

Randomized Sampling-based Motion Planning Methods

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Lecture 06

B4M36UIR – Artificial Intelligence in Robotics



Overview of the Lecture

- Part 1 – Randomized Sampling-based Motion Planning Methods
 - Sampling-Based Methods
 - Probabilistic Road Map (PRM)
 - Characteristics
 - Rapidly Exploring Random Tree (RRT)
- Part 2 – Optimal Sampling-based Motion Planning Methods
 - Optimal Motion Planners
 - Rapidly-exploring Random Graph (RRG)



Part I

Part 1 – Sampling-based Motion Planning



Outline

- Sampling-Based Methods
- Probabilistic Road Map (PRM)
- Characteristics
- Rapidly Exploring Random Tree (RRT)

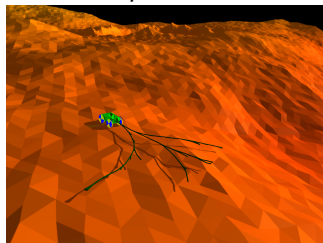


Sampling-based Motion Planning

- Avoids explicit representation of the obstacles in \mathcal{C} -space

- A “black-box” function is used to evaluate a configuration q is a collision free, e.g.,
- Based on geometrical models and testing collisions of the models
- In 2D or 3D shape of the robot and environment can be represented as sets of triangles, i.e., tessellated models
- Collision test – an intersection of triangles

E.g., using RAPID library <http://gamma.cs.unc.edu/OBB/>



- It creates a discrete representation of \mathcal{C}_{free}
- Configurations in \mathcal{C}_{free} are sampled randomly and connected to a roadmap (**probabilistic roadmap**)
- Rather than full completeness they provides **probabilistic completeness** or resolution completeness

Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)



Probabilistic Roadmaps

A discrete representation of the continuous \mathcal{C} -space generated by randomly sampled configurations in \mathcal{C}_{free} that are connected into a graph.

- **Nodes** of the graph represent admissible configuration of the robot.
- **Edges** represent a feasible path (trajectory) between the particular configurations.

Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)

Having the graph, the final path (trajectory) is found by a graph search technique.

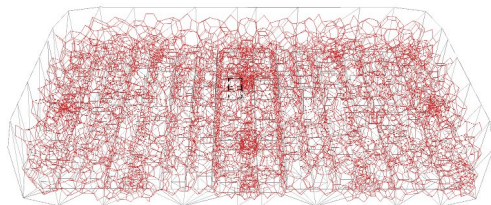


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Incremental Sampling and Searching

- Single query sampling-based algorithms incrementally create a search graph (roadmap)
 1. **Initialization** – $G(V, E)$ an undirected search graph, V may contain q_{start} , q_{goal} and/or other points in \mathcal{C}_{free}
 2. **Vertex selection method** – choose a vertex $q_{cur} \in V$ for expansion
 3. **Local planning method** – for some $q_{new} \in \mathcal{C}_{free}$, attempt to construct a path $\tau : [0, 1] \rightarrow \mathcal{C}_{free}$ such that $\tau(0) = q_{cur}$ and $\tau(1) = q_{new}$, τ must be checked to ensure it is collision free
 - If τ is not a collision-free, go to Step 2
 4. **Insert an edge in the graph** – Insert τ into E as an edge from q_{cur} to q_{new} and insert q_{new} to V if $q_{new} \notin V$
 5. **Check for a solution** – Determine if G encodes a solution, e.g., single search tree or graph search
 6. **Repeat to Step 2** – iterate unless a solution has been found or a termination condition is satisfied

LaValle, S. M.: Planning Algorithms (2006), Chapter 5.4



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Probabilistic Roadmap Strategies

Multi-Query – roadmap based

- Generate a single roadmap that is then used for planning queries several times.
- An representative technique is **Probabilistic RoadMap (PRM)**

Kavraki, L., Svestka, P., Latombe, J.-C., Overmars, M. H.B (1996): Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces. T-RO.

Single-Query – incremental

- For each planning problem constructs a new roadmap to characterize the subspace of \mathcal{C} -space that is relevant to the problem.
 - Rapidly-exploring Random Tree – RRT *LaValle, 1998*
 - Expansive-Space Tree – EST *Hsu et al., 1997*
 - Sampling-based Roadmap of Trees – SRT
(combination of multiple-query and single-query approaches)
Plaku et al., 2005



