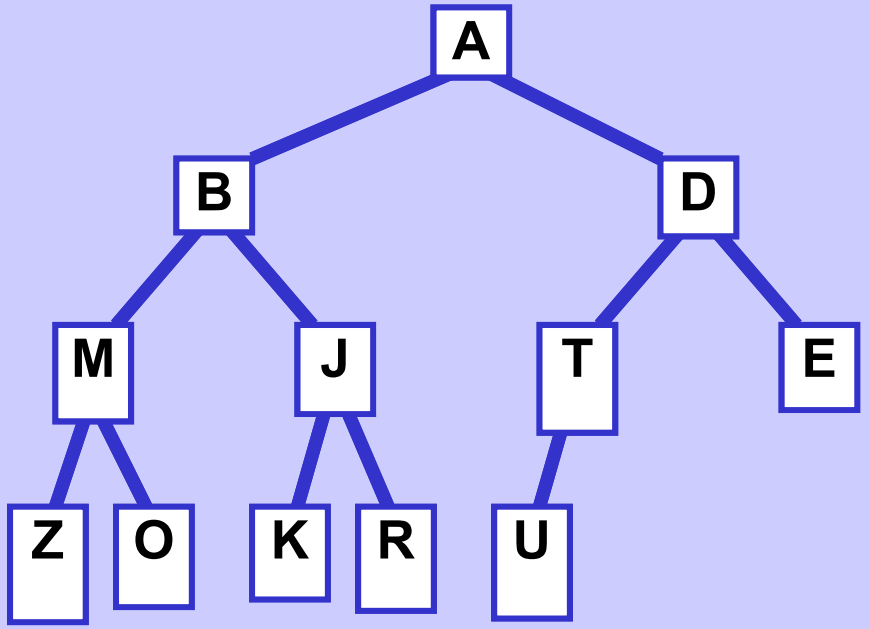
Merge! ..... " if (in[i1] <= in[i2]) { out[j] = in[i1]; …"

When the two compared and merged elements are equal, merge the left one first.

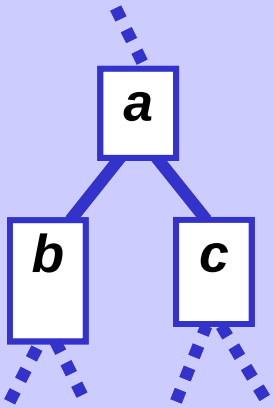
**Merge sort is stable.**

# Heap sort

## Heap



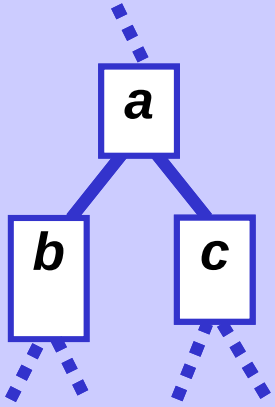

## Heap property.




$$a \leq b \;\&\&\; a \leq c$$

# Heap sort

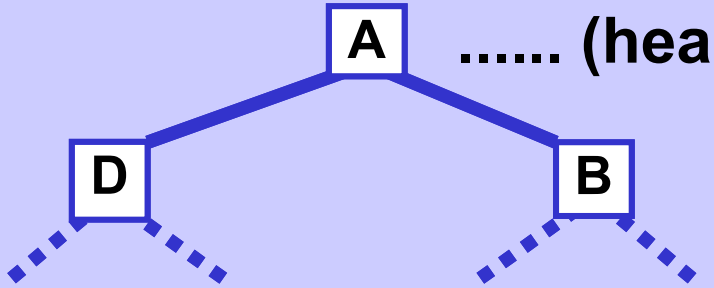
## Terminology




*a* ..... predecessor, parent of *b* *c*

*b* , *c* ..... successor, child of *a*

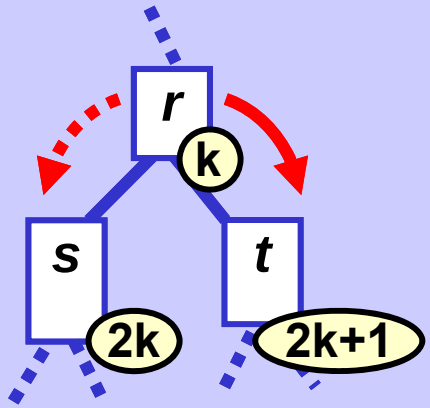
---



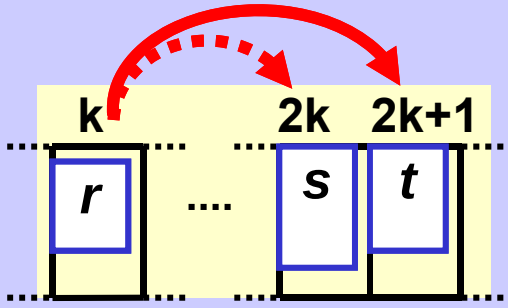

..... (heap) top

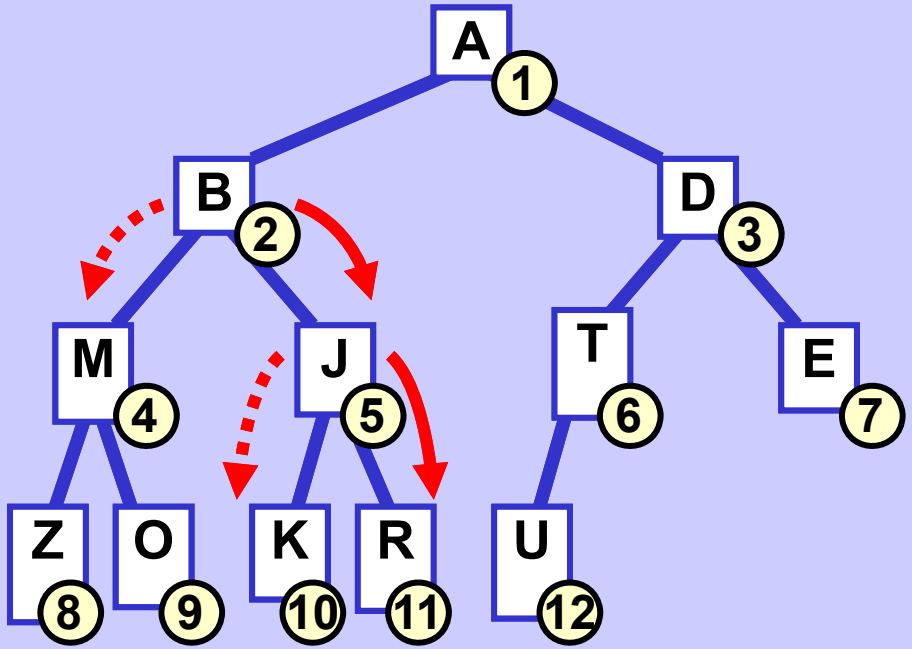
# Heap sort

## Heap stored in an array

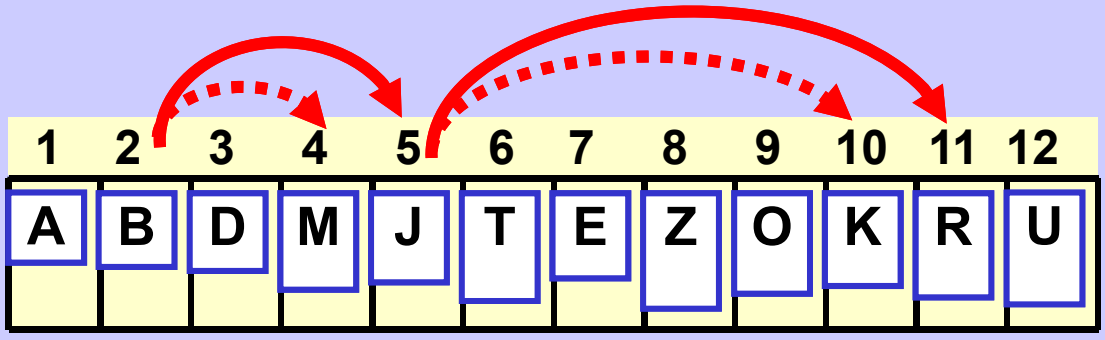



children



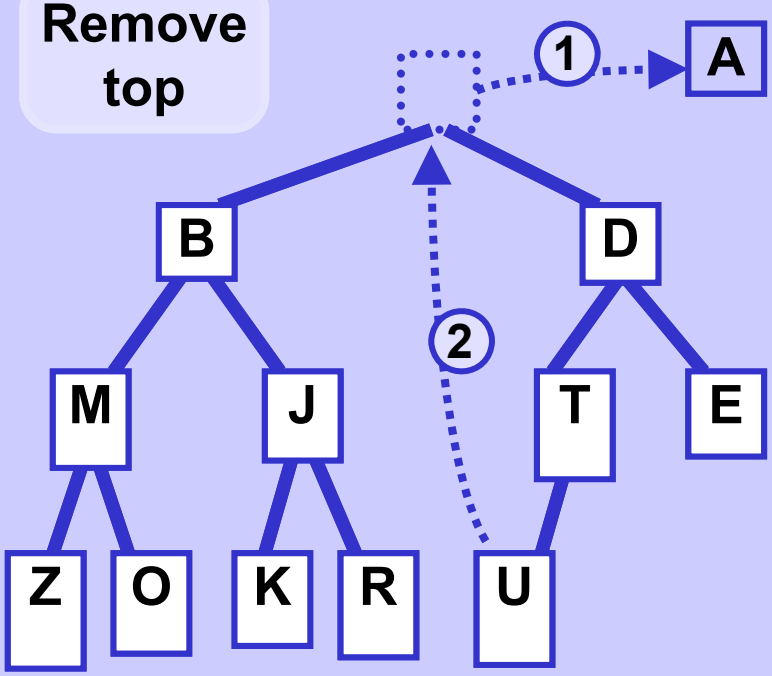

children



# Heap repair

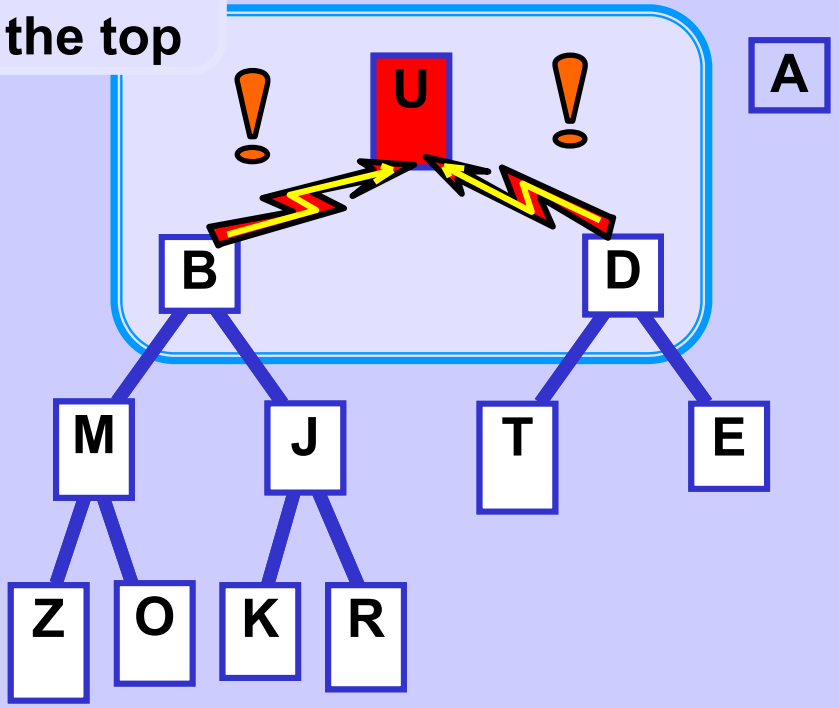
## Top removed (1)

