

Randomized Sampling-based Motion Planning Methods

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Lecture 06

B4M36UIR – Artificial Intelligence in Robotics

(Randomized) Sampling-based Motion Planning

- It uses an explicit representation of the obstacles in C -space
 - A “black-box” function is used to evaluate a configuration q is a collision-free, e.g.,
 - Based on geometrical models and testing collisions of the models
 - In 2D or 3D shape of the robot and environment can be represented as sets of triangles, i.e., tessellated models
 - Collision test – an intersection of triangles
E.g., using RAPID library <http://gamma.cs.unc.edu/OBB/>
- Creates a discrete representation of C_{free}
- Configurations in C_{free} are sampled randomly and connected to a roadmap (probabilistic roadmap)
- Rather than full completeness they provide probabilistic completeness or resolution completeness
Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)



Probabilistic Roadmap Strategies

- Multi-Query** – roadmap based
 - Generate a single roadmap that is then used for planning queries several times.
 - An representative technique is **Probabilistic RoadMap (PRM)**
Kavraki, L., Svestka, P., Latombe, J.-C., Overmars, M. H.B (1996): Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces. T-RO.
- Single-Query** – incremental
 - For each planning problem constructs a new roadmap to characterize the subspace of C -space that is relevant to the problem.
 - Rapidly-exploring Random Tree – RRT
LaValle, 1998
 - Expansive-Space Tree – EST
Hsu et al., 1997
 - Sampling-based Roadmap of Trees – SRT
(combination of multiple-query and single-query approaches)
Plaku et al., 2005

Overview of the Lecture

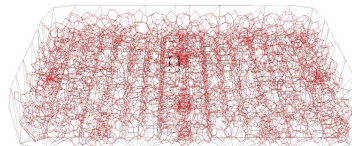
- Part 1 – Randomized Sampling-based Motion Planning Methods
 - Sampling-Based Methods
 - Probabilistic Road Map (PRM)
 - Characteristics
 - Rapidly Exploring Random Tree (RRT)
- Part 2 – Optimal Sampling-based Motion Planning Methods
 - Optimal Motion Planners
 - Rapidly-exploring Random Graph (RRG)

Probabilistic Roadmaps

A discrete representation of the continuous C -space generated by randomly sampled configurations in C_{free} that are connected into a graph

- Nodes of the graph represent admissible configuration of the robot
- Edges represent a feasible path (trajectory) between the particular configurations

Probabilistic complete algorithms: with increasing number of samples an admissible solution would be found (if exists)




Having the graph, the final path (trajectory) is found by a graph search technique

Multi-Query Strategy

Build a roadmap (graph) representing the environment

- Learning phase
 - Sample n points in C_{free}
 - Connect the random configurations using a local planner
- Query phase
 - Connect start and goal configurations with the PRM
E.g., using a local planner
 - Use the graph search to find the path

 Probabilistic Roadmaps for Path Planning in High Dimensional Configuration Spaces
Lydia E. Kavraki and Petr Svestka and Jean-Claude Latombe and Mark H. Overmars,
IEEE Transactions on Robotics and Automation, 12(4):566–580, 1996.

First planner that demonstrates ability to solve general planning problems in more than 4-5 dimensions.

