Grid and Graph based Path Planning Methods

Jan Faigl

Department of Computer Science
Faculty of Electrical Engineering
Czech Technical University in Prague

Lecture 04

B4M36UIR – Artificial Intelligence in Robotics

Overview of the Lecture

- Part 1 – Grid and Graph based Path Planning Methods

  - Grid-based Planning

  - DT for Path Planning

  - Graph Search Algorithms

  - D* Lite

Part I

# Part 1 – Grid and Graph based Path Planning Methods

Outline

- Grid-based Planning

- DT for Path Planning

- Graph Search Algorithms

- D* Lite

Grid-based Planning

- A subdivision of $\mathcal{C}_{free}$ into smaller cells
- **Grow obstacles** can be simplified by growing borders by a diameter of the robot
- Construction of the planning graph $G = (V, E)$ for V as a set of cells and E as the **neighbor-relations**
  - 4-neighbors and 8-neighbors
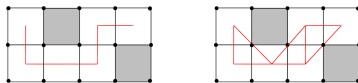
- A grid map can be constructed from the so-called occupancy grid maps

  E.g., using thresholding
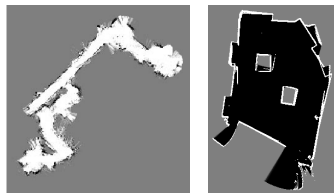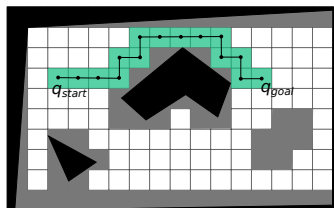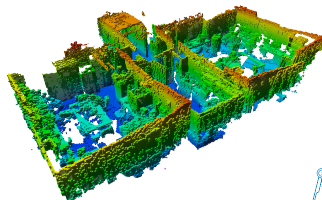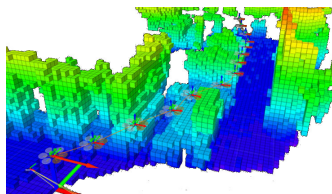
Grid-based Environment Representations

- Hiearchical planning
  - Coarse resolution and re-planning on finer resolution

  Holte, R. C. et al. (1996): Hierarchical A *: searching abstraction hierarchies efficiently. AAAI.

- Octree can be used for the map representation

- In addition to squared (or rectangular) grid a hexagonal grid can be used

- 3D grid maps – **octomap**

  `https://octomap.github.io`

– Memory grows with the size of the environment

– Due to limited resolution it may fail in narrow passages of $\mathcal{C}_{free}$

Example of Simple Grid-based Planning

- Wave-front propagation using path simplication

- Initial map with a robot and goal
- Obstacle growing
- Wave-front propagation – "flood fill"
- Find a path using a navigation function
- Path simplification
  - "Ray-shooting" technique combined with **Bresenham's line algorithm**
  - The path is a sequence of "key" cells for avoiding obstacles

Example of Simple Grid-based Planning

- Wave-front propagation using path simplication

- Initial map with a robot and goal
- Obstacle growing
- Wave-front propagation – "flood fill"
- Find a path using a navigation function
- Path simplification
    - "Ray-shooting" technique combined with **Bresenham's line algorithm**
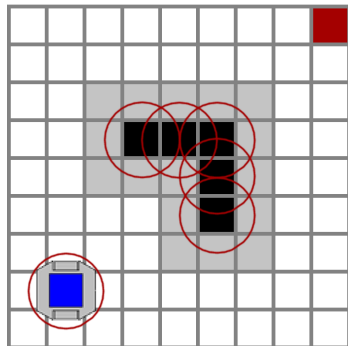   - The path is a sequence of "key" cells for avoiding obstacles

Example of Simple Grid-based Planning

- Wave-front propagation using path simplication

- Initial map with a robot and goal
- Obstacle growing
- Wave-front propagation – "flood fill"
- Find a path using a navigation function
- Path simplification
    - "Ray-shooting" technique combined with **Bresenham's line algorithm**
    - The path is a sequence of "key" cells for avoiding obstacles

| 8	7	6	5	4	3	2	1	0
8	7	6	5	4	3	2	1	1
8	7						2	2
8	8						3	3
9	9						4	4
10	10	10	10				5	5
11	11	11	10				6	6
	12	11	10	9	8	7	7	7
		11	10	9	8	8	8	8