① Remove top




② last ⟶ first

③ Put at the top



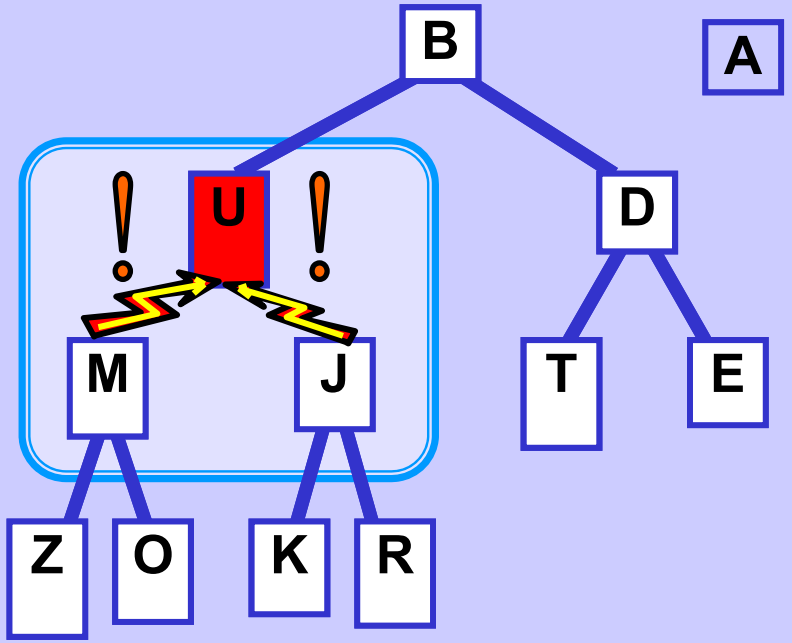

U > B, U > D, B < D

⇒ swap B ⟷ U

# Heap repair

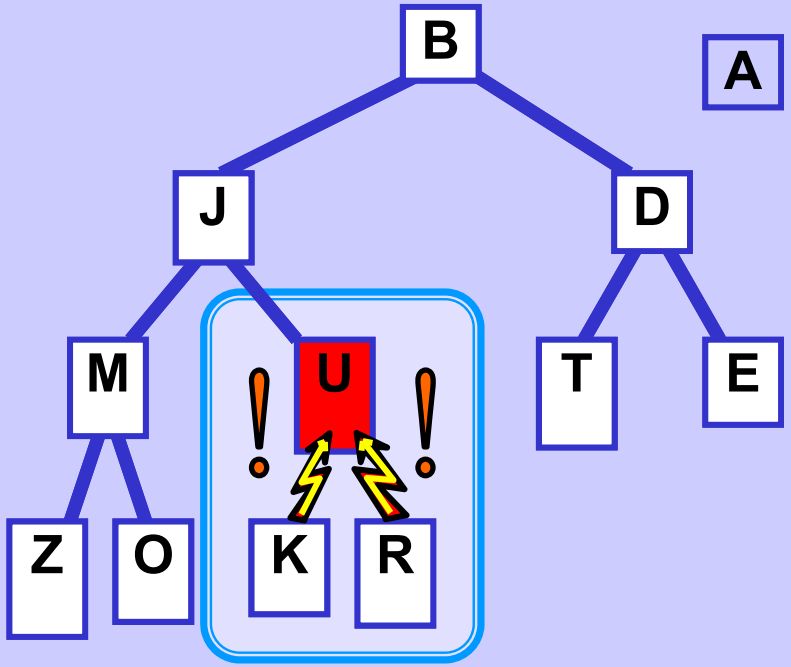
## Top removed (2)

③ **Put at the top - cont...**




$U > M,\ U > J,\ \underline{J < M}$

$\Rightarrow$ swap $J \leftrightarrow U$



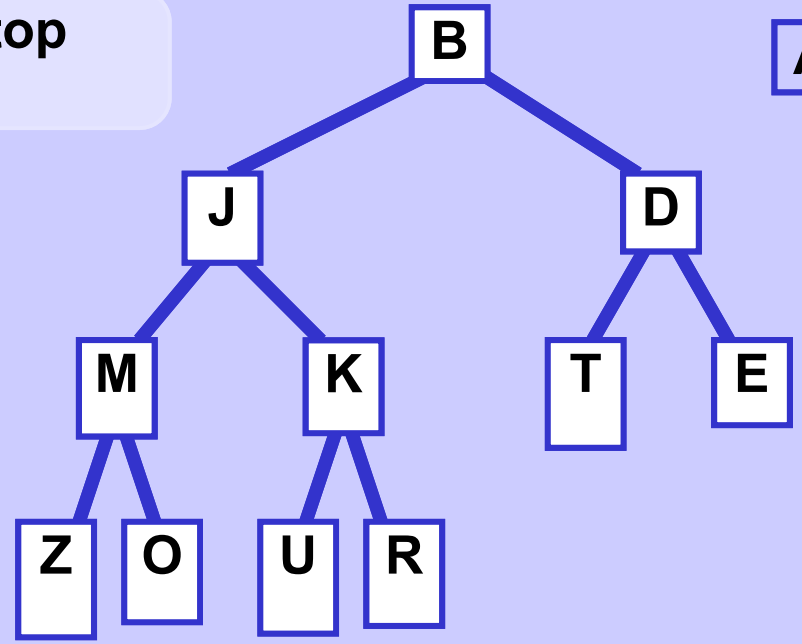

$U > K,\ U > R,\ \underline{K < R}$

$\Rightarrow$ swap $K \leftrightarrow U$

# Heap repair

## Top removed (3)

③ Put at the top - done.

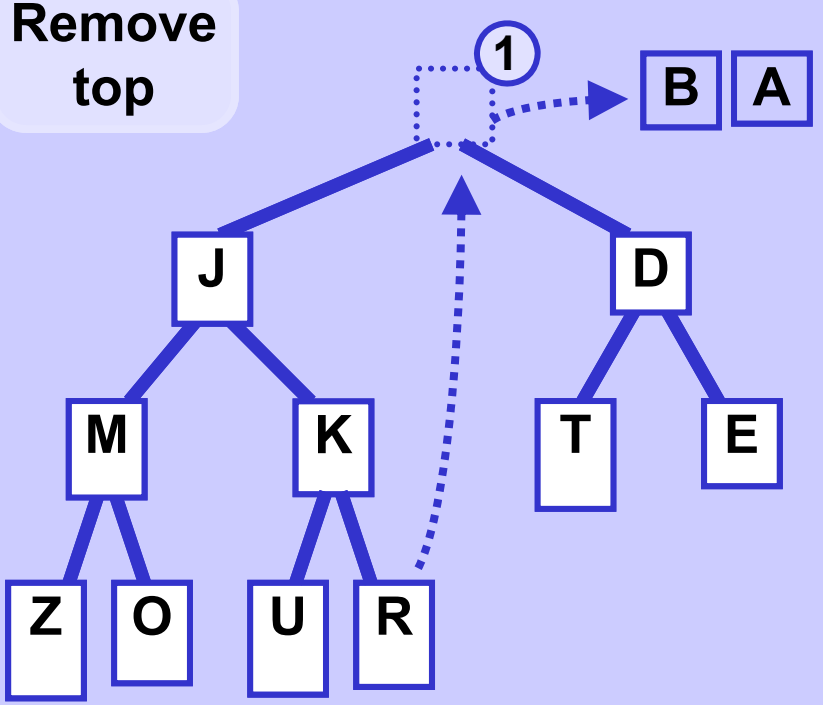



New heap

# Heap repair

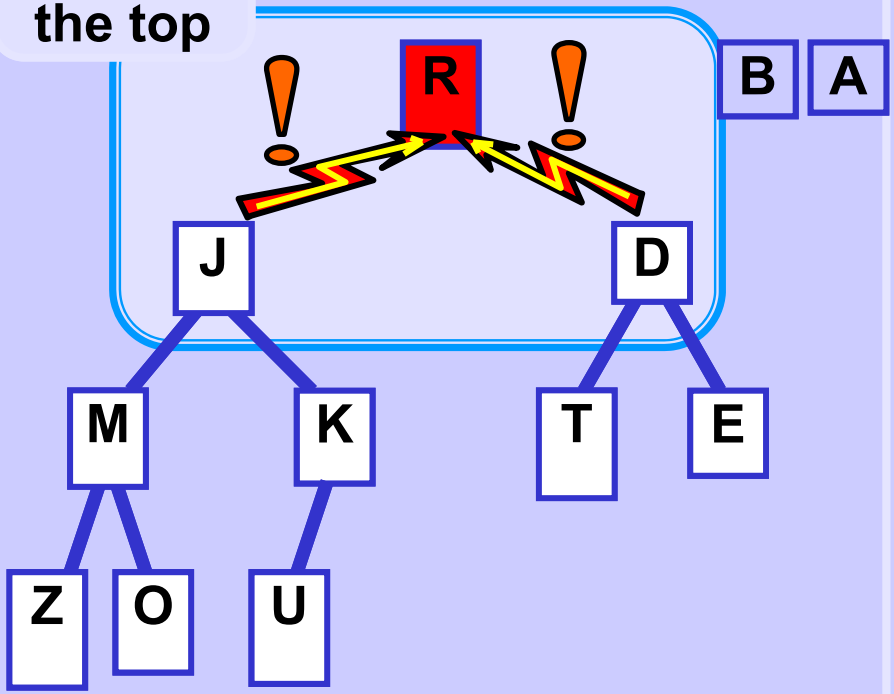
## Top removed II (1)

① Remove top




② last ⟶ first

③ Put at the top



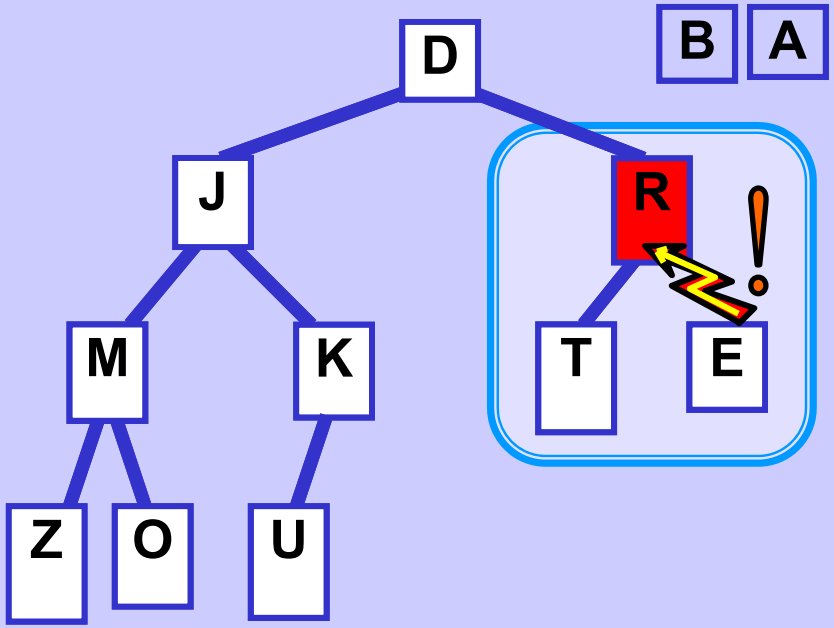

R > J, R > D, D < J

⇒ swap D ⟷ R

# Heap repair

## Top removed II (2)

③ Put at the top - cont.

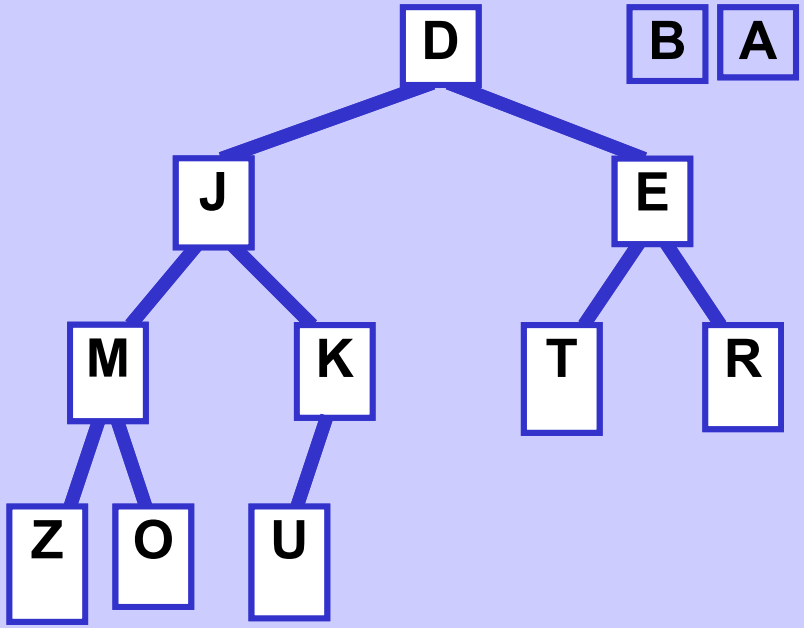



$R < T,\ R > E$

$\Rightarrow$ swap $E \leftrightarrow R$

## Top removed II (3)

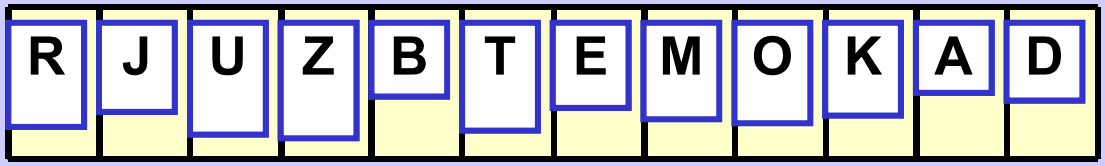
③ Put at the top - done.