Part I

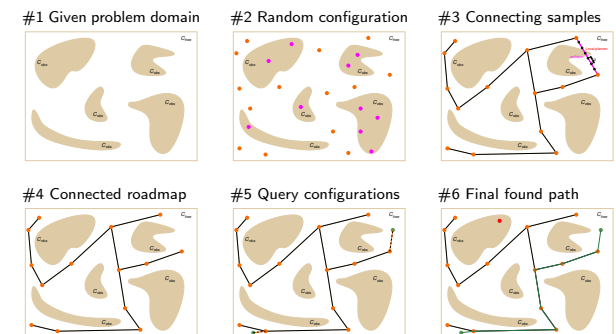
Part 1 – Sampling-based Motion Planning

Incremental Sampling and Searching

- Single query sampling-based algorithms incrementally create a search graph (roadmap)
 - Initialization** – $G(V, E)$ an undirected search graph, V may contain q_{start}, q_{goal} and/or other points in C_{free}
 - Vertex selection method** – choose a vertex $q_{cur} \in V$ for expansion
 - Local planning method** – for some $q_{new} \in C_{free}$, attempt to construct a path $\tau : [0, 1] \rightarrow C_{free}$ such that $\tau(0) = q_{cur}$ and $\tau(1) = q_{new}$, τ must be checked to ensure it is collision free
 - If τ is not a collision-free, go to Step 2
 - Insert an edge in the graph** – Insert τ into E as an edge from q_{cur} to q_{new} and insert q_{new} to V if $q_{new} \notin V$
 - Check for a solution** – Determine if G encodes a solution, e.g., single search tree or graph search
 - Repeat to Step 2** – iterate unless a solution has been found or a termination condition is satisfied

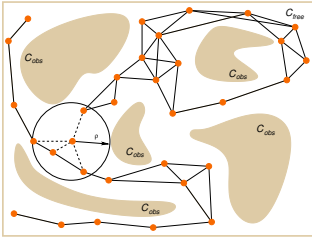
LaValle, S. M.: Planning Algorithms (2006), Chapter 5.4

PRM Construction



Practical PRM

- Incremental construction
- Connect nodes in a radius ρ
- Local planner tests collisions up to selected resolution δ
- Path can be found by Dijkstra's algorithm



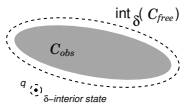
What are the properties of the PRM algorithm?

We need a couple of more formalisms.

Probabilistic Completeness 1/2

First, we need **robustly feasible path planning problem** $(C_{free}, q_{init}, Q_{goal})$

- $q \in C_{free}$ is **δ -interior state** of C_{free} if the closed ball of radius δ centered at q lies entirely inside C_{free}



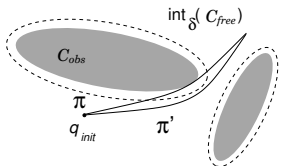
- **δ -interior** of C_{free} is $\text{int}_\delta(C_{free}) = \{q \in C_{free} | \mathcal{B}_{\delta}(q) \subseteq C_{free}\}$
A collection of all δ -interior states

- A collision free path π has **strong δ -clearance**, if π lies entirely inside $\text{int}_\delta(C_{free})$
- $(C_{free}, q_{init}, Q_{goal})$ is **robustly feasible** if a solution exists and it is a feasible path with **strong δ -clearance**, for $\delta > 0$

Asymptotic Optimality 2/4

- A collision-free path $\pi : [0, s] \rightarrow C_{free}$ has **weak δ -clearance** if there exists a path π' that has **strong δ -clearance** and homotopy ψ with $\psi(0) = \pi$, $\psi(1) = \pi'$, and for all $\alpha \in (0, 1]$ there exists $\delta_\alpha > 0$ such that $\psi(\alpha)$ has strong δ -clearance

Weak δ -clearance does not require points along a path to be at least a distance δ away from obstacles



- A path π with a weak δ -clearance
- π' lies in $\text{int}_\delta(C_{free})$ and it is the same homotopy class as π

Path Planning Problem Formulation

- Path planning problem is defined by a triplet

$$\mathcal{P} = (C_{free}, q_{init}, Q_{goal}),$$

- $C_{free} = \text{cl}(C \setminus C_{obs})$, $C = (0, 1)^d$, for $d \in \mathbb{N}$, $d \geq 2$
- $q_{init} \in C_{free}$ is the initial configuration (condition)
- Q_{goal} is the goal region defined as an open subspace of C_{free}

- Function $\pi : [0, 1] \rightarrow \mathbb{R}^d$ of **bounded variation** is called:

- **path** if it is continuous;
- **collision-free path** if it is path and $\pi(\tau) \in C_{free}$ for $\tau \in [0, 1]$;
- **feasible** if it is collision-free path, and $\pi(0) = q_{init}$ and $\pi(1) \in \text{cl}(Q_{goal})$.