Example of Simple Grid-based Planning

- Wave-front propagation using path simplication

- Initial map with a robot and goal
- Obstacle growing
- Wave-front propagation – "flood fill"
- Find a path using a navigation function
- Path simplification
  - "Ray-shooting" technique combined with **Bresenham's line algorithm**
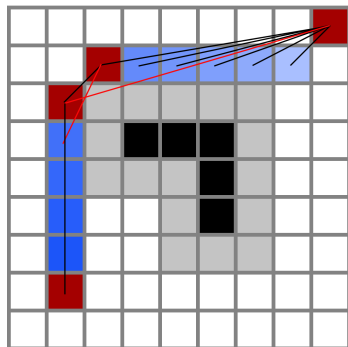 - The path is a sequence of "key" cells for avoiding obstacles
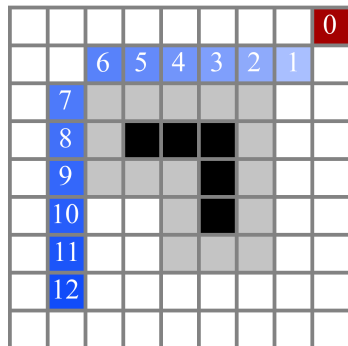
# Example of Simple Grid-based Planning

- Wave-front propagation using path simplication

- Initial map with a robot and goal
- Obstacle growing
- Wave-front propagation – "flood fill"
- Find a path using a navigation function
- Path simplification
    - "Ray-shooting" technique combined with **Bresenham's line algorithm**
    - The path is a sequence of "key" cells for avoiding obstacles

Path Simplification
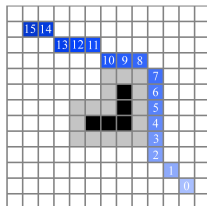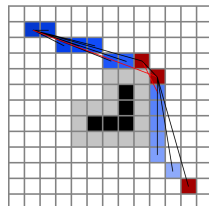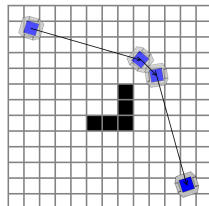
- The initial path is found in a grid using 4-neighborhood

- The rayshoot cast a line into a grid and possible collisions of the robot with obstacles are checked

- The "farthest" cells without collisions are used as "turn" points

- The final path is a sequence of straight line segments
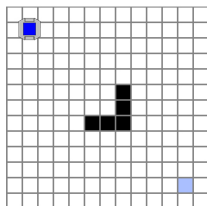
Initial and goal locations

Obstacle growing,
wave-front propagation

Ray-shooting

Simplified path

Bresenham's Line Algorithm

- Filling a grid by a line with avoding float numbers
- A line from (x_0, y_0) to (x_1, y_1) is given by $y = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0$

```cpp
1   CoordsVector& bresenham(const Coords& pt1, const
            Coords& pt2, CoordsVector& line)
2   {
3       // The pt2 point is not added into line
4       int x0 = pt1.c; int y0 = pt1.r;
5       int x1 = pt2.c; int y1 = pt2.r;
6       Coords p;
7       int dx = x1 - x0;
8       int dy = y1 - y0;
9       int steep = (abs(dy) >= abs(dx));
10      if (steep) {
11          SWAP(x0, y0);
12          SWAP(x1, y1);
13          dx = x1 - x0; // recompute Dx, Dy
14          dy = y1 - y0;
15      }
16      int xstep = 1;
17      if (dx < 0) {
18          xstep = -1;
19          dx = -dx;
20      }
21      int ystep = 1;
22      if (dy < 0) {
23          ystep = -1;
24          dy = -dy;
25      }
26      int twoDy = 2 * dy;
27      int twoDyTwoDx = twoDy - 2 * dx; //2*Dy - 2*Dx
28      int e = twoDy - dx; //2*Dy - Dx
29      int y = y0;
30      int xDraw, yDraw;
31      for (int x = x0; x != x1; x += xstep) {
32          if (steep) {
33              xDraw = y;
34              yDraw = x;
35          } else {
36              xDraw = x;
37              yDraw = y;
38          }
39          p.c = xDraw;
40          p.r = yDraw;
41          line.push_back(p); // add to the line
42          if (e > 0) {
43              e += twoDyTwoDx; //E += 2*Dy - 2*Dx
44              y = y + ystep;
45          } else {
46              e += twoDy; //E += 2*Dy
47          }
48      }
49      return line;
50  }
```