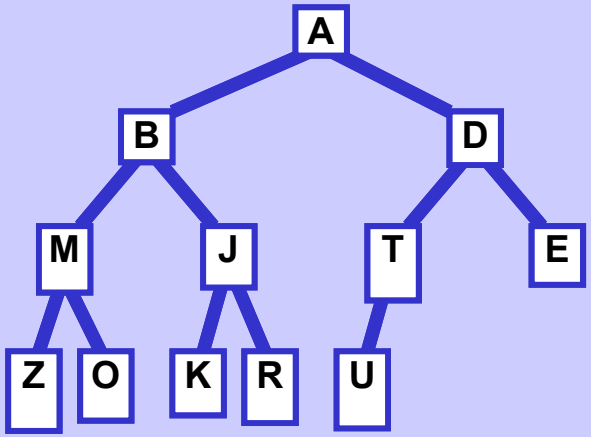

New heap

# Heap sort

**I** Unsorted

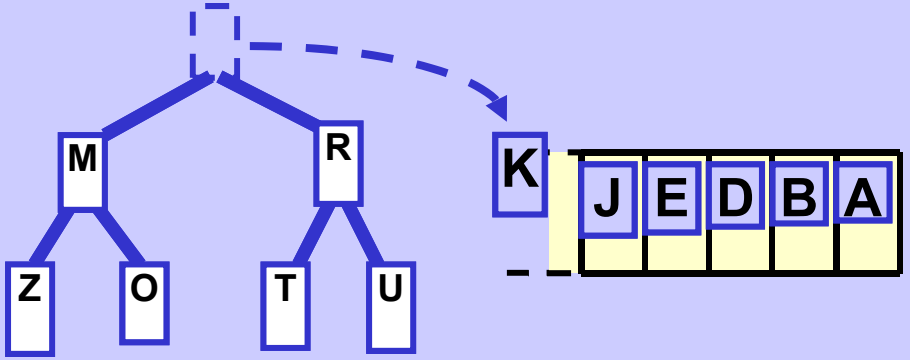



**II** Make a heap

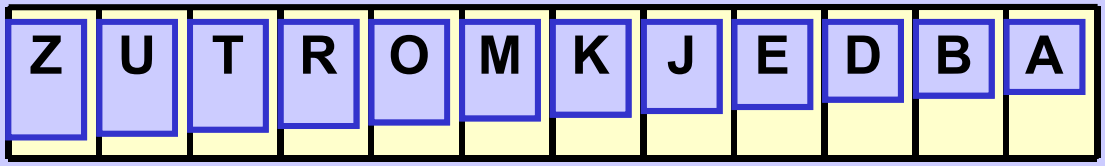



**III**

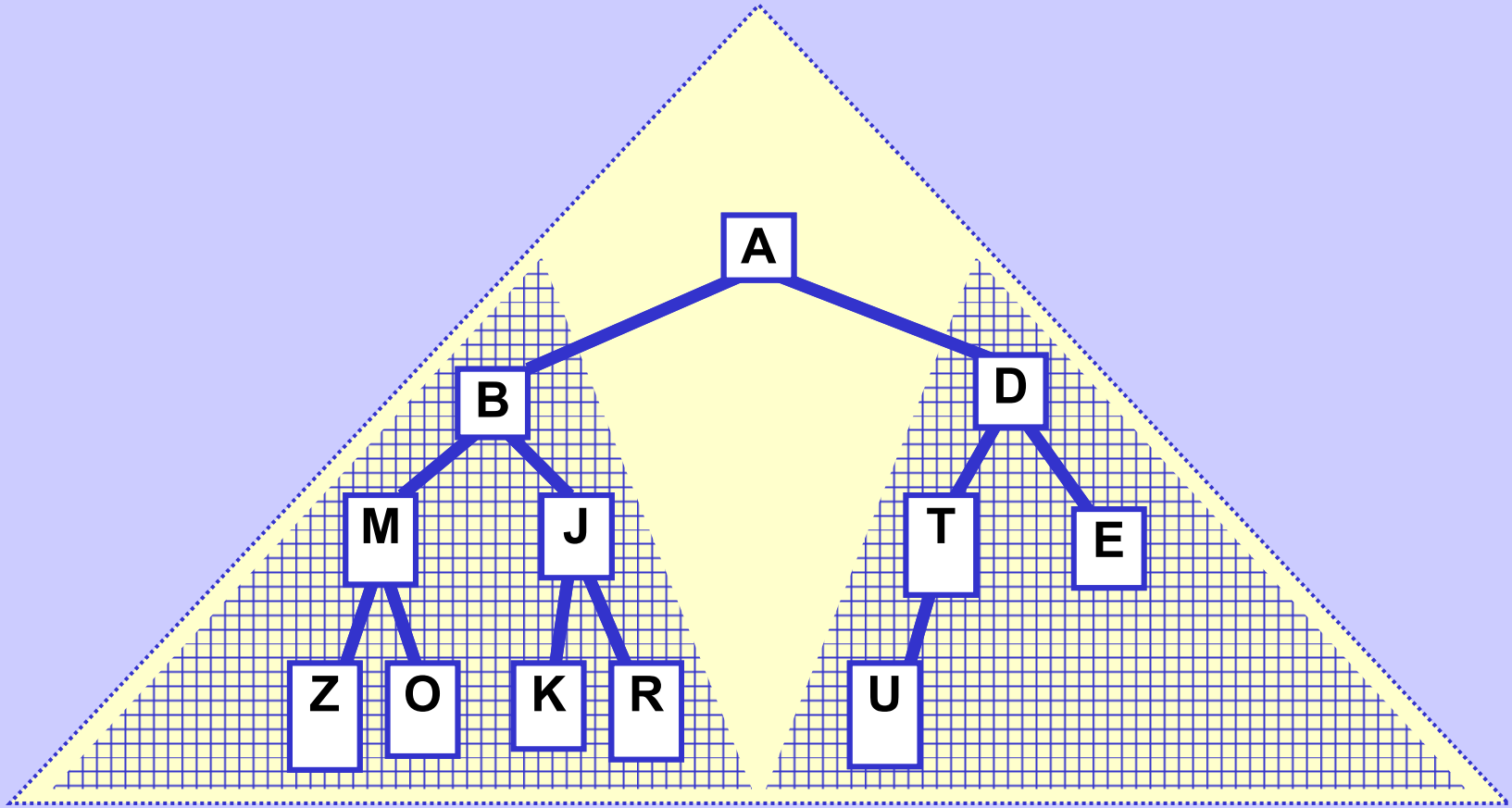
```
for (i = 0; i < n; i++)
    a[i] = “remove top”;
```

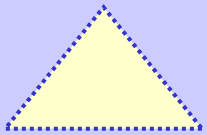



**IV** Sorted

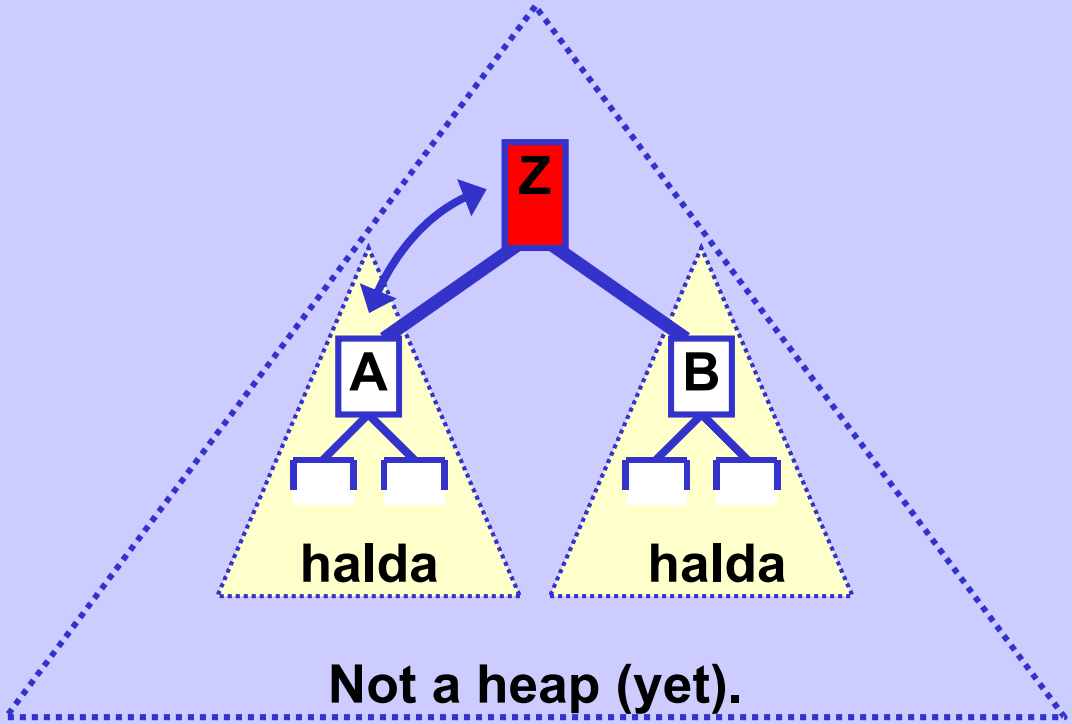


# Recursive property of "Being a heap"




 Is a heap $\Rightarrow$  is a heap and  is a heap.

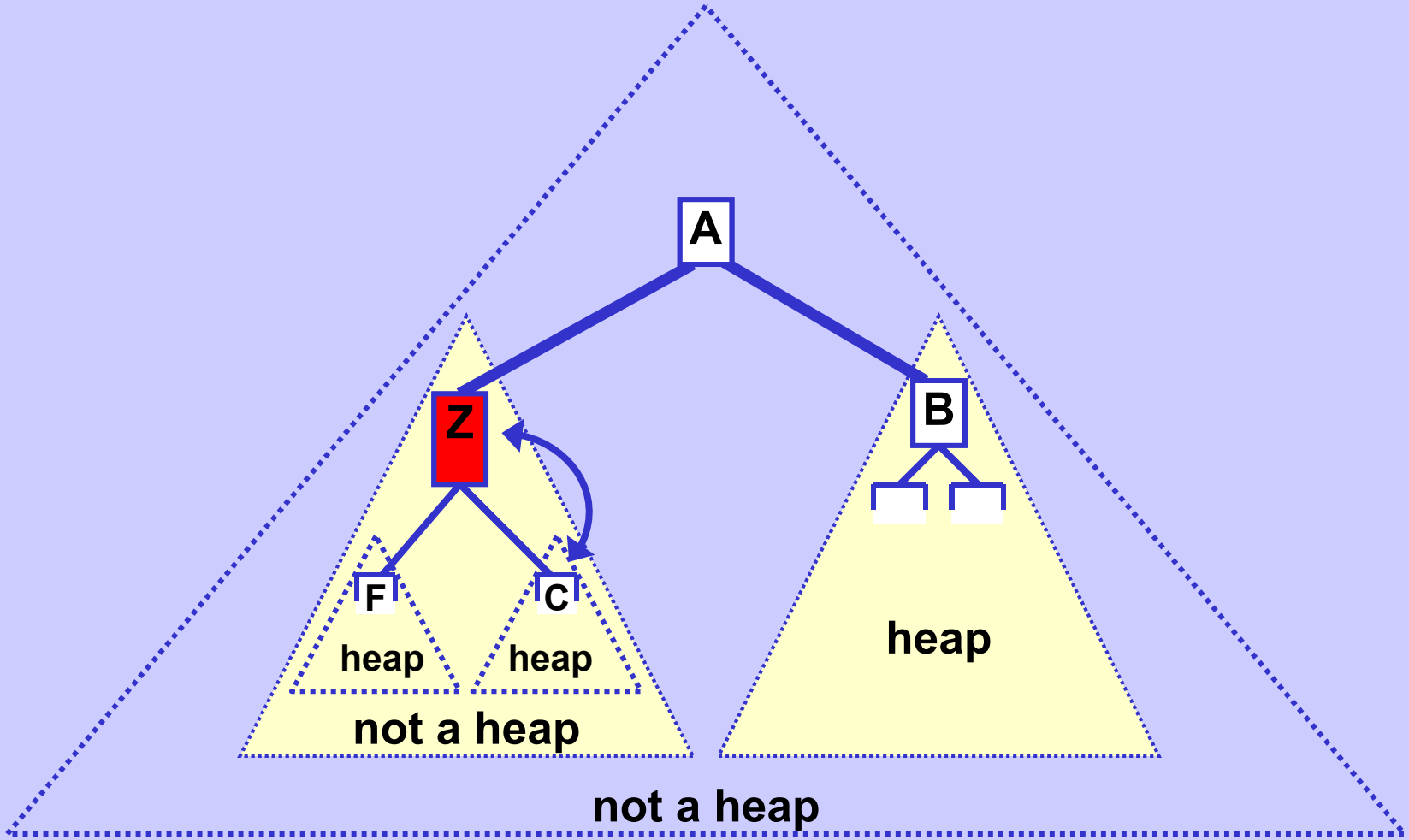
# Make one bigger heap from two smaller ones