- A function π with the total variation $\text{TV}(\pi) < \infty$ is said to have bounded variation, where $\text{TV}(\pi)$ is the total variation

$$\text{TV}(\pi) = \sup_{(n \in \mathbb{N}, 0 = \tau_0 < \tau_1 < \dots < \tau_n = 1)} \sum_{i=1}^n |\pi(\tau_i) - \pi(\tau_{i-1})|$$

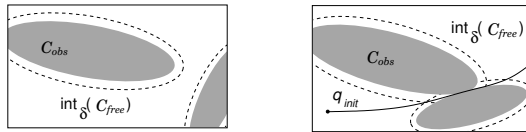
- The total variation $\text{TV}(\pi)$ is de facto a path length

Probabilistic Completeness 2/2

An algorithm \mathcal{ALG} is **probabilistically complete** if, for any **robustly feasible path planning problem** $\mathcal{P} = (C_{free}, q_{init}, Q_{goal})$

$$\lim_{n \rightarrow \infty} \text{Pr}(\mathcal{ALG} \text{ returns a solution to } \mathcal{P}) = 1.$$

- It is a “relaxed” notion of completeness
- Applicable only to problems with a **robust solution**



We need some space, where random configurations can be sampled

Asymptotic Optimality 3/4

- It is applicable with a **robust optimal solution** that can be obtained as a limit of robust (non-optimal) solutions
- A collision-free path π^* is **robustly optimal solution** if it has **weak δ -clearance** and for any sequence of collision free paths $\{\pi_n\}_{n \in \mathbb{N}}$, $\pi_n \in C_{free}$ such that $\lim_{n \rightarrow \infty} \pi_n = \pi^*$,

$$\lim_{n \rightarrow \infty} c(\pi_n) = c(\pi^*)$$

There exists a path with strong δ -clearance, and π^* is of the lower cost.

- Weak δ -clearance implies robustly feasible solution problem
(thus, probabilistic completeness)

Path Planning Problem

- **Feasible path planning:**

For a path planning problem $(C_{free}, q_{init}, Q_{goal})$

- Find a feasible path $\pi : [0, 1] \rightarrow C_{free}$ such that $\pi(0) = q_{init}$ and $\pi(1) \in \text{cl}(Q_{goal})$, if such path exists
- Report failure if no such path exists

- **Optimal path planning:**

The optimality problem asks for a feasible path with the minimum cost

For $(C_{free}, q_{init}, Q_{goal})$ and a cost function $c : \Sigma \rightarrow \mathbb{R}_{\geq 0}$

- Find a feasible path π^* such that $c(\pi^*) = \min\{c(\pi) : \pi \text{ is feasible}\}$
- Report failure if no such path exists

The cost function is assumed to be monotonic and bounded, i.e., there exists k_c such that $c(\pi) \leq k_c \text{TV}(\pi)$

Asymptotic Optimality 1/4

Asymptotic optimality relies on a notion of **weak δ -clearance**

Notice, we use strong δ -clearance for probabilistic completeness

- Function $\psi : [0, 1] \rightarrow C_{free}$ is called **homotopy**, if $\psi(0) = \pi_1$ and $\psi(1) = \pi_2$ and $\psi(\tau)$ is collision-free path for all $\tau \in [0, 1]$
- A collision-free path π_1 is **homotopic** to π_2 if there exists homotopy function ψ

A path homotopic to π can be continuously transformed to π through C_{free}

Asymptotic Optimality 4/4

An algorithm \mathcal{ALG} is **asymptotically optimal** if, for any path planning problem $\mathcal{P} = (C_{free}, q_{init}, Q_{goal})$ and cost function c that admit a robust optimal solution with the finite cost c^*

$$\text{Pr} \left(\left\{ \lim_{i \rightarrow \infty} Y_i^{\mathcal{ALG}} = c^* \right\} \right) = 1$$

- $Y_i^{\mathcal{ALG}}$ is the extended random variable corresponding to the minimum-cost solution included in the graph returned by \mathcal{ALG} at the end of iteration i

Properties of the PRM Algorithm

- Completeness for the standard PRM has not been provided when it was introduced
- A simplified version of the PRM (called sPRM) has been mostly studied
- sPRM is probabilistically complete

What are the differences between PRM and sPRM?