Multi-Query Strategy

Build a roadmap (graph) representing the environment

1. Learning phase

1.1 Sample n points in C_{free}

1.2 Connect the random configurations using a local planner

2. Query phase

2.1 Connect start and goal configurations with the PRM

E.g., using a local planner

2.2 Use the graph search to find the path



Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces

Lydia E. Kavraki and Petr Svestka and Jean-Claude Latombe and Mark H. Overmars,

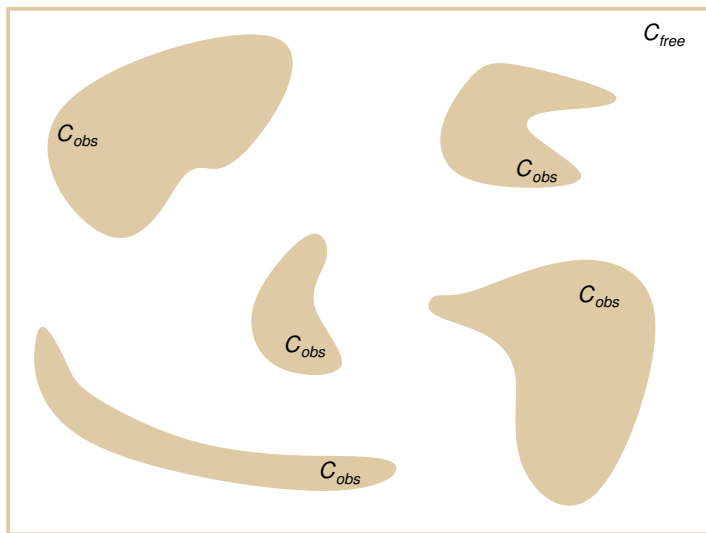
IEEE Transactions on Robotics and Automation, 12(4):566–580, 1996.

First planner that demonstrates ability to solve general planning problems in more than 4-5 dimensions.