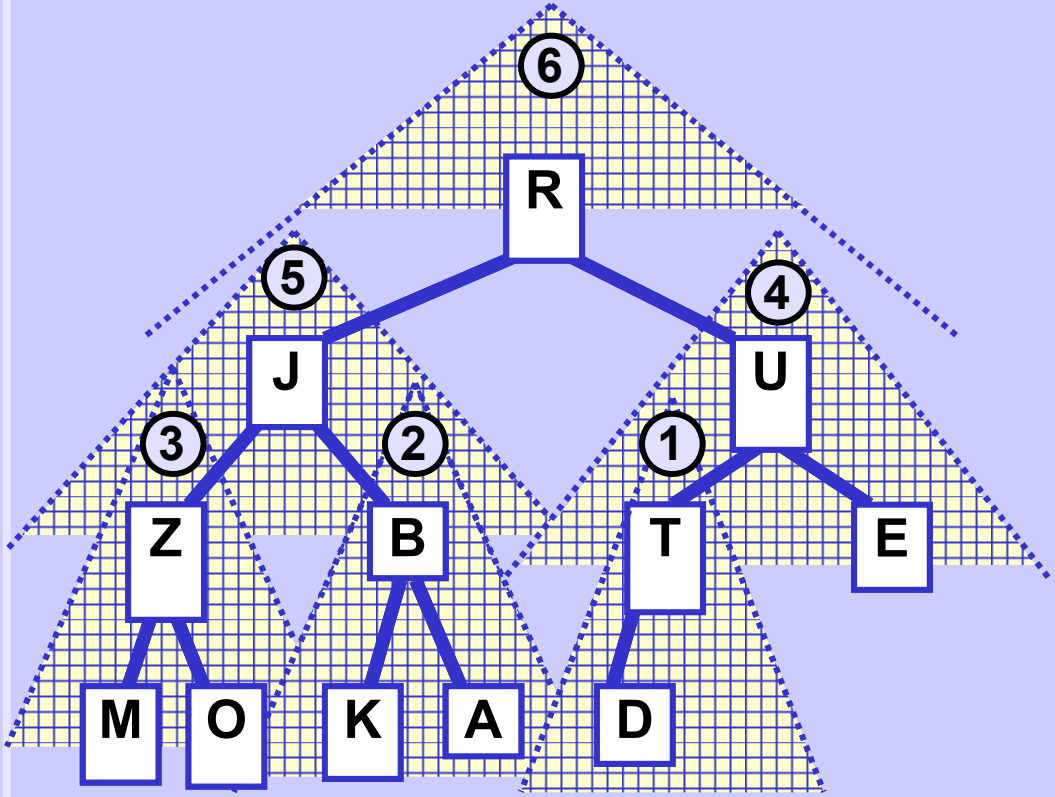

$Z > A$ or $Z > B$

$\Rightarrow$ swap: $Z \leftrightarrow \min(A,B)$

# Make one bigger heap from two smaller ones



# Create a heap




Make a heap in ① ...

... make a heap in ② ...

... make a heap in ③ ...

... make a heap in ④ ...

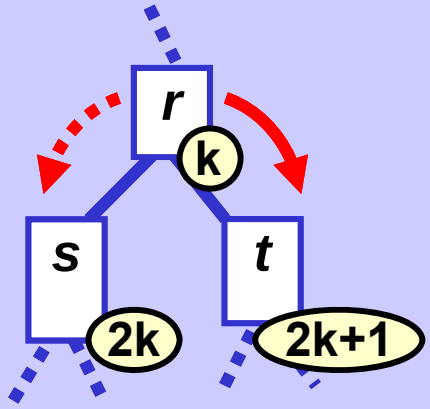
... make a heap in ⑤ ...

... make a heap in ⑥ ...

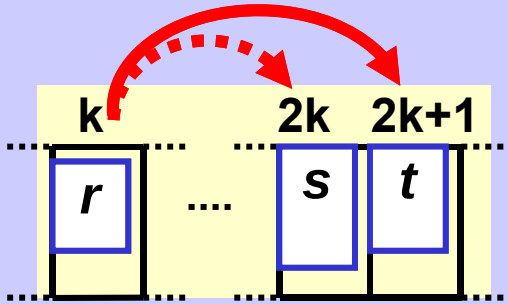
... and the whole heap is complete.

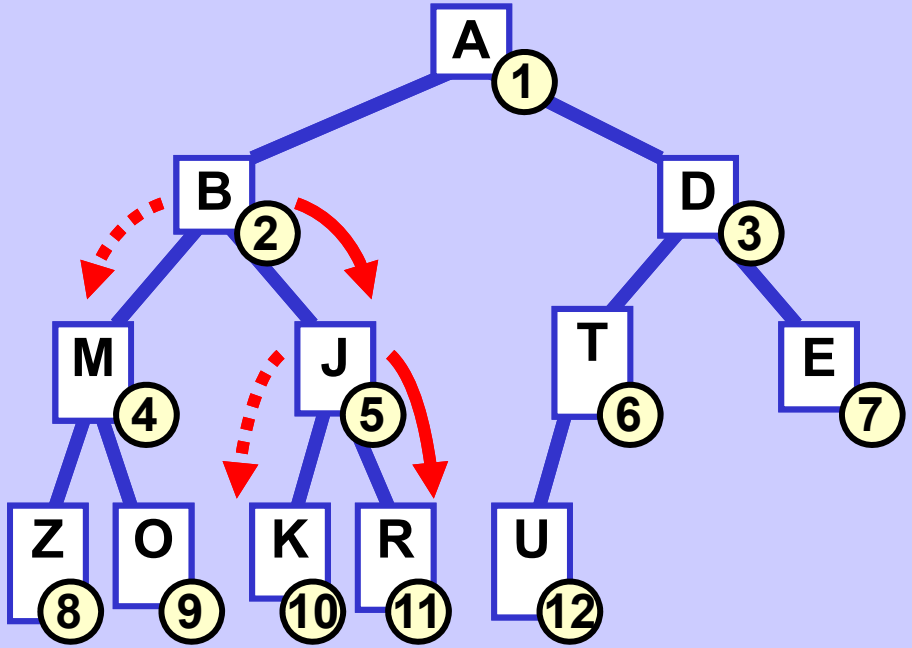
# Heap in an array

## Heap stored in an array

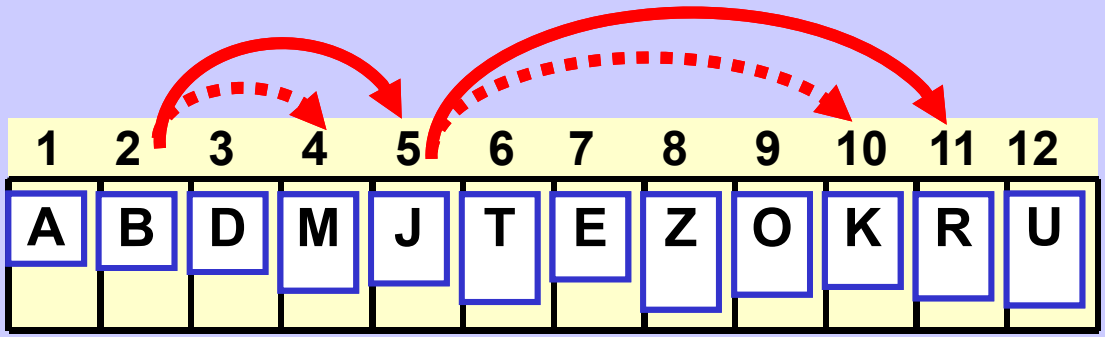



children




children



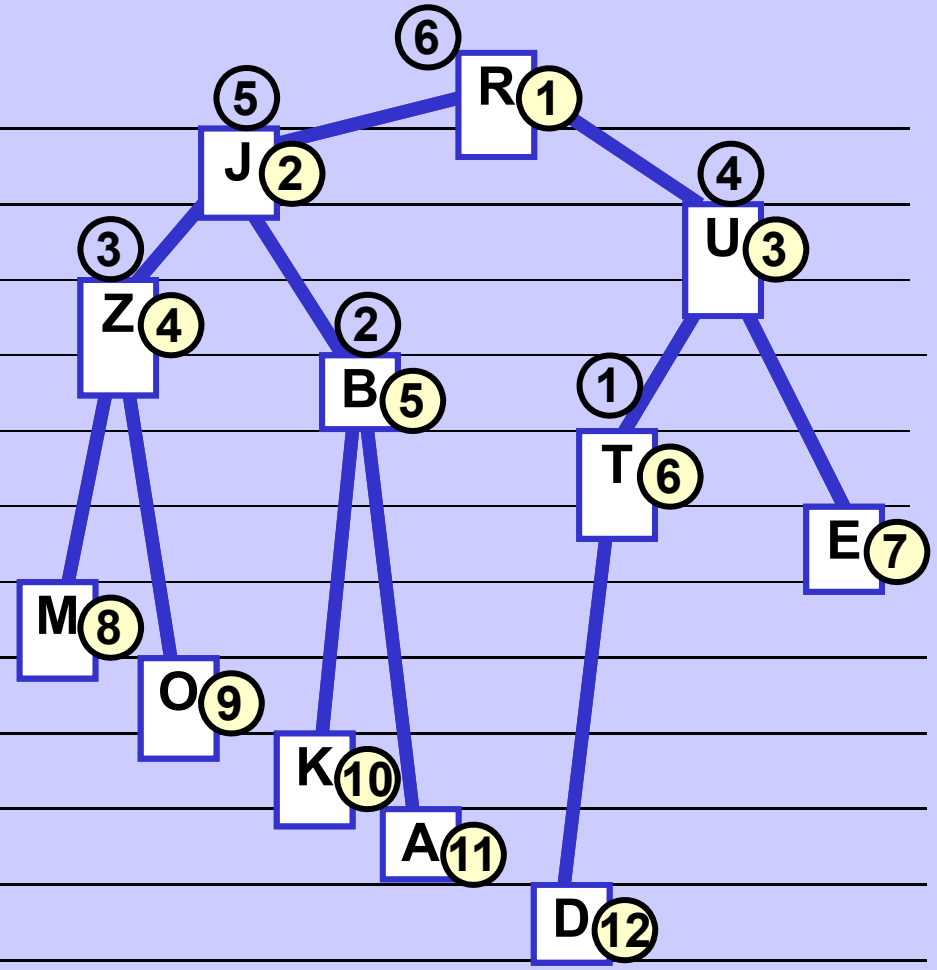
# Heap in an array

## Array

| | | |
|---|---|---|
| 6 | 1 | R |
| 5 | 2 | J |
| 4 | 3 | U |
| 3 | 4 | Z |
| 2 | 5 | B |
| 1 | 6 | T |
| | 7 | E |
| | 8 | M |
| | 9 | O |
| | 10 | K |
| | 11 | A |
| | 12 | D |