PRM vs simplified PRM (sPRM)

Algorithm 1: PRM

Vstup: q_{init} , number of samples n , radius ρ
Výstup: PRM – $G = (V, E)$

```

V ← ∅; E ← ∅;
for i = 0, ..., n do
    qrand ← SampleFree;
    U ← Near(G = (V, E), qrand, ρ);
    V ← V ∪ {qrand};
    foreach u ∈ U, with increasing ||u - qrand|| do
        if qrand and u are not in the same connected component of G = (V, E) then
            if CollisionFree(qrand, u) then
                E ← E ∪ {(qrand, u), (u, qrand)};
return G = (V, E);
    
```

Algorithm 2: sPRM

Vstup: q_{init} , number of samples n , radius ρ
Výstup: PRM – $G = (V, E)$

```

V ← {qinit} ∪ {SampleFree};
foreach v ∈ V do
    U ← Near(G = (V, E), v, ρ) \ {v};
    foreach u ∈ U do
        if CollisionFree(v, u) then
            E ← E ∪ {(v, u), (u, v)};
return G = (V, E);
    
```

There are several ways for the set U of vertices to connect them

- k -nearest neighbors to v
- variable connection radius ρ as a function of n

PRM – Properties

■ sPRM (simplified PRM)

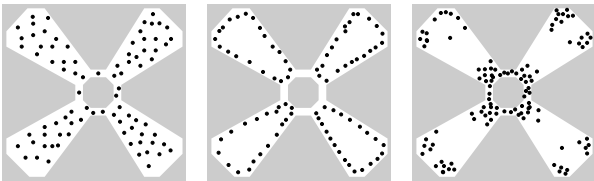
- **Probabilistically complete and asymptotically optimal**
 - Processing complexity $O(n^2)$
 - Query complexity $O(n^2)$
 - Space complexity $O(n^2)$
 - Heuristics practically used are usually not probabilistic complete
 - k -nearest sPRM is not probabilistically complete
 - variable radius sPRM is not probabilistically complete
- Based on analysis of Karaman and Frazzoli*

PRM algorithm:

- + Has very simple implementation
- + Completeness (for sPRM)
- Differential constraints (car-like vehicles) are not straightforward

Comments about Random Sampling 1/2

- Different sampling strategies (distributions) may be applied



- Notice, one of the main issue of the randomized sampling-based approaches is the narrow passage
- Several modifications of sampling based strategies have been proposed in the last decades

Comments about Random Sampling 2/2

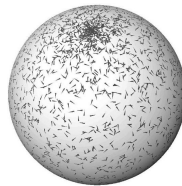
- A solution can be found using only a few samples.

Do you know the Oracleum? (from Alice in Wonderland)

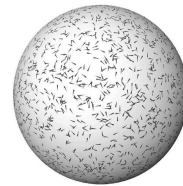
- Sampling strategies are important

- Near obstacles
- Narrow passages
- Grid-based
- Uniform sampling must be carefully considered

James J. Kuffner (2004): Effective Sampling and Distance Metrics for 3D Rigid Body Path Planning. ICRA.



Naïve sampling



Uniform sampling of SO(3) using Euler angles

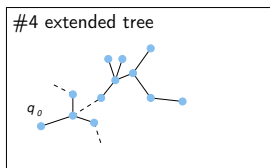
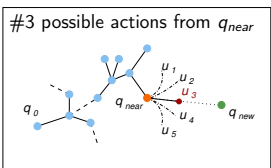
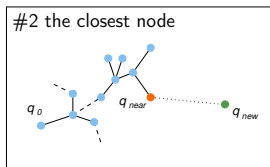
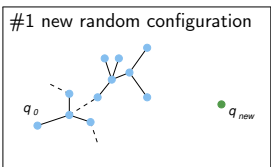
Rapidly Exploring Random Tree (RRT)

Single-Query algorithm

- It incrementally builds a graph (tree) towards the goal area.
It does not guarantee precise path to the goal configuration.
- 1. Start with the initial configuration q_0 , which is a root of the constructed graph (tree)
- 2. Generate a new random configuration q_{new} in C_{free}
- 3. Find the closest node q_{near} to q_{new} in the tree
E.g., using KD-tree implementation like ANN or FLANN libraries
- 4. Extend q_{near} towards q_{new}
Extend the tree by a small step, but often a direct control $u \in U$ that will move robot the position closest to q_{new} is selected (applied for δt)
- 5. Go to Step 2, until the tree is within a sufficient distance from the goal configuration