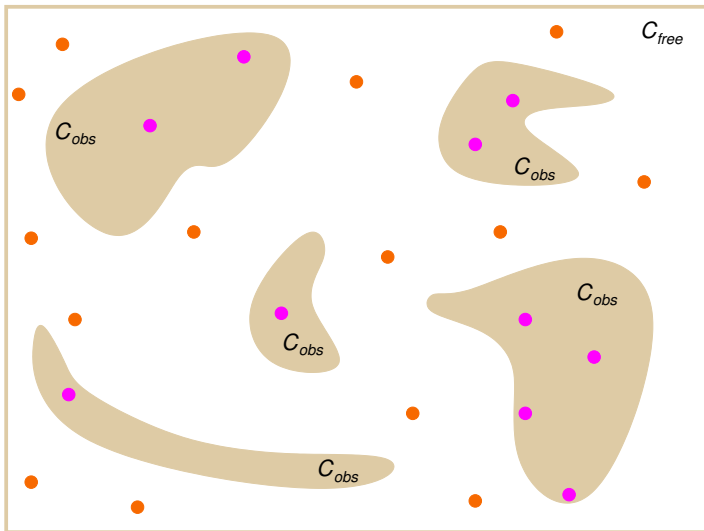
PRM Construction

Given problem domain



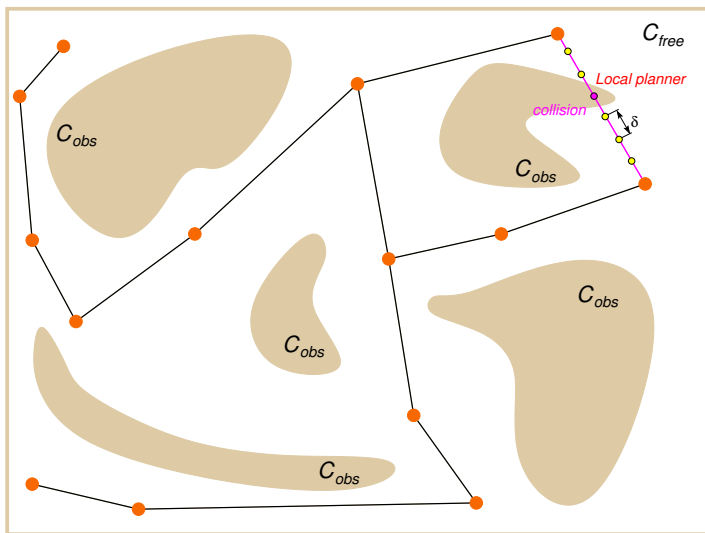
PRM Construction

Random configuration



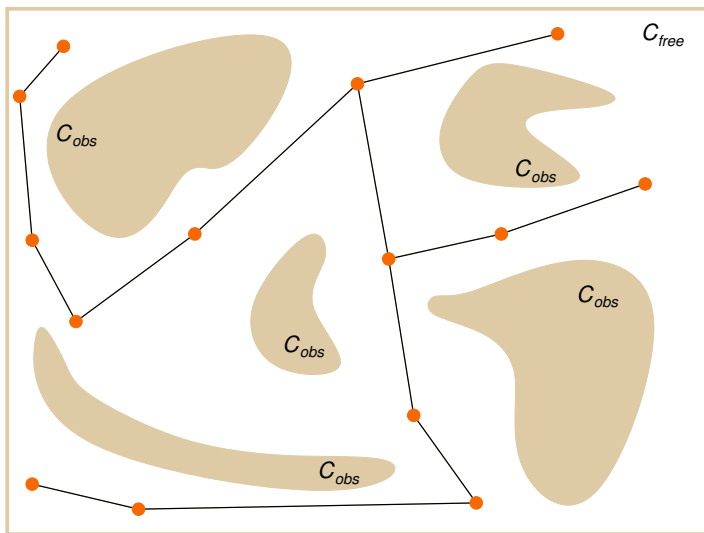
PRM Construction

Connecting random samples



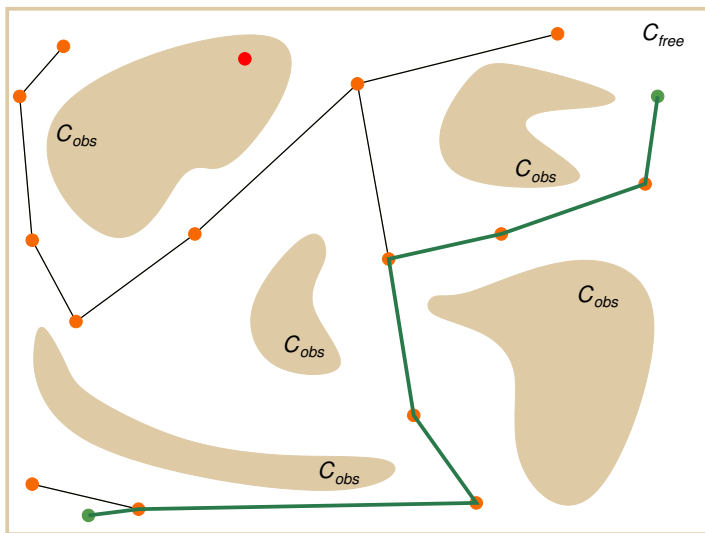
PRM Construction

Connected roadmap



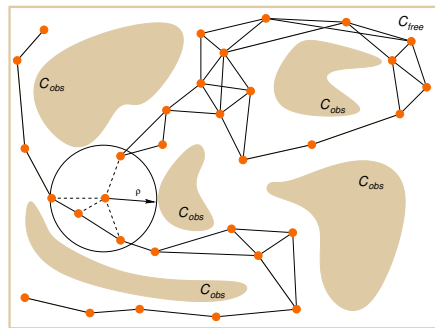
PRM Construction

Final found path



Practical PRM

- Incremental construction
- Connect nodes in a radius ρ
- Local planner tests collisions up to selected resolution δ
- Path can be found by Dijkstra's algorithm



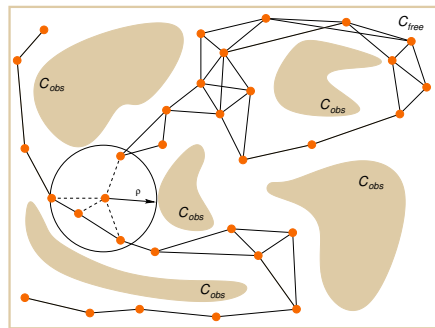
What are the properties of the PRM algorithm?

We need a couple of more formalism.



Practical PRM

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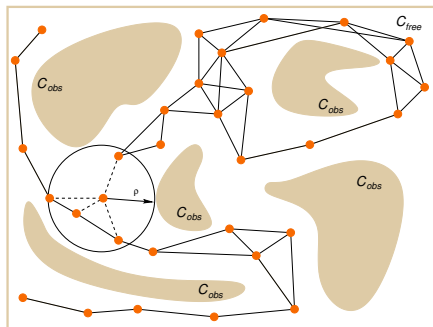
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Practical PRM

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What are the properties of the PRM algorithm?

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Path Planning Problem Formulation

- Path planning problem is defined by a triplet

$$\mathcal{P} = (\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal}),$$

- $\mathcal{C}_{free} = \text{cl}(\mathcal{C} \setminus \mathcal{C}_{obs})$, $\mathcal{C} = (0, 1)^d$, for $d \in \mathbb{N}$, $d \geq 2$
 - $q_{init} \in \mathcal{C}_{free}$ is the initial configuration (condition)
 - \mathcal{G}_{goal} is the goal region defined as an open subspace of \mathcal{C}_{free}
- Function $\pi : [0, 1] \rightarrow \mathbb{R}^d$ of *bounded variation* is called :
 - **path** if it is continuous;
 - **collision-free path** if it is path and $\pi(\tau) \in \mathcal{C}_{free}$ for $\tau \in [0, 1]$;
 - **feasible** if it is collision-free path, and $\pi(0) = q_{init}$ and $\pi(1) \in \text{cl}(\mathcal{Q}_{goal})$.

- A function π with the total variation $\text{TV}(\pi) < \infty$ is said to have bounded variation, where $\text{TV}(\pi)$ is the total variation

$$\text{TV}(\pi) = \sup_{\{n \in \mathbb{N}, 0 = \tau_0 < \tau_1 < \dots < \tau_n = s\}} \sum_{i=1}^n |\pi(\tau_i) - \pi(\tau_{i-1})|$$

- The total variation $\text{TV}(\pi)$ is de facto a path length.



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- **Characteristics**
- Rapidly Exploring Random Tree (RRT)



Path Planning Problem

■ Feasible path planning:

For a path planning problem $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$

- Find a feasible path $\pi : [0, 1] \rightarrow \mathcal{C}_{free}$ such that $\pi(0) = q_{init}$ and $\pi(1) \in \text{cl}(\mathcal{Q}_{goal})$, if such path exists.
- Report failure if no such path exists.

■ Optimal path planning:

The optimality problem ask for a feasible path with the minimum cost.