Outline

- Grid-based Planning

- DT for Path Planning

- Graph Search Algorithms

- D* Lite

Distance Transform based Path Planning

- For a given goal location and grid map compute a navigational function using *wave-front* algorithm, i.e., a kind of *potential field*
    - The value of the goal cell is set to 0 and all other free cells are set to some very high value
    - For each free cell compute a number of cells towards the goal cell
    - It uses 8-neighbors and distance is the Euclidean distance of the centers of two cells, i.e., EV=1 for orthogonal cells or $EV = \sqrt{2}$ for diagonal cells
    - The values are iteratively computed until the values are changed
    - The value of the cell c is computed as

$$cost(c) = \min_{i=1}^{8} \left(cost(c_i) + EV_{c_i,c} \right),$$

    where c_i is one of the neighboring cells from 8-neighborhood of the cell c

- The algorithm provides a cost map of the path distance from any free cell to the goal cell

- The path is then used following the gradient of the cell cost

Jarvis, R. (2004): Distance Transform Based Visibility Measures for Covert Path Planning in Known but Dynamic Environments

Example – Distance Transform based Path Planning

Distance Transform Path Planning

Algorithm 1: Distance Transform for Path Planning

---

for $y := 0$ to $yMax+1$ **do**
    for $x := 0$ to $xMax+1$ **do**
        if *goal [x,y]* **then**
            | cell $[x,y] := 0$;
        else
            | cell $[x,y] := xMax * y$ Max;

repeat
    for $y := 2$ to $yMax$ **do**
        for $x := 2$ to $xMax$ **do**
            if *not blocked [x,y]* **then**
                | cell $[x,y] := \min ($cell$[x\text{-}1,y]+1, $cell$[x\text{-}1,y\text{-}1]+\sqrt{2},$cell$[x,y\text{-}1]+1, $cell$[x+1,y\text{-}1]+\sqrt{2},$cell $[x,y])$;

    for $y := yMax\text{-}1 downto 1$ **do**
        for $x := xMax\text{-}1$ downto 1 **do**
            if *not blocked [x,y]* **then**
                | cell$[x,y] := \min($cell$[x+1,y]+1,$cell$[x+1,y+1]+\sqrt{2},$cell$[x,y+1]+1,$cell$[x\text{-}1,y+1]+\sqrt{2},$cell$[x,y])$;

until *no change*;

Distance Transform based Path Planning – Impl. 1/2

```
 1   Grid& DT::compute(Grid& grid) const
 2   {
 3       static const double DIAGONAL = sqrt(2);
 4       static const double ORTOGONAL = 1;
 5       const int H = map.H;
 6       const int W = map.W;
 7       assert(grid.H == H and grid.W == W, "size");
 8       bool anyChange = true;
 9       int counter = 0;
10       while (anyChange) {
11           anyChange = false;
12           for (int r = 1; r < H - 1; r++) {
13               for (int c = 1; c < W - 1; c++) {
14                   if (map[r][c] != FREESPACE) {
15                       continue;
16                   } //obstacle detected
17                   double t[4];
18                   t[0] = grid[r - 1][c - 1] + DIAGONAL;
19                   t[1] = grid[r - 1][c] + ORTOGONAL;
20                   t[2] = grid[r - 1][c + 1] + DIAGONAL;
21                   t[3] = grid[r][c - 1] + ORTOGONAL;
22                   double pom = grid[r][c];
23                   for (int i = 0; i < 4; i++) {
24                       if (pom > t[i]) {
25                           pom = t[i];
26                           anyChange = true;
27                       }
28                   }
29                   if (anyChange) {
30                       grid[r][c] = pom;
31                   }
32               }
33           }
35           for (int r = H - 2; r >= 0; r--) {
36               for (int c = W - 2; c > 0; c--) {
37                   if (map[r][c] != FREESPACE) {
38                       continue;
39                   } //obstacle detected
40                   double t[4];
41                   t[1] = grid[r + 1][c] + ORTOGONAL;
42                   t[0] = grid[r + 1][c + 1] + DIAGONAL;
43                   t[3] = grid[r][c + 1] + ORTOGONAL;
44                   t[2] = grid[r + 1][c - 1] + DIAGONAL;
45                   double pom = grid[r][c];
46                   bool s = false;
47                   for (int i = 0; i < 4; i++) {
48                       if (pom > t[i]) {
49                           pom = t[i];
50                           s = true;
51                       }
52                   }
53                   if (s) {
54                       anyChange = true;
55                       grid[r][c] = pom;
56                   }
57               }
58           }
59           counter++;
60       } //end while any change
61       return grid;
62   }
```