Or terminates after dedicated running time

RRT Construction



RRT Algorithm

- Motivation is a single query and *control-based* path finding
- It incrementally builds a graph (tree) towards the goal area

Algorithm 3: Rapidly Exploring Random Tree (RRT)

Vstup: q_{init} , number of samples n
Výstup: Roadmap $G = (V, E)$

```

V ← {qinit}; E ← ∅;
for i = 1, ..., n do
    qrand ← SampleFree;
    qnearest ← Nearest(G = (V, E), qrand);
    qnew ← Steer(qnearest, qrand);
    if CollisionFree(qnearest, qnew) then
        V ← V ∪ {qnew}; E ← E ∪ {(qnearest, qnew)};
return G = (V, E);
    
```

Extend the tree by a small step, but often a direct control $u \in U$ that will move robot to the position closest to q_{new} is selected (applied for δt)



Rapidly-exploring random trees: A new tool for path planning

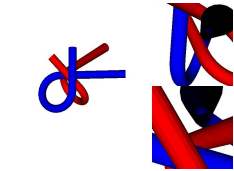
S. M. LaValle,

Technical Report 98-11, Computer Science Dept., Iowa State University, 1998

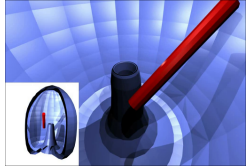
Properties of RRT Algorithms

- Rapidly explores the space
 q_{new} will more likely be generated in large not yet covered parts
- Allows considering kinodynamic/dynamic constraints (during the expansion)
- Can provide trajectory or a sequence of direct control commands for robot controllers
- A collision detection test is usually used as a “black-box”
E.g., RAPID, Bullet libraries
- Similarly to PRM, RRT algorithms have poor performance in narrow passage problems
- RRT algorithms provides feasible paths
It can be relatively far from optimal solution, e.g., according to the length of the path
- Many variants of RRT have been proposed

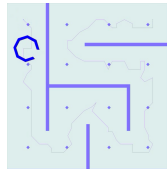
RRT – Examples 1/2



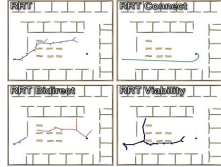
Alpha puzzle benchmark



Bugtrap benchmark



Apply rotations to reach the goal



Variants of RRT algorithms

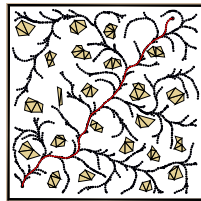
Courtesy of V. Vonásek and P. Vaněk

Control-Based Sampling

- Select a configuration q from the tree T of the current configurations

- Pick a control input $\vec{u} = (v, \varphi)$ and integrate system (motion) equation over a short period

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \varphi \end{pmatrix} = \int_t^{t+\Delta t} \begin{pmatrix} v \cos \phi \\ v \sin \phi \\ \frac{v}{L} \tan \varphi \end{pmatrix} dt$$



- If the motion is collision-free, add the endpoint to the tree

E.g., considering k configurations for $k\delta t = dt$

RRT and Quality of Solution 1/2

- Let Y_i^{RRT} be the cost of the best path in the RRT at the end of iteration i
- Y_i^{RRT} converges to a random variable

$$\lim_{i \rightarrow \infty} Y_i^{RRT} = Y_\infty^{RRT}$$

- The random variable Y_∞^{RRT} is sampled from a distribution with zero mass at the optimum, and

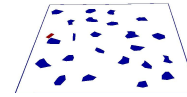
$$Pr\{Y_\infty^{RRT} > c^*\} = 1$$

Karaman and Frazzoli, 2011

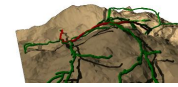
- The best path in the RRT converges to a sub-optimal solution almost surely

RRT – Examples 2/2

- Planning for a car-like robot

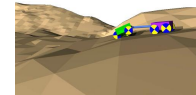


- Planning on a 3D surface



- Planning with dynamics

(friction forces)