For $(\mathcal{C}_{free}, q_{init}, \mathcal{Q}_{goal})$ and a cost function $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$

- Find a feasible path π^* such that $c(\pi^*) = \min\{c(\pi) : \pi \text{ is feasible}\}$.
- Report failure if no such path exists.

The cost function is assumed to be monotonic and bounded, i.e., there exists k_c such that $c(\pi) \leq k_c \text{TV}(\pi)$.



Path Planning Problem

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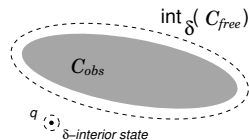
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Probabilistic Completeness 1/2

First, we need **robustly feasible path planning problem**
 $(\mathcal{C}_{free}, q_{init}, Q_{goal})$.

- $q \in \mathcal{C}_{free}$ is **δ -interior state of \mathcal{C}_{free}** if the closed ball of radius δ centered at q lies entirely inside \mathcal{C}_{free} .



- **δ -interior** of \mathcal{C}_{free} is $\text{int}_{\delta}(\mathcal{C}_{free}) = \{q \in \mathcal{C}_{free} \mid \mathcal{B}_{q,\delta} \subseteq \mathcal{C}_{free}\}$.

A collection of all δ -interior states.

- A collision free path π has **strong δ -clearance**, if π lies entirely inside $\text{int}_{\delta}(\mathcal{C}_{free})$.
- $(\mathcal{C}_{free}, q_{init}, Q_{goal})$ is **robustly feasible** if a solution exists and it is a feasible path with **strong δ -clearance**, for $\delta > 0$.

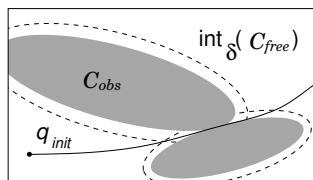
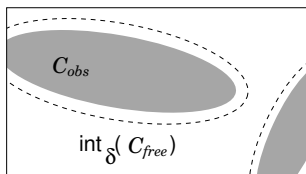


Probabilistic Completeness 2/2

An algorithm \mathcal{ALG} is **probabilistically complete** if, for any *robustly feasible path planning problem* $\mathcal{P} = (C_{free}, q_{init}, Q_{goal})$

$$\lim_{n \rightarrow \infty} Pr(\mathcal{ALG} \text{ returns a solution to } \mathcal{P}) = 1.$$

- It is a “*relaxed*” notion of completeness
- Applicable only to problems with a **robust solution**.



We need some space, where random configurations can be sampled



Asymptotic Optimality 1/4

Asymptotic optimality relies on a notion of **weak δ -clearance**

Notice, we use strong δ -clearance for probabilistic completeness

- Function $\psi : [0, 1] \rightarrow \mathcal{C}_{free}$ is called **homotopy**, if $\psi(0) = \pi_1$ and $\psi(1) = \pi_2$ and $\psi(\tau)$ is collision-free path for all $\tau \in [0, 1]$.
- A collision-free path π_1 is **homotopic** to π_2 if there exists homotopy function ψ .

A path homotopic to π can be continuously transformed to π through \mathcal{C}_{free} .



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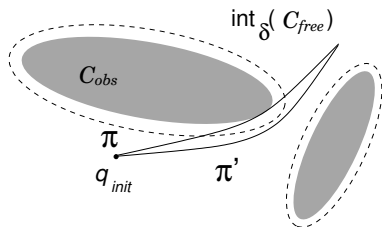
A path homotopic to π can be continuously transformed to π through \mathcal{C}_{free} .



Asymptotic Optimality 2/4

- A collision-free path $\pi : [0, s] \rightarrow \mathcal{C}_{free}$ has **weak δ -clearance** if there exists a path π' that has **strong δ -clearance** and homotopy ψ with $\psi(0) = \pi$, $\psi(1) = \pi'$, and for all $\alpha \in (0, 1]$ there exists $\delta_\alpha > 0$ such that $\psi(\alpha)$ has strong δ -clearance.

Weak δ -clearance does not require points along a path to be at least a distance δ away from obstacles.



- A path π with a weak δ -clearance
- π' lies in $\text{int}_\delta(\mathcal{C}_{free})$ and it is the same homotopy class as π



Asymptotic Optimality 3/4

- It is applicable with a **robust optimal solution** that can be obtained as a limit of robust (non-optimal) solutions.
- A collision-free path π^* is **robustly optimal solution** if it has *weak δ -clearance* and for any sequence of collision free paths $\{\pi_n\}_{n \in \mathbb{N}}$, $\pi_n \in \mathcal{C}_{free}$ such that $\lim_{n \rightarrow \infty} \pi_n = \pi^*$,

$$\lim_{n \rightarrow \infty} c(\pi_n) = c(\pi^*).$$

There exists a path with strong δ -clearance, and π^ is homotopic to such path and π^* is of **the lower cost**.*

- Weak δ -clearance implies robustly feasible solution problem
(*thus, probabilistic completeness*)



Asymptotic Optimality 4/4

An algorithm \mathcal{ALG} is **asymptotically optimal** if, for any path planning problem $\mathcal{P} = (\mathcal{C}_{free}, q_{init}, Q_{goal})$ and cost function c that admit a robust optimal solution with the finite cost c^*

$$Pr \left(\left\{ \lim_{i \rightarrow \infty} Y_i^{\mathcal{ALG}} = c^* \right\} \right) = 1.$$

- $Y_i^{\mathcal{ALG}}$ is the extended random variable corresponding to the minimum-cost solution included in the graph returned by \mathcal{ALG} at the end of iteration i .