## Array elements ordered randomly

## Not a heap

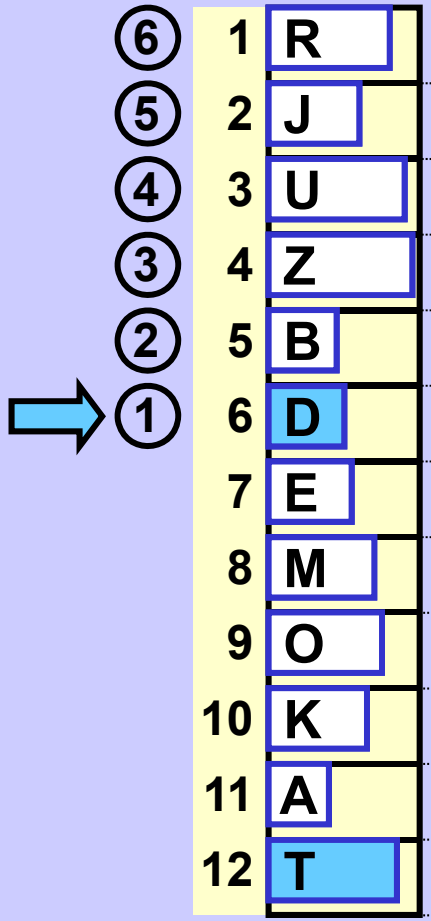


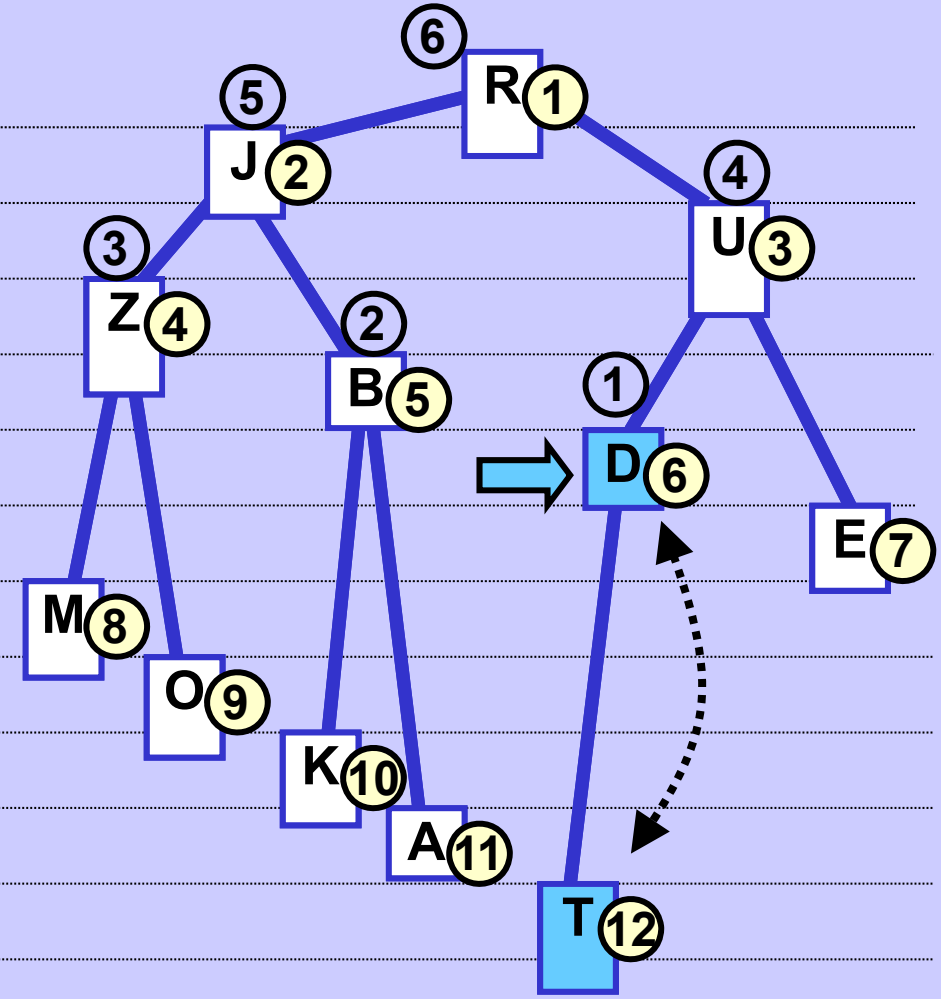
# Heap in an array

Moves

Currently created heap

## Array



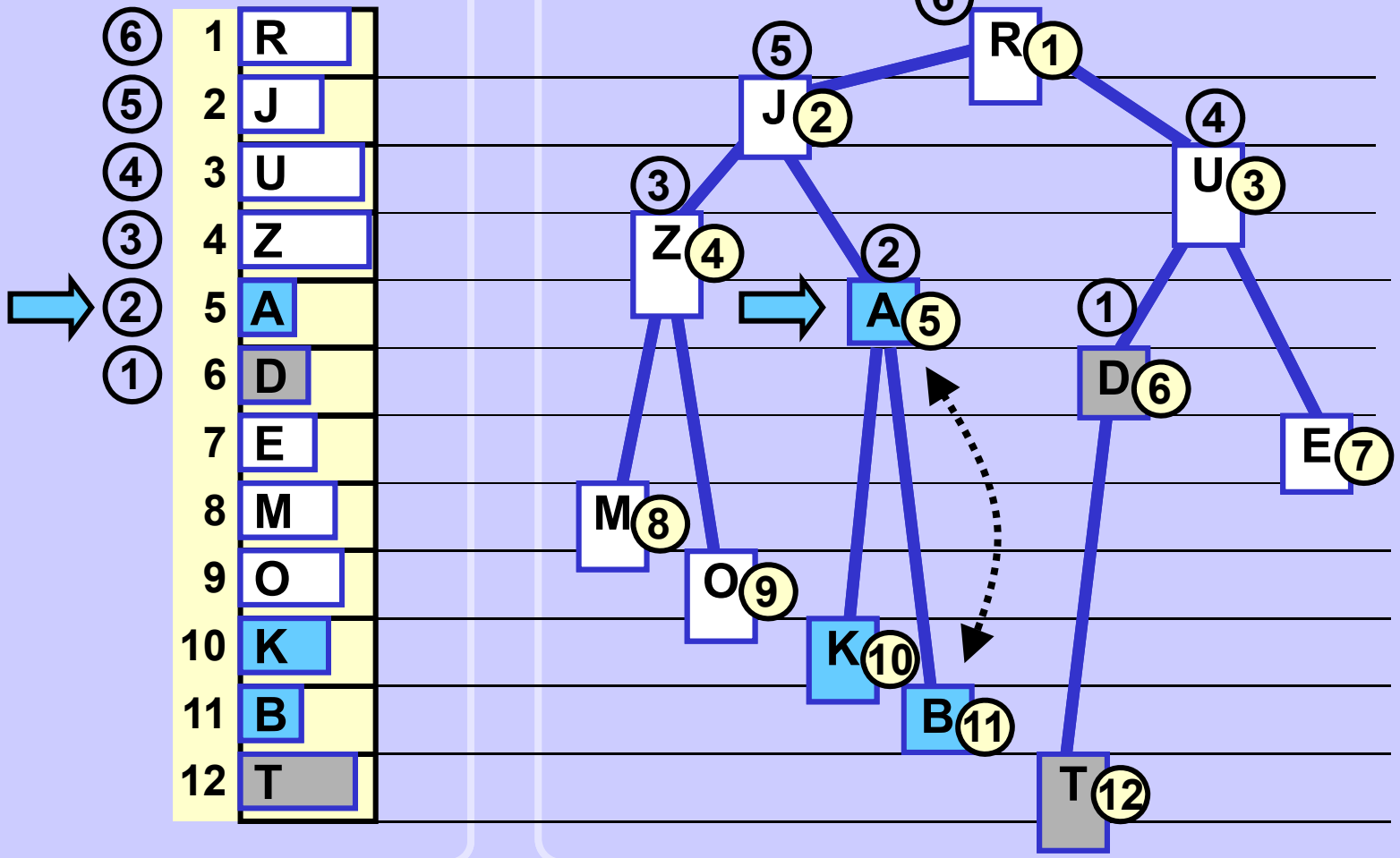
# Heap in an array

Earlier heap(s)

Moves

Currently created heap

## Array



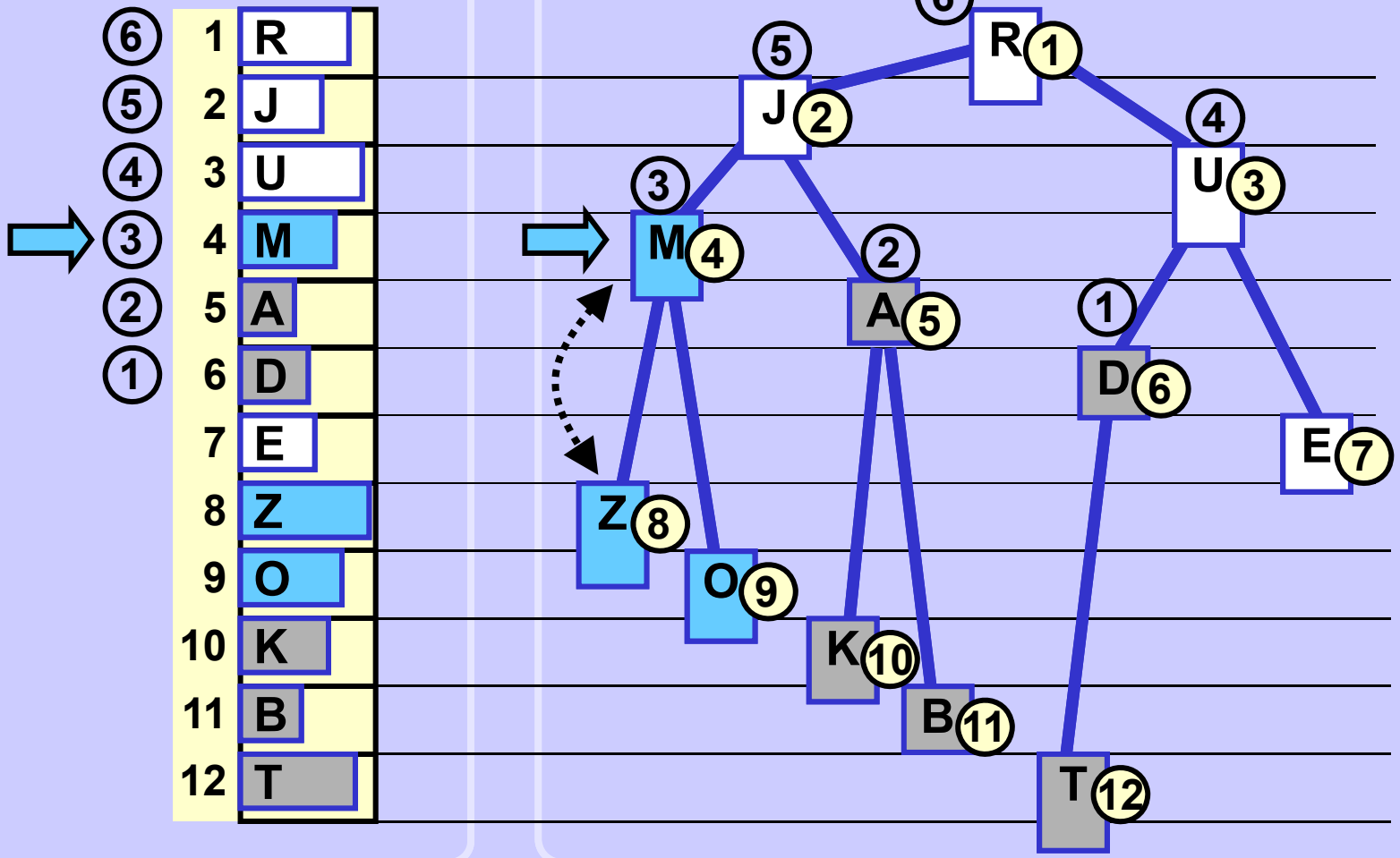
# Heap in an array

Earlier heap(s)