Courtesy of V. Vonásek and P. Vaněk

Car-Like Robot

- Configuration

$$\vec{x} = \begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$$

position and orientation

- Controls

$$\vec{u} = \begin{pmatrix} v \\ \varphi \end{pmatrix}$$

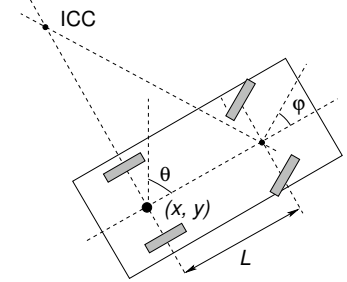
forward velocity, steering angle

- System equation

$$\dot{x} = v \cos \phi$$

$$\dot{y} = v \sin \phi$$

$$\dot{\phi} = \frac{v}{L} \tan \varphi$$



Kinematic constraints $\dim(\vec{u}) < \dim(\vec{x})$

Differential constraints on possible \dot{q} :

$$\dot{x} \sin(\phi) - \dot{y} \cos(\phi) = 0$$

Sampling-Based Motion Planning

- PRM and RRT are theoretically probabilistic complete
- They provide a feasible solution without quality guarantee

Despite that, they are successfully used in many practical applications

- In 2011, a systematical study of the asymptotic behavior of randomized sampling-based planners has been published

It shows, that in some cases, they converge to a non-optimal value with a probability 1

- Based on the study, new algorithms have been proposed: RRG and optimal RRT (RRT*)

Karaman, S., Frazzoli, E. (2011): Sampling-based algorithms for optimal motion planning. IJRR.



<http://seerac.scripts.mit.edu/rtrtstar>

Part II

Part 2 – Optimal Sampling-based Motion Planning Methods

RRT and Quality of Solution 2/2

- RRT does not satisfy a necessary condition for the asymptotic optimality

- For $0 < R < \inf_{q \in Q_{goal}} \|q - q_{init}\|$, the event $\{\lim_{n \rightarrow \infty} Y_n^{RRT} = c^*\}$ occurs only if the k -th branch of the RRT contains vertices outside the R -ball centered at q_{init} for infinitely many k

See Appendix B in Karaman&Frazzoli, 2011

- It is required the root node will have infinitely many subtrees that extend at least a distance ϵ away from q_{init}

The sub-optimality is caused by disallowing new better paths to be discovered

Rapidly-exploring Random Graph (RRG)

Algorithm 4: Rapidly-exploring Random Graph (RRG)

Vstup: q_{init} , number of samples n

Výstup: $G = (V, E)$

```

V ← ∅; E ← ∅;
for i = 0, ..., n do
    qrand ← SampleFree;
    qnearest ← Nearest(G = (V, E), qrand);
    qnew ← Steer(qnearest, qrand);
    if CollisionFree(qnearest, qnew) then
        Qnear ← Near(G = (V, E), qnew, min{γRRG(log(card(V))/card(V))1/d, η});
        V ← V ∪ {qnew};
        E ← E ∪ {(qnearest, qnew), (qnew, qnearest)};
        foreach qnear ∈ Qnear do
            if CollisionFree(qnear, qnew) then
                E ← E ∪ {(qrand, qnear), (qrand, qnew)};
return G = (V, E);
    
```

Proposed by Karaman and Frazzoli (2011). Theoretical results are related to properties of Random Geometric Graphs (RGG) introduced by Gilbert (1961) and further studied by Penrose (1999).

RRG Expansions

- At each iteration, RRG tries to connect new sample to all vertices in the r_n ball centered at it.