Properties of the PRM Algorithm

- Completeness for the standard PRM has not been provided when it was introduced
- A simplified version of the PRM (called sPRM) has been mostly studied
- sPRM is probabilistically complete

What are the differences between PRM and sPRM?



PRM vs simplified PRM (sPRM)

Algorithm 1: PRM**Input:** q_{init} , number of samples n , radius ρ **Output:** PRM – $G = (V, E)$

```

 $V \leftarrow \emptyset; E \leftarrow \emptyset;$ 
for  $i = 0, \dots, n$  do
     $q_{rand} \leftarrow \text{SampleFree};$ 
     $U \leftarrow \text{Near}(G = (V, E), q_{rand}, \rho);$ 
     $V \leftarrow V \cup \{q_{rand}\};$ 
    foreach  $u \in U$ , with increasing
     $\|u - q_r\|$  do
        if  $q_{rand}$  and  $u$  are not in the
        same connected component of
         $G = (V, E)$  then
            if  $\text{CollisionFree}(q_{rand}, u)$ 
            then
                 $E \leftarrow E \cup$ 
                 $\{(q_{rand}, u), (u, q_{rand})\};$ 

```

return $G = (V, E);$

Algorithm 2: sPRM**Input:** q_{init} , number of samples n ,
radius ρ **Output:** PRM – $G = (V, E)$

```

 $V \leftarrow \{q_{init}\} \cup$ 
 $\{\text{SampleFree}_i\}_{i=1, \dots, n-1}; E \leftarrow \emptyset;$ 
foreach  $v \in V$  do
     $U \leftarrow \text{Near}(G = (V, E), v, \rho) \setminus \{v\};$ 
    foreach  $u \in U$  do
        if  $\text{CollisionFree}(v, u)$  then
             $E \leftarrow E \cup \{(v, u), (u, v)\};$ 

```

return $G = (V, E);$

There are several ways for the set U of vertices to connect them

- k -nearest neighbors to v
- variable connection radius ρ as a function of n



PRM – Properties

- **sPRM** (simplified PRM)
 - **Probabilistically complete and asymptotically optimal**
 - Processing complexity $O(n^2)$
 - Query complexity $O(n^2)$
 - Space complexity $O(n^2)$
- Heuristics practically used are usually not probabilistic complete
 - k -nearest sPRM is not probabilistically complete
 - variable radius sPRM is not probabilistically complete

Based on analysis of Karaman and Frazzoli

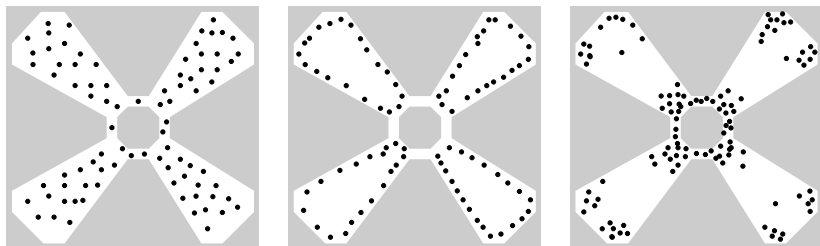
PRM algorithm:

- + Has very simple implementation
- + Completeness (for sPRM)
- Differential constraints (car-like vehicles) are not straightforward



Comments about Random Sampling 1/2

- Different sampling strategies (distributions) may be applied



- Notice, one of the main issue of the randomized sampling-based approaches is the narrow passage
- Several modifications of sampling based strategies have been proposed in the last decades



Comments about Random Sampling 2/2

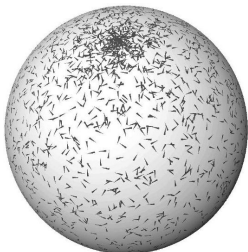
- A solution can be found using only a few samples.

Do you know the Oracleum? (from Alice in Wonderland)

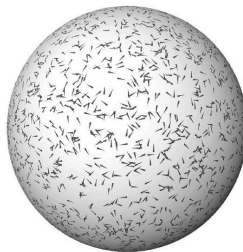
- Sampling strategies are important

- Near obstacles
- Narrow passages
- Grid-based
- Uniform sampling must be carefully considered.

James J. Kuffner (2004):, [Effective Sampling and Distance Metrics for 3D Rigid Body Path Planning.](#), ICRA.



Naïve sampling



Uniform sampling of $SO(3)$ using Euler angles



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- Probabilistic Road Map (PRM)
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Rapidly Exploring Random Tree (RRT)

Single-Query algorithm

- It incrementally builds a graph (tree) towards the goal area.

It does not guarantee precise path to the goal configuration.