Distance Transform based Path Planning – Impl. 2/2

- The path is retrived by following the minimal value towards the goal using `min8Point()`

```
1   Coords& min8Point(const Grid& grid, Coords& p)
2   {
3       double min = std::numeric_limits<double>::max();
4       const int H = grid.H;
5       const int W = grid.W;
6       Coords t;
7
8       for (int r = p.r - 1; r <= p.r + 1; r++) {
9           if (r < 0 or r >= H) { continue; }
10          for (int c = p.c - 1; c <= p.c + 1; c++) {
11              if (c < 0 or c >= W) { continue; }
12              if (min > grid[r][c]) {
13                  min = grid[r][c];
14                  t.r = r; t.c = c;
15              }
16          }
17      }
18      p = t;
19      return p;
20  }
```

```
22  CoordsVector& DT::findPath(const Coords& start,
        const Coords& goal, CoordsVector& path)
23  {
24      static const double DIAGONAL = sqrt(2);
25      static const double ORTOGONAL = 1;
26      const int H = map.H;
27      const int W = map.W;
28      Grid grid(H, W, H*W); // H*W max grid value
29      grid[goal.r][goal.c] = 0;
30      compute(grid);
31
32      if (grid[start.r][start.c] >= H*W) {
33          WARN("Path has not been found");
34      } else {
35          Coords pt = start;
36          while (pt.r != goal.r or pt.c != goal.c) {
37              path.push_back(pt);
38              min8Point(grid, pt);
39          }
40          path.push_back(goal);
41      }
42      return path;
43  }
```

DT Example
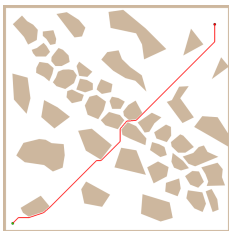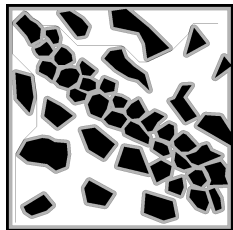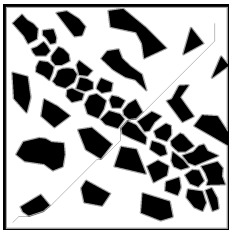
$\delta = 10$ cm, $L = 27.2$ m

$\delta = 30$ cm, $L = 42.8$ m

Outline

- Grid-based Planning

- DT for Path Planning

- Graph Search Algorithms

- D* Lite

Graph Search Algorithms

- The grid can be considered as a graph and the path can be found using graph search algorithms
- The search algorithms working on a graph are of general use, e.g.
  - Breadth-first search (BSD)
  - Depth first search (DFS)
  - Dijsktra's algorithm,
  - A* algorithm and its variants
- There can be grid based speedups techniques, e.g.,
  - Jump Search Algorithm (JPS) and JPS+
- There are many search algorithm for on-line search, incremental search and with any-time and real-time properties, e.g.,
  - Lifelong Planning A* (LPA*)

    Koenig, S., Likhachev, M. and Furcy, D. (2004): Lifelong Planning A*. AIJ.
  - E-Graphs – Experience graphs

    Phillips, M. et al. (2012): E-Graphs: Bootstrapping Planning with Experience Graphs. RSS.

Examples of Graph/Grid Search Algorithms

A* (general)

Dijkstra's Algorithm

- Disjktra's algorithm determines paths as iterative update of the cost of the shortest path to the particular nodes

  Edsger W. Dijkstra, 1956

  - Let start with the initial cell (node) with the cost set to 0 and update all successors

  - Select the node
    - with a path from the initial node
    - and has a lower cost
  - Repeat until there is a reachable node
    - I.e., a node with a path from the initial node
    - has a cost and parent (*green nodes*).