- The ball of radius

$$r(\text{card}(V)) = \min \left\{ \gamma_{RRG} \left(\frac{\log(\text{card}(V))}{\text{card}(V)} \right)^{1/d}, \eta \right\}$$

where

- η is the constant of the local steering function
- $\gamma_{RRG} > \gamma_{RRG}^* = 2(1 + 1/d)^{1/d} (\mu(C_{free})/\xi_d)^{1/d}$
 - d – dimension of the space;
 - $\mu(C_{free})$ – Lebesgue measure of the obstacle-free space;
 - ξ_d – volume of the unit ball in d -dimensional Euclidean space.
- The connection radius decreases with n
- The rate of decay \approx the average number of connections attempted is proportional to $\log(n)$

RRG Properties

- Probabilistically complete
- Asymptotically optimal
- Complexity is $O(\log n)$

(per one sample)

- Computational efficiency and optimality

- Attempt connection to $\Theta(\log n)$ nodes at each iteration;

in average

- Reduce volume of the “connection” ball as $\log(n)/n$;
- Increase the number of connections as $\log(n)$

Other Variants of the Optimal Motion Planning

- PRM*** – it follows standard PRM algorithm where connections are attempted between roadmap vertices that are within connection radius r as a function of n

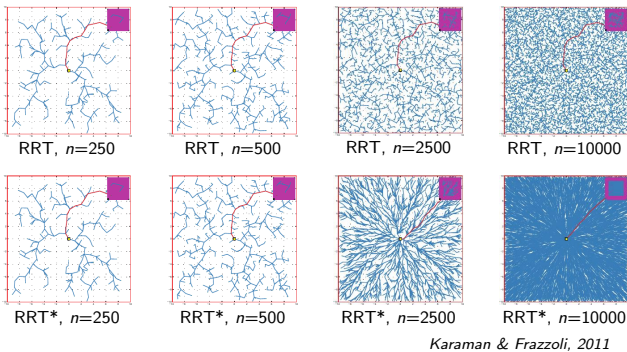
$$r(n) = \gamma_{PRM}(\log(n)/n)^{1/d}$$

- RRT*** – a modification of the RRG, where cycles are avoided

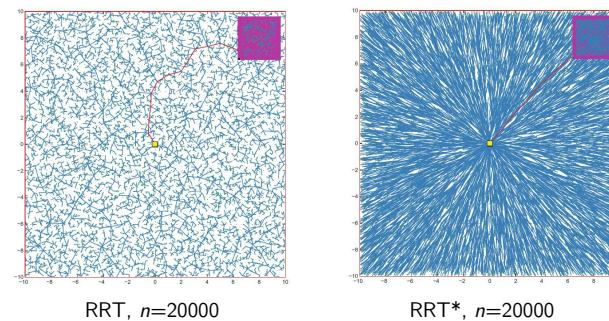
A tree version of the RRG

- A tree roadmap allows to consider non-holonomic dynamics and kinodynamic constraints
- It is basically RRG with “rerouting” the tree when a better path is discovered

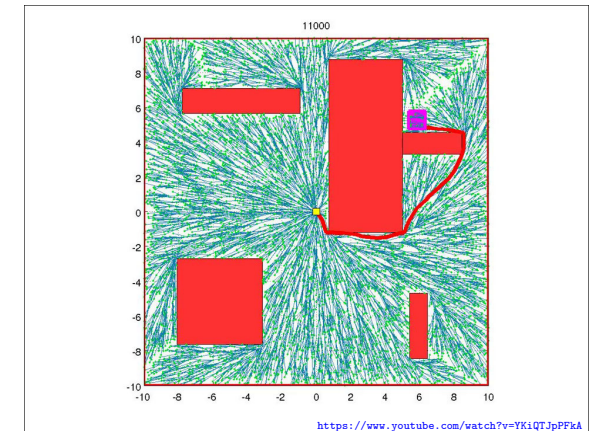
Example of Solution 1/3



Example of Solution 2/3



Example of Solution 3/3



Overview of Randomized Sampling-based Algorithms

Algorithm	Probabilistic Completeness	Asymptotic Optimality
sPRM	✓	✗
k-nearest sPRM	✗	✗
RRT	✓	✗
RRG	✓	✓
PRM*	✓	✓
RRT*	✓	✓

Notice, k -nearest variants of RRG, PRM*, and RRT* are complete and optimal as well

Summary of the Lecture

Topics Discussed

- Randomized Sampling-based Methods
- Probabilistic Road Map (PRM)
- Characteristics of path planning problems
- Random sampling
- Rapidly Exploring Random Tree (RRT)
- Optimal sampling-based motion planning
- Rapidly-exploring Random Graph (RRG)

- Next: Multi-Goal Motion Planning and Multi-Goal Path Planning