Moves

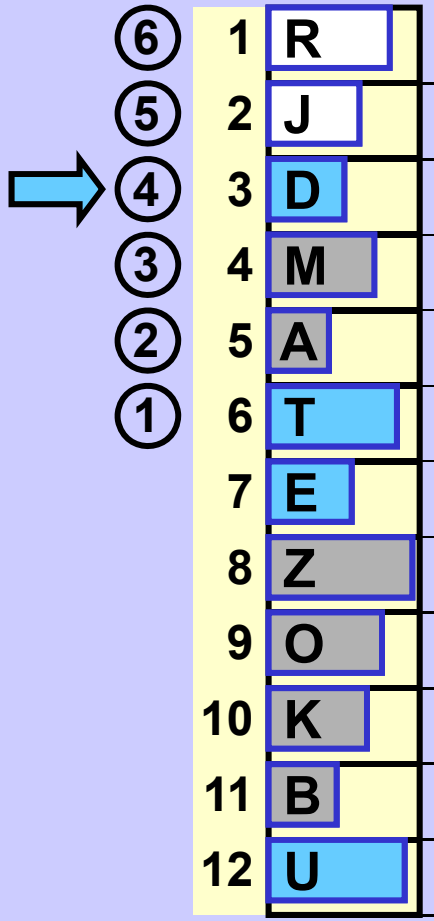
Currently created heap

## Array



# Heap in an array

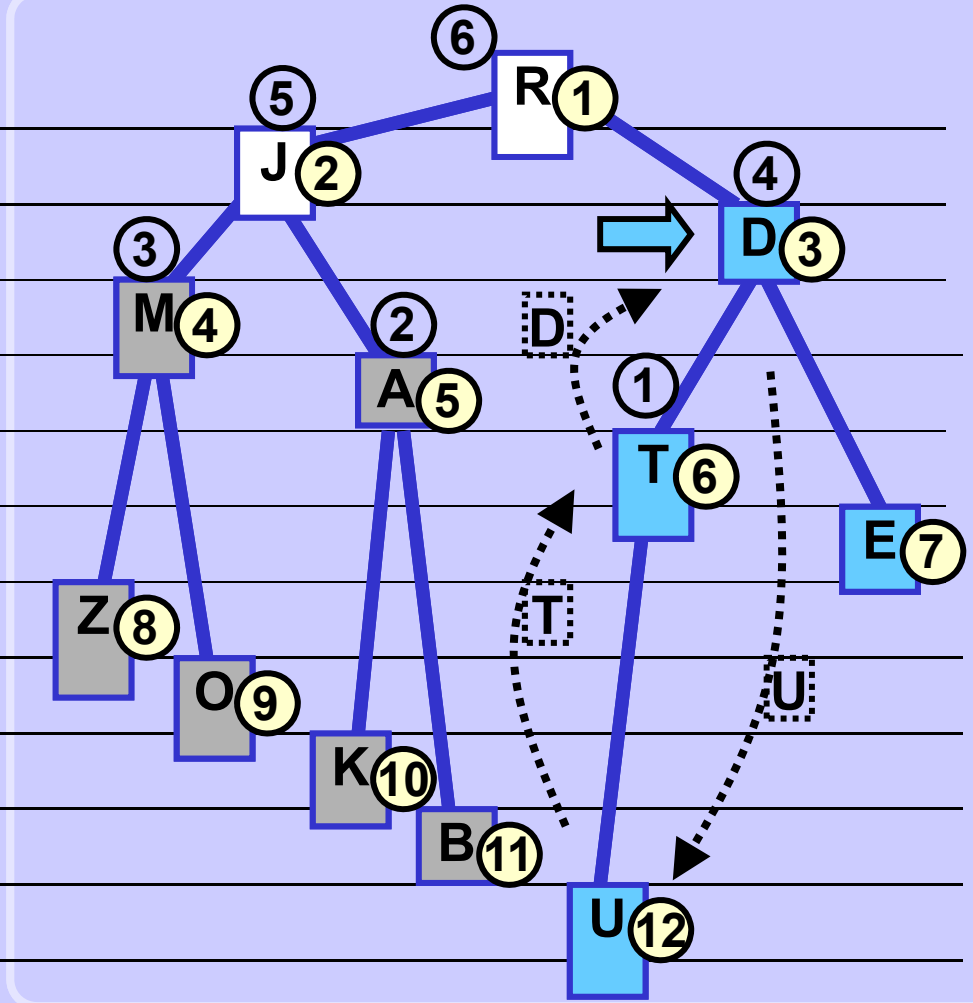
## Array




Earlier heap(s)

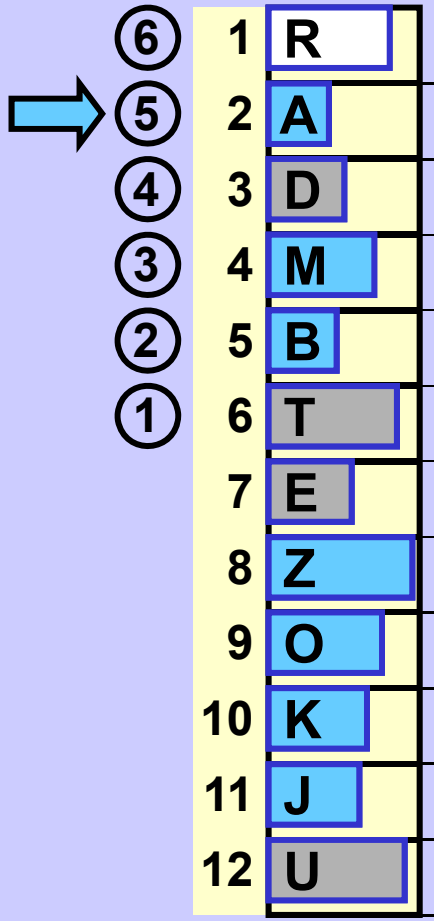
Moves

Currently created heap



# Heap in an array

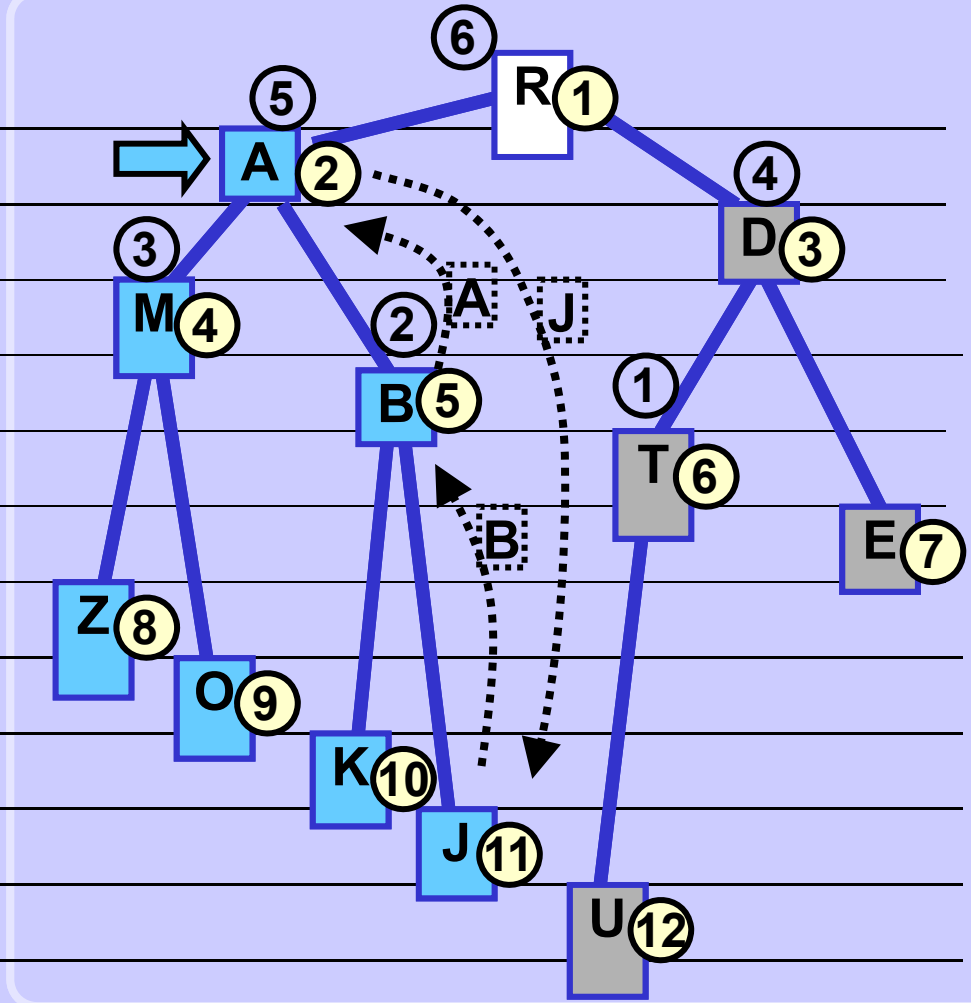
## Array



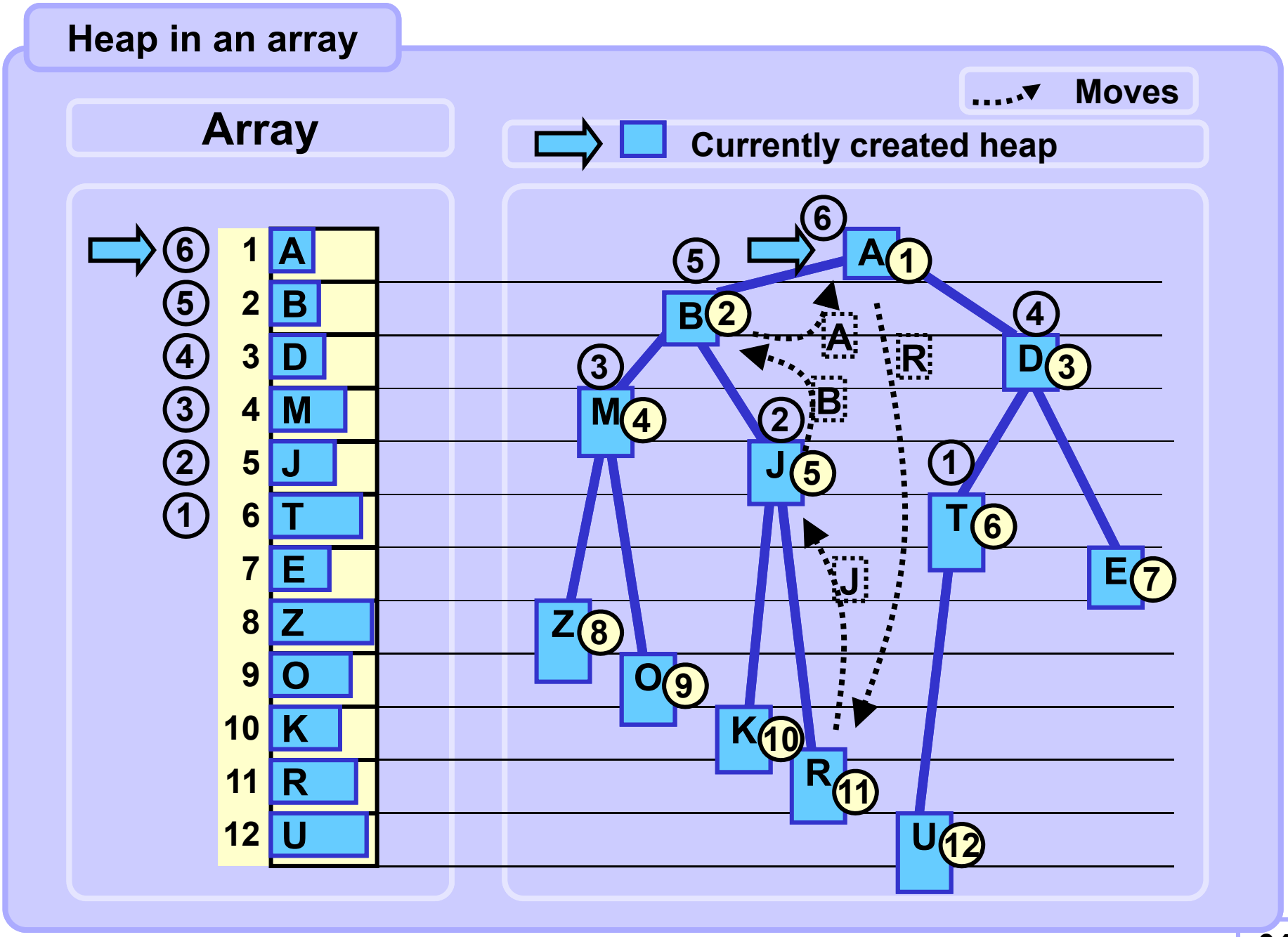

Earlier heap(s)