1. Start with the initial configuration q_0 , which is a root of the constructed graph (tree)

2. Generate a new random configuration q_{new} in \mathcal{C}_{free}

3. Find the closest node q_{near} to q_{new} in the tree

E.g., using KD-tree implementation like ANN or FLANN libraries

4. Extend q_{near} towards q_{new}

Extend the tree by a small step, but often a direct control $u \in \mathcal{U}$ that will move robot the position closest to q_{new} is selected (applied for δt).

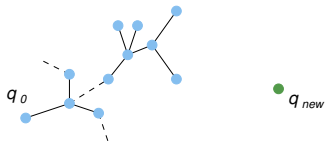
5. Go to Step 2, until the tree is within a sufficient distance from the goal configuration

Or terminates after dedicated running time.

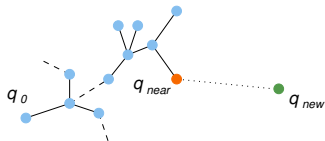


RRT Construction

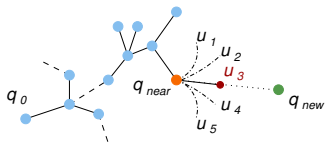
#1 new random configuration



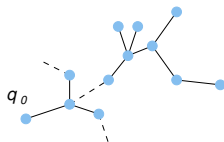
#2 the closest node



#3 possible actions from q_near



#4 extended tree



RRT Algorithm

- Motivation is a single query and *control-based* path finding
- It incrementally builds a graph (tree) towards the goal area.

Algorithm 3: Rapidly Exploring Random Tree (RRT)

Input: q_{init} , number of samples n

Output: Roadmap $G = (V, E)$

$V \leftarrow \{q_{init}\}; E \leftarrow \emptyset;$

for $i = 1, \dots, n$ **do**

$q_{rand} \leftarrow \text{SampleFree};$

$q_{nearest} \leftarrow \text{Nearest}(G = (V, E), q_{rand});$

$q_{new} \leftarrow \text{Steer}(q_{nearest}, q_{rand});$

if $\text{CollisionFree}(q_{nearest}, q_{new})$ **then**

$V \leftarrow V \cup \{x_{new}\}; E \leftarrow E \cup \{(x_{nearest}, x_{new})\};$

return $G = (V, E);$

Extend the tree by a small step, but often a direct control $u \in \mathcal{U}$ that will move robot to the position closest to q_{new} is selected (applied for dt).



Rapidly-exploring random trees: A new tool for path planning

S. M. LaValle,

Technical Report 98-11, Computer Science Dept., Iowa State University, 1998



Properties of RRT Algorithms

- Rapidly explores the space

q_{new} will more likely be generated in large not yet covered parts.

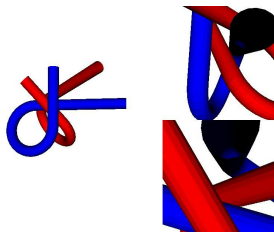
- Allows considering kinodynamic/dynamic constraints (during the expansion).
- Can provide trajectory or a sequence of direct control commands for robot controllers.
- A collision detection test is usually used as a “black-box”.

E.g., RAPID, Bullet libraries.

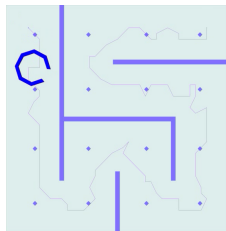
- Similarly to PRM, RRT algorithms have poor performance in narrow passage problems.
- RRT algorithms provides feasible paths.
It can be relatively far from optimal solution, e.g., according to the length of the path.
- Many variants of RRT have been proposed.



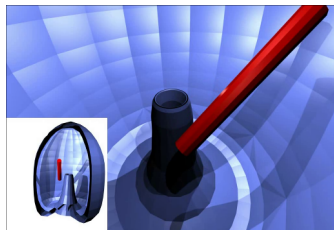
RRT – Examples 1/2



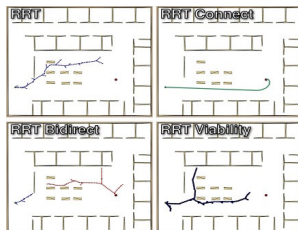
Alpha puzzle benchmark



Apply rotations to reach the goal



Bugtrap benchmark



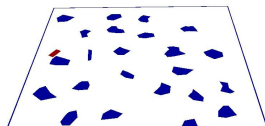
Variants of RRT algorithms

Courtesy of V. Vonásek and P. Vaněk

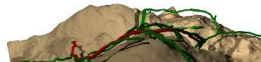


RRT – Examples 2/2

- Planning for a car-like robot

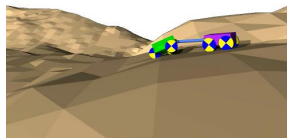


- Planning on a 3D surface



- Planning with dynamics

(friction forces)