  The cost of nodes can only decrease (edge cost is positive). Therefore, for a node with the currently lowest cost, there cannot be a shorter path from the initial node.
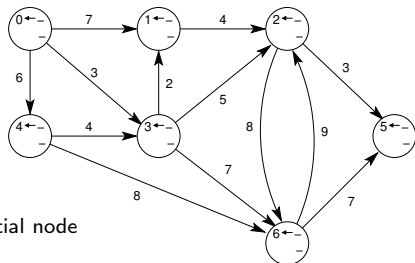
Dijkstra's Algorithm

■ Disktra's algorithm determines paths as iterative update of the cost of the shortest path to the particular nodes

Edsger W. Dijkstra, 1956

■ **Let start with the initial cell (node) with the cost set to 0 and update all successors**

■ Select the node
  ■ with a path from the initial node
  ■ and has a lower cost

■ Repeat until there is a reachable node
  ■ I.e., a node with a path from the initial node
  ■ has a cost and parent (*green nodes*).

The cost of nodes can only decrease (edge cost is positive). Therefore, for a node with the currently lowest cost, there cannot be a shorter path from the initial node.
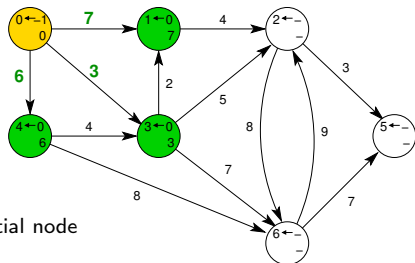
Dijkstra's Algorithm

- Dijsktra's algorithm determines paths as iterative update of the cost of the shortest path to the particular nodes

    Edsger W. Dijkstra, 1956

    - Let start with the initial cell (node) with the cost set to 0 and update all successors

    - Select the node
        - with a path from the initial node
        - and has a lower cost
    - Repeat until there is a reachable node
        - I.e., a node with a path from the initial node
        - has a cost and parent (*green nodes*).

    The cost of nodes can only decrease (edge cost is positive). Therefore, for a node with the currently lowest cost, there cannot be a shorter path from the initial node.
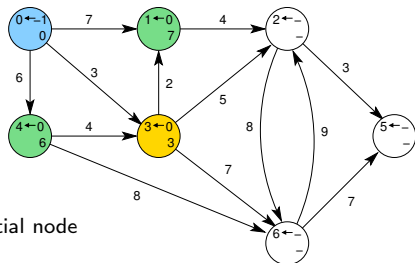
Dijkstra's Algorithm

- Disjktra's algorithm determines paths as iterative update of the cost of the shortest path to the particular nodes

  Edsger W. Dijkstra, 1956

  - Let start with the initial cell (node) with the cost set to 0 and update all successors
  - Select the node
    - with a path from the initial node
    - and has a lower cost
  - Repeat until there is a reachable node
    - I.e., a node with a path from the initial node
    - has a cost and parent (*green nodes*).

  The cost of nodes can only decrease (edge cost is positive). Therefore, for a node with the currently lowest cost, there cannot be a shorter path from the initial node.
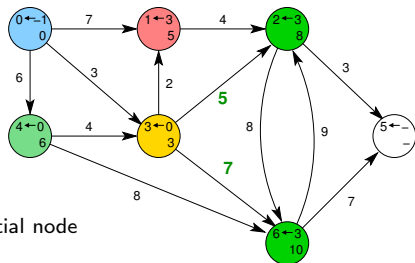
Example (cont.)

1: After the expansion, the shortest path to the node 2 is over the node 3
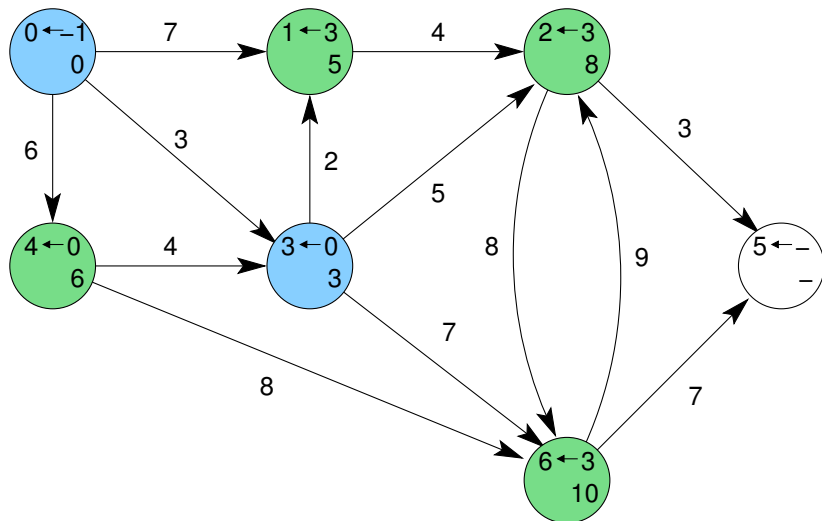
Example (cont.)