Moves

Currently created heap

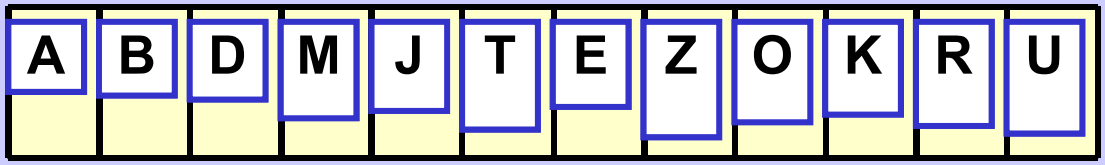


# Heap in an array

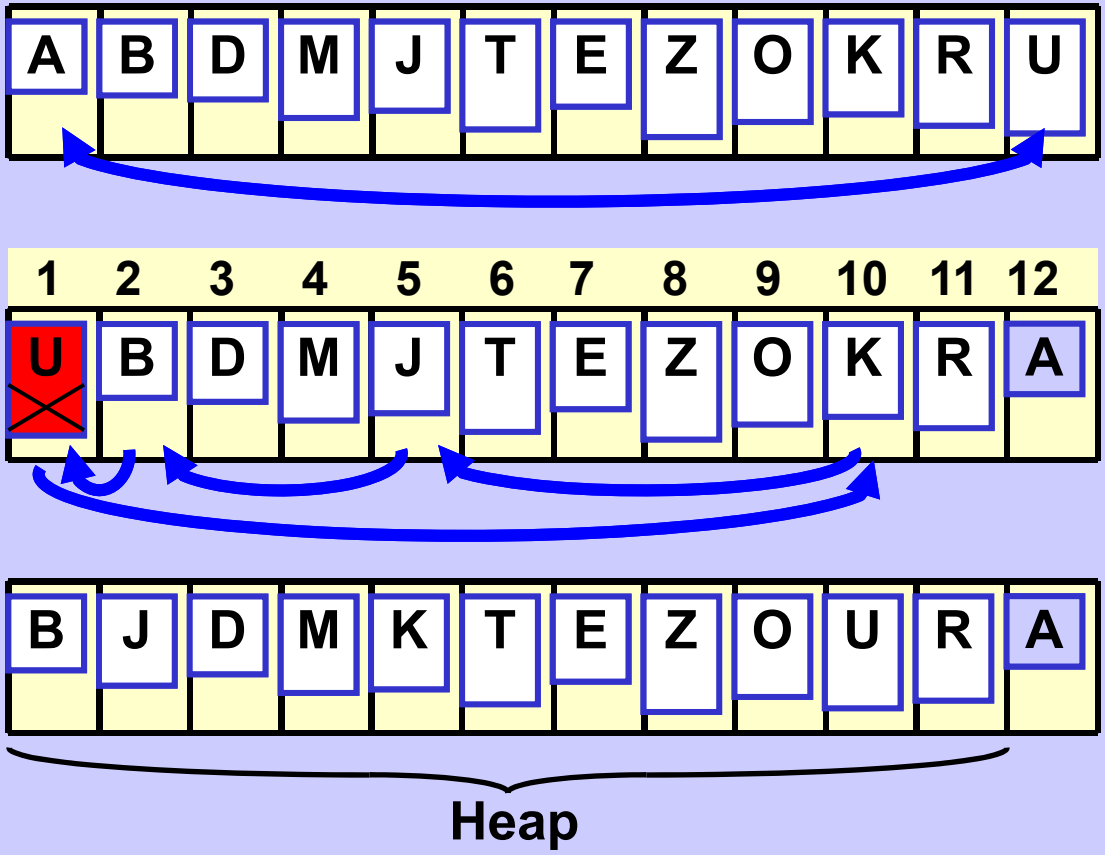


# Heap sort

Heap

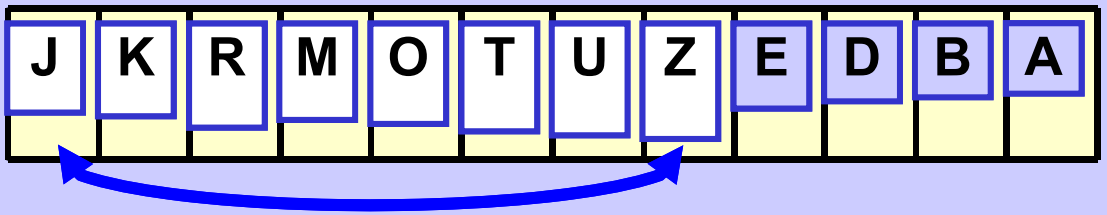


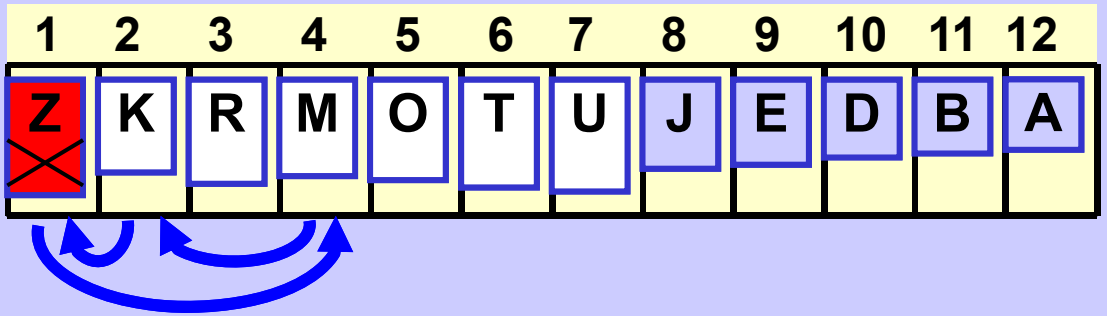

Step1 1

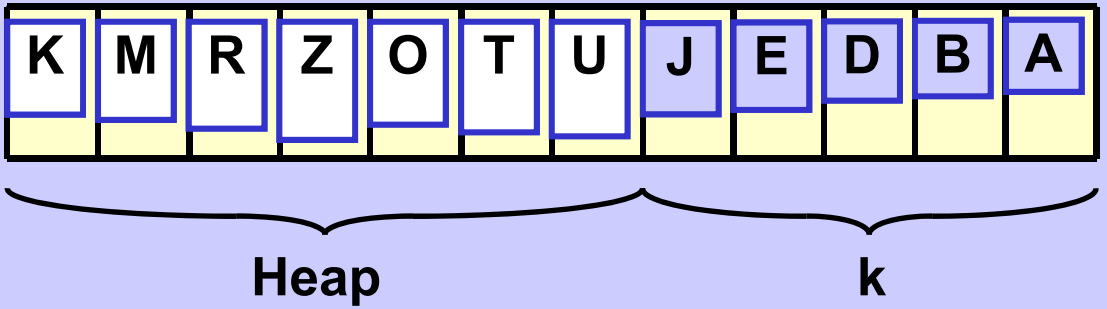


# Heap sort

## Step k



## Heap sort

```
# beware!  array is arr[1] ... arr[n]

def heapSort (arr):
    n = len(arr)-1

    # create a heap
    for i in range(n//2, -1, -1): # progress backwards!
        repairTop(arr, i, n)

    for i in range(n, 1, -1):     # progress backwards!
        swap(arr, 1, i)
        repairTop(arr, 1, i-1)
```

## Heap sort

```
def repairTop (arr, top, bottom):
    i = top     # arr[2*i] and arr[2*i+1]
    j = i*2     # are successors of arr[i]
    topVal = arr[top]

    # try to find a successor < topVal
    if j < bottom and arr[j] > arr[j+1]: j += 1

    # while successors < topVal move successors up
    while j <= bottom and topVal > arr[j]:
        arr[i] = arr[j]
        i = j; j = j*2      # move to next suceessor
        if j < bottom and arr[j] > arr[j+1]: j += 1

    # put topVal to its correct place
    arr[i] = topVal
```

# Heap sort

repairTop operation worst case ... $\log_2(n)$ (n = heap size)

make a heap ... n/2 repairTop calls

$\log_2(n/2) + \log_2(n/2+1) + \ldots + \log_2(n) \leq (n/2)(\log_2(n)) = \underline{\underline{O(n \cdot \log_2(n))}}$

sort the heaps ... n-1 repairTop calls, worst case:

$\log_2(n) + \log_2(n-1) + \ldots + 1 \leq n \cdot \log_2(n) = O(n \cdot \log_2(n))$

surprisingly, also the best case = $\underline{\underline{\Theta(n \cdot \log_2(n))}}$

total ... make a heap + sort the heap = $\underline{\underline{\Theta(n \cdot \log_2(n))}}$

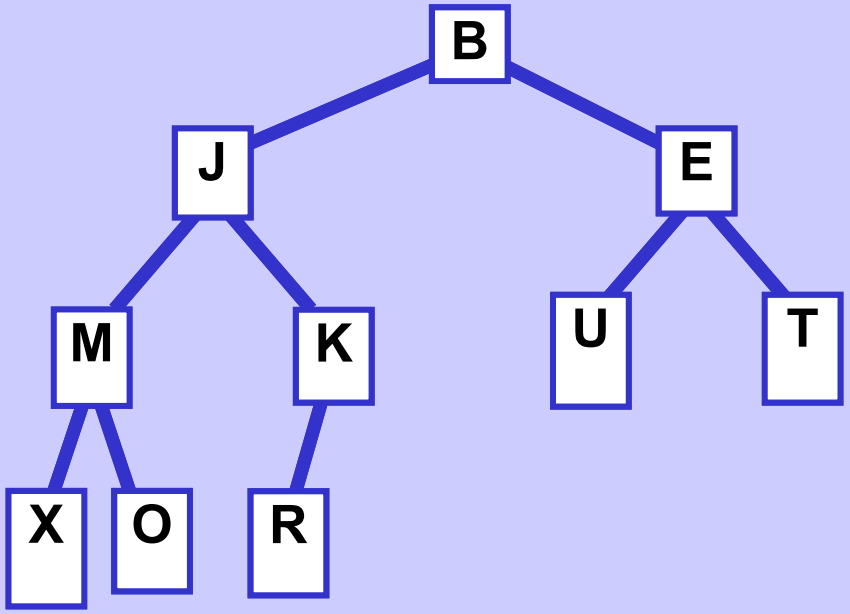
Asymptotic complexity of Heap sort is $\Theta(n \cdot \log_2(n))$.

Heap sort is not stable.

# Priority queue

**Operations**
- **Insert or Enqueue**
- **Front, Top, Peek -- read topmost element**
- **Dequeue, Pop, Poll -- delete topmost element.**

**The element with the smallest value (biggest value in max-heaps) of all elements in the heap is always at the top.**




**Priority queue might be implemented using a heap.**

**Officially:**

**" A *binary* heap".**

# Priority queue implemented with binary heap -- operations

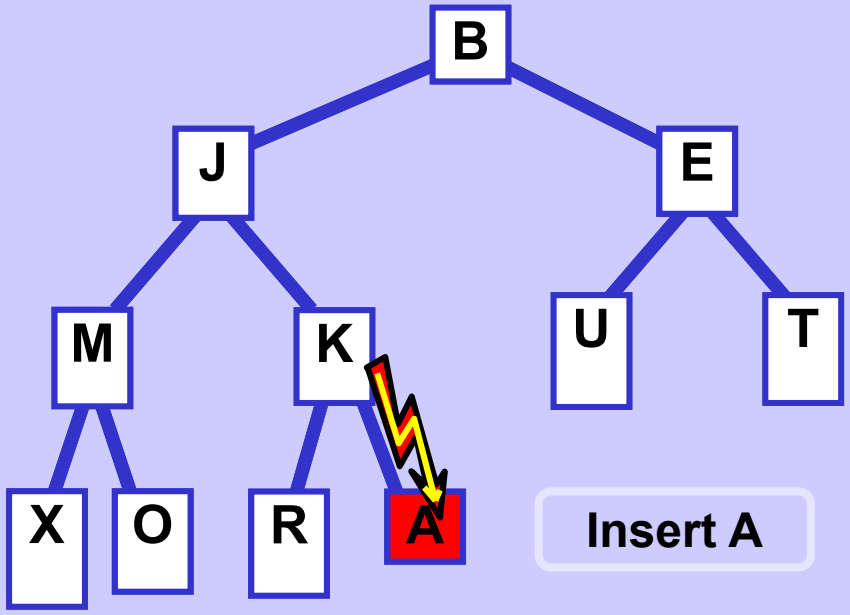
Read the topmost element (Front, Top, Peek, ...) .