Courtesy of V. Vonásek and P. Vaněk



Car-Like Robot

■ Configuration

$$\vec{x} = \begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$$

position and orientation

■ Controls

$$\vec{u} = \begin{pmatrix} v \\ \varphi \end{pmatrix}$$

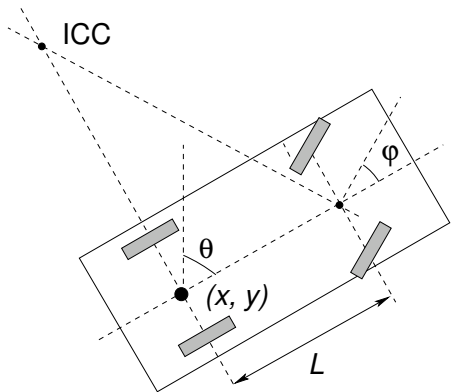
forward velocity, steering angle

■ System equation

$$\dot{x} = v \cos \phi$$

$$\dot{y} = v \sin \phi$$

$$\dot{\phi} = \frac{v}{L} \tan \varphi$$



Kinematic constraints $\dim(\vec{u}) < \dim(\vec{x})$

Differential constraints on possible \dot{q} :

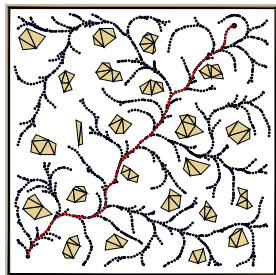
$$\dot{x} \sin(\phi) - \dot{y} \cos(\phi) = 0$$



Control-Based Sampling

- Select a configuration q from the tree T of the current configurations
- Pick a control input $\vec{u} = (v, \phi)$ and integrate system (motion) equation over a short period

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \varphi \end{pmatrix} = \int_t^{t+\Delta t} \begin{pmatrix} v \cos \phi \\ v \sin \phi \\ \frac{v}{L} \tan \phi \end{pmatrix} dt$$



- If the motion is collision-free, add the endpoint to the tree

E.g., considering k configurations for $k\delta t = dt$.



Part II

Part 2 – Optimal Sampling-based Motion Planning Methods



Outline

- Optimal Motion Planners
- Rapidly-exploring Random Graph (RRG)



Sampling-Based Motion Planning

- PRM and RRT are theoretically probabilistic complete
- They provide a feasible solution without quality guarantee

Despite of that, they are successfully used in many practical applications

- In 2011, a systematical study of the asymptotic behaviour of randomized sampling-based planners has been published

It shows, that in some cases, they converge to a non-optimal value with a probability 1.

- Based on the study, new algorithms have been proposed: **RRG** and optimal RRT (**RRT***)

Karaman, S., Frazzoli, E. (2011): Sampling-based algorithms for optimal motion planning. IJRR.



<http://sertac.scripts.mit.edu/rrtstar>



RRT and Quality of Solution 1/2

- Let Y_i^{RRT} be the cost of the best path in the RRT at the end of iteration i .
- Y_i^{RRT} converges to a random variable

$$\lim_{i \rightarrow \infty} Y_i^{RRT} = Y_{\infty}^{RRT}.$$

- The random variable Y_{∞}^{RRT} is sampled from a distribution with zero mass at the optimum, and

$$Pr[Y_{\infty}^{RRT} > c^*] = 1.$$

Karaman and Frazzoli, 2011

- The best path in the RRT converges to a sub-optimal solution almost surely.



RRT and Quality of Solution 2/2

- RRT does not satisfy a necessary condition for the asymptotic optimality
 - For $0 < R < \inf_{q \in Q_{goal}} \|q - q_{init}\|$, the event $\{\lim_{n \rightarrow \infty} Y_n^{RRT} = c^*\}$ occurs only if the k -th branch of the RRT contains vertices outside the R -ball centered at q_{init} for infinitely many k .

See Appendix B in Karaman&Frazzoli, 2011

- It is required the root node will have infinitely many subtrees that extend at least a distance ϵ away from q_{init}
 - The sub-optimality is caused by disallowing new better paths to be discovered.*



Outline

- Optimal Motion Planners
- Rapidly-exploring Random Graph (RRG)



Rapidly-exploring Random Graph (RRG)

Algorithm 4: Rapidly-exploring Random Graph (RRG)

Input: q_{init} , number of samples n

Output: $G = (V, E)$

$V \leftarrow \emptyset; E \leftarrow \emptyset;$

for $i = 0, \dots, n$ **do**

$q_{rand} \leftarrow \text{SampleFree};$

$q_{nearest} \leftarrow \text{Nearest}(G = (V, E), q_{rand});$

$q_{new} \leftarrow \text{Steer}(q_{nearest}, q_{rand});$

if $\text{CollisionFree}(q_{nearest}, q_{new})$ **then**

$Q_{near} \leftarrow \text{Near}(G =$

$(V, E), q_{new}, \min\{\gamma_{RRG}(\log(\text{card}(V))/\text{card}(V))^{1/d}, \eta\});$

$V \leftarrow V \cup \{q_{new}\};$

$E \leftarrow E \cup \{(q_{nearest}, q_{new}), (q_{new}, q_{nearest})\};$

foreach $q_{near} \in Q_{near}$ **do**

if $\text{CollisionFree}(q_{near}, q_{new})$ **then**

$E \leftarrow E \cup \{(q_{rand}, u), (u, q_{rand})\};$

return $G = (V, E);$

Proposed by Karaman and Frazzoli (2011). Theoretical results are related to properties of [Random Geometric Graphs \(RGG\)](#) introduced by Gilbert (1961) and further studied by Penrose (1999).