2: There is not shorter path to the node 2 over the node 1

$1: 4 + 5 = 9 > 8!$

Example (cont.)

3: After the expansion, there is a new path to the node 5

Example (cont.)

4: The path does not improve for further expansions

Dijkstra's Algorithm – Impl.

```
1   dij->nodes[dij->start_node].cost = 0; // init
2   void *pq = pq_alloc(dij->num_nodes); // set priority queue
3   int cur_label;
4   pq_push(pq, dij->start_node, 0);
5   while ( !pq_is_empty(pq) && pq_pop(pq, &cur_label)) {
6      node_t *cur = &(dij->nodes[cur_label]); // remember the current node
7      for (int i = 0; i < cur->edge_count; ++i) { // all edges of cur
8         edge_t *edge = &(dij->graph->edges[cur->edge_start + i]);
9         node_t *to = &(dij->nodes[edge->to]);
10        const int cost = cur->cost + edge->cost;
11        if (to->cost == -1) { // node to has not been visited
12           to->cost = cost;
13           to->parent = cur_label;
14           pq_push(pq, edge->to, cost); // put node to the queue
15        } else if (cost < to->cost) { // node already in the queue
16           to->cost = cost; // test if the cost can be reduced
17           to->parent = cur_label; // update the parent node
18           pq_update(pq, edge->to, cost); // update the priority queue
19        }
20     } // loop for all edges of the cur node
21  } // priority queue empty
22  pq_free(pq); // release memory
```
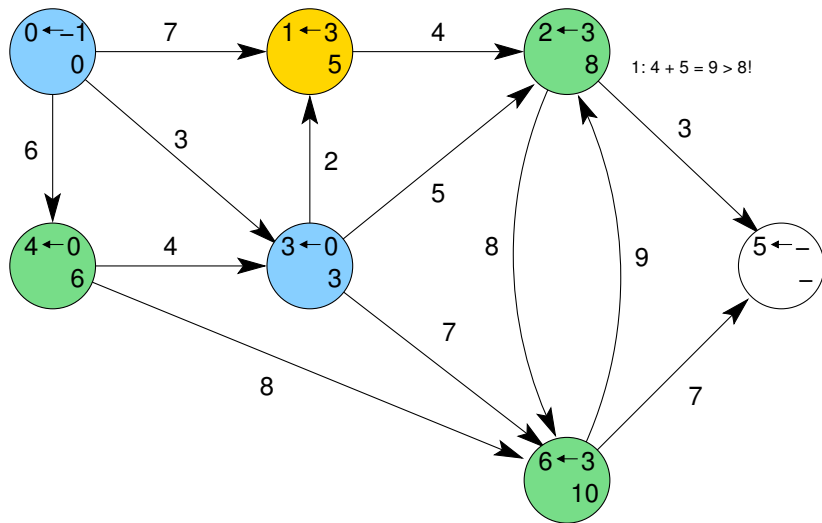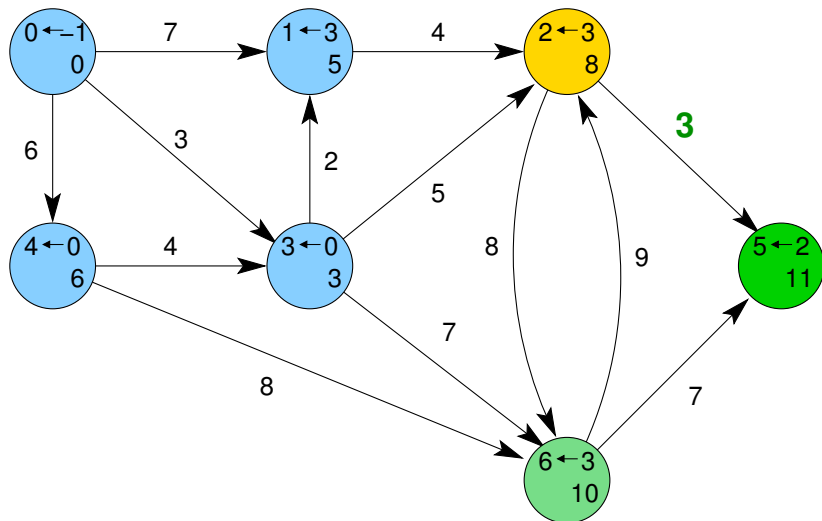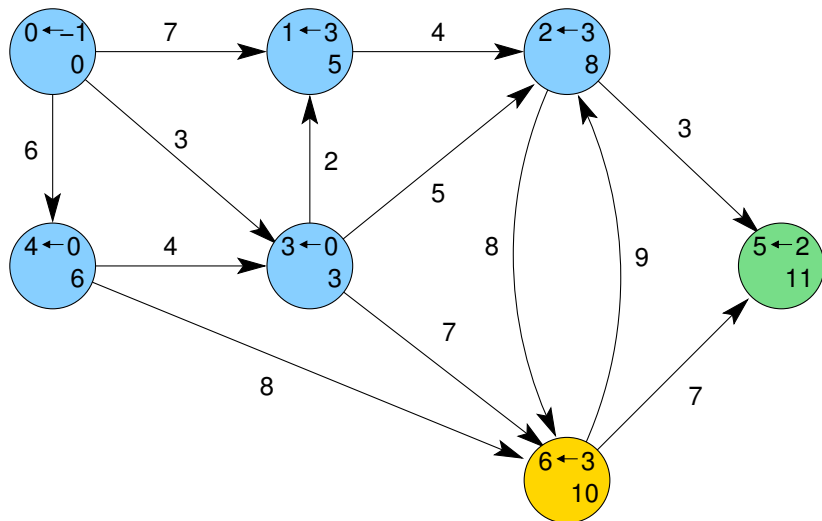
A* Algorithm