Obvious.

Delete the topmost element (Dequeue, Pop, Poll, ...) =
Remove the top and repair the heap.

As before.

Insert an element to the queue (Insert, Enqueue, ... )




Insert A

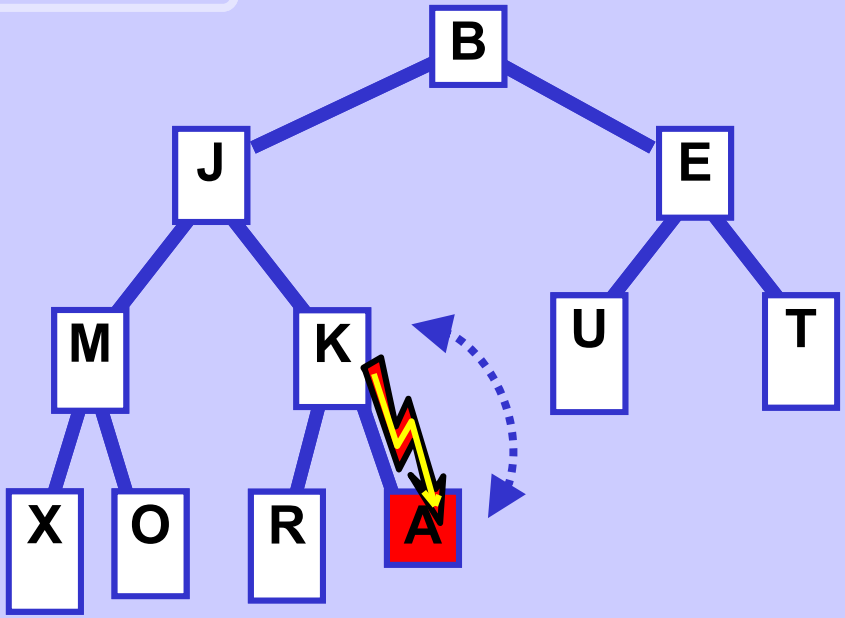
Insert the element
at the end of the queue
(end of the heap).

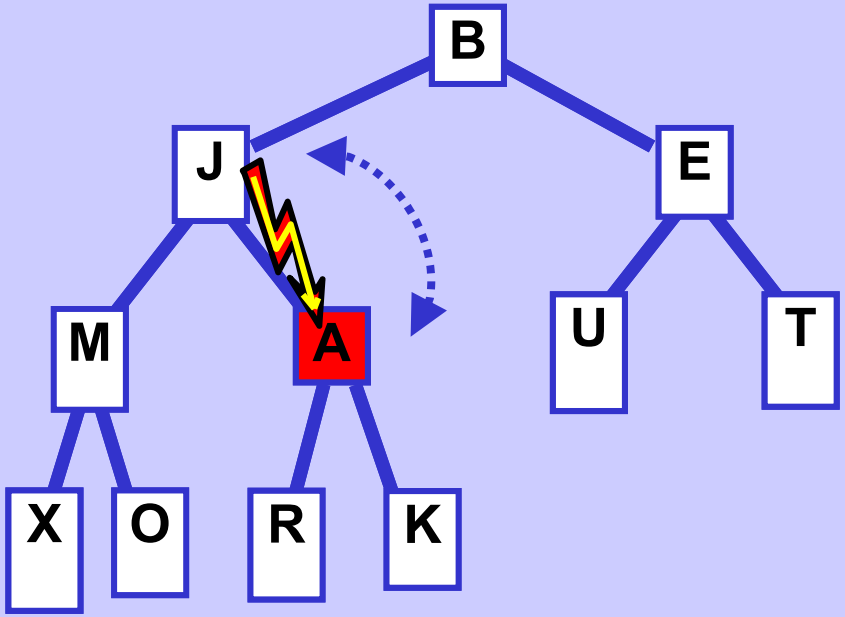
In most cases, this
violates the heap property

and the heap has to be
repaired.

# Priority queue implemented with binary heap -- Insert

## Insert A



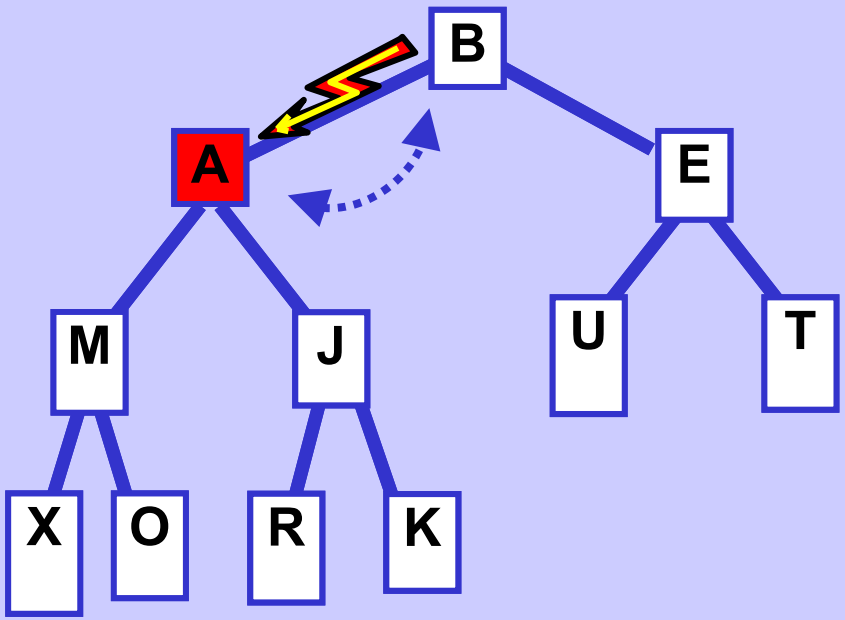



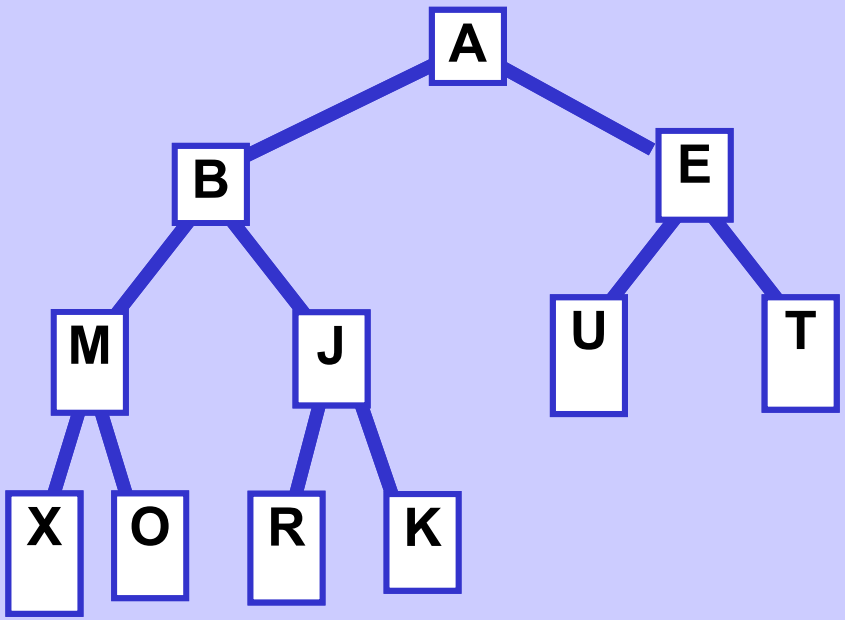

Heap property is violated,
swap the element with its parent.

Heap property is still violated,
swap the element with its parent.

# Priority queue implemented with binary heap -- Insert

## Inserting A



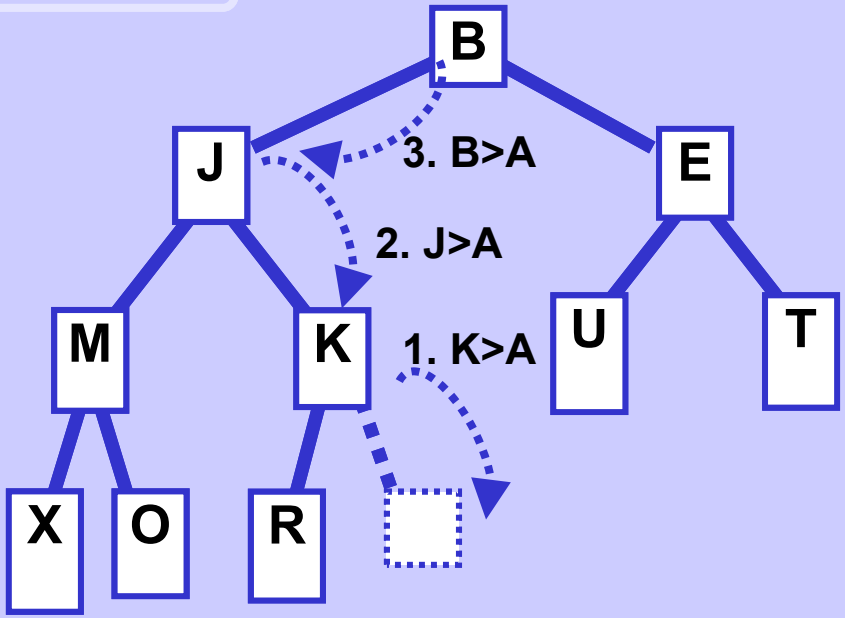



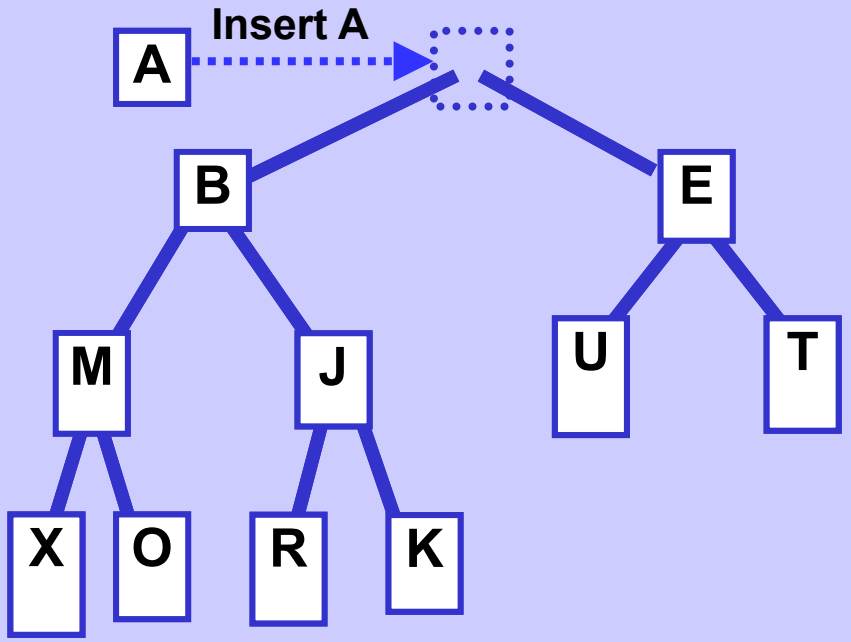

Heap property is still violated, swap the element with its parent.

Heap property is respected, the inserted element has found its place in the queue (heap).

# Binary heap -- Insert element more effectively

## Insert A




Do not insert the element at the end of the queue.
First, find its place and while searching move down other elements encountered in the search.

Finally, store the inserted element at its correct position.

## Binary heap – Insert

```
# beware!  array is arr[1] ... arr[n]
# bottom == ndx of last elem
def heapInsert(arr, x, bottom):
    bottom += 1   # expand the heap space
    j = bottom
    i = j/2       # parent index

    while i > 0 and arr[i] > x:
        arr[j] = arr[i]        # move elem down the heap
        j = i; i /= 2          # move indices up the heap

    arr[i] = x              # put inserted elem to its place
    return bottom
```

## Insert -- Complexity

Inserting represents a traversal in a binary tree from a leaf to the root in the worst case. Therefore, the Insert complexity is $O(\log_2(n))$ .