RRG Expansions

- At each iteration, RRG tries to connect new sample to the all vertices in the r_n ball centered at it.
- The ball of radius

$$r(\text{card}(V)) = \min \left\{ \gamma_{RRG} \left(\frac{\log(\text{card}(V))}{\text{card}(V)} \right)^{1/d}, \eta \right\}$$

where

- η is the constant of the local steering function
- $\gamma_{RRG} > \gamma_{RRG}^* = 2(1 + 1/d)^{1/d} (\mu(C_{free})/\xi_d)^{1/d}$
 - d – dimension of the space;
 - $\mu(C_{free})$ – Lebesgue measure of the obstacle-free space;
 - ξ_d – volume of the unit ball in d -dimensional Euclidean space.
- The connection radius decreases with n
- The rate of decay \approx the average number of connections attempted is proportional to $\log(n)$



RRG Properties

- Probabilistically complete
- Asymptotically optimal
- Complexity is $O(\log n)$

(per one sample)

- Computational efficiency and optimality

- Attempt connection to $\Theta(\log n)$ nodes at each iteration;

in average

- Reduce volume of the “connection” ball as $\log(n)/n$;
- Increase the number of connections as $\log(n)$.



Other Variants of the Optimal Motion Planning

- **PRM*** – it follows standard PRM algorithm where connections are attempted between roadmap vertices that are within connection radius r as a function of n

$$r(n) = \gamma_{PRM}(\log(n)/n)^{1/d}$$

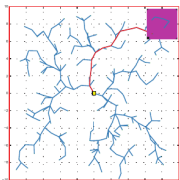
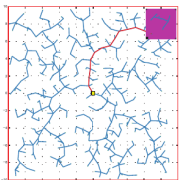
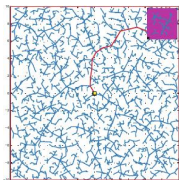
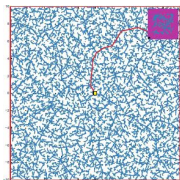
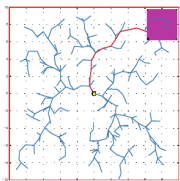
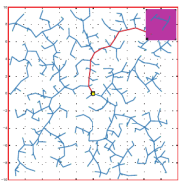
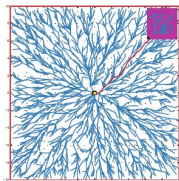
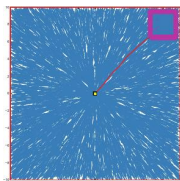
- **RRT*** – a modification of the RRG, where cycles are avoided

A tree version of the RRG

- A tree roadmap allows to consider non-holonomic dynamics and kinodynamic constraints.
- It is basically RRG with “rerouting” the tree when a better path is discovered.



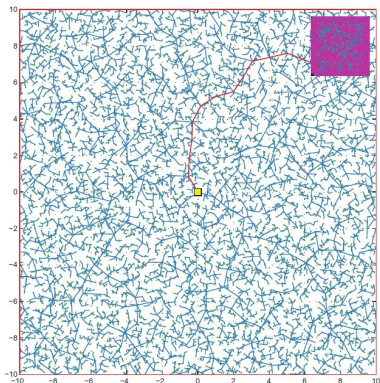
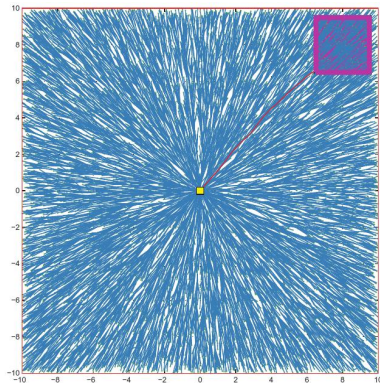
Example of Solution 1/2

RRT, $n=250$ RRT, $n=500$ RRT, $n=2500$ RRT, $n=10000$ RRT*, $n=250$ RRT*, $n=500$ RRT*, $n=2500$ RRT*, $n=10000$

Karaman & Frazzoli, 2011



Example of Solution 2/2

RRT, $n=20000$ RRT*, $n=20000$ 

Overview of Randomized Sampling-based Algorithms

Algorithm	Probabilistic Completeness	Asymptotic Optimality
sPRM	✓	✗
k-nearest sPRM	✗	✗
RRT	✓	✗
RRG	✓	✓
PRM*	✓	✓
RRT*	✓	✓

Notice, k-nearest variants of RRG, PRM, and RRT* are complete and optimal as well.*



Summary of the Lecture



Topics Discussed

- Randomized Sampling-based Methods
 - Probabilistic Road Map (PRM)
 - Characteristics of path planning problems
 - Random sampling
 - Rapidly Exploring Random Tree (RRT)
 - Optimal sampling-based motion planning
 - Rapidly-exploring Random Graph (RRG)
- Next: Multi-Goal Motion Planning and Multi-Goal Path Planning



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- **Next: Multi-Goal Motion Planning and Multi-Goal Path Planning**