- A* uses a user-defined h-values (heuristic) to focus the search

    Peter Hart, Nils Nilsson, and Bertram Raphael, 1968

    - Prefer expansion of the node n with the lowest value

    $$f(n) = g(n) + h(n),$$

    where $g(n)$ is the cost (path length) from the start to n and $h(n)$ is the estimated cost from n to the goal

- h-values approximate the goal distance from particular nodes

- **Admissiblity condition** – heuristic always underestimate the remaining cost to reach the goal

    - Let $h^*(n)$ be the true cost of the optimal path from n to the goal
    - Then $h(n)$ is **admissible** if for all n: $h(n) \leq h^*(n)$
    - E.g., Euclidean distance is admissible
        - A straight line will always be the shortest path

- Dijkstra's algorithm – $h(n) = 0$

A* Implementation Notes

- The most costly operations of A* are
  - Insert and lookup an element in the **closed list**
  - Insert element and get minimal element (according to $f()$ value) from the **open list**
- The **closed list** can be efficiently implemented as a **hash set**
- The **open list** is usually implemented as a **priority queue**, e.g.,
  - Fibonacii heap, binomial heap, k-level bucket
  - **binary heap** is usually sufficient ($O(\log n)$)
- Forward A*
  1. Create a search tree and initiate it with the start location
  2. Select generated but not yet expanded state s with the smallest f-value, $f(s) = g(s) + h(s)$
  3. Stop if s is the goal
  4. Expand the state s
  5. Goto Step 2

Similar to Dijkstra's algorithm but it used $f(s)$ with heuristic $h(s)$ instead of pure $g(s)$

Dijsktra's vs A* vs Jump Point Search (JPS)


Dijkstra's Algorithm

# Jump Point Search Algorithm for Grid-based Path Planning

- **Jump Point Search** (JPS) algorithm is based on a macro operator that identifies and selectively expands only certain nodes (**jump points**)

  Harabor, D. and Grastien, A. (2011): Online Graph Pruning for Pathfinding on Grid Maps. AAAI.

- Natural neighbors after neighbor prunning with forced neighbors because of obstacle
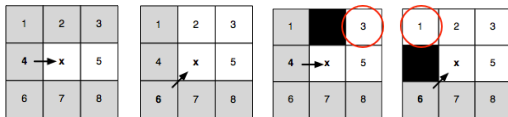
- Intermediate nodes on a path connecting two jump points are never expanded
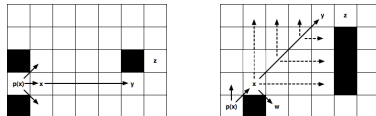
  - No preprocessing and no memory overheads while it speeds up A*

    https://harablog.wordpress.com/2011/09/07/jump-point-search/

- JPS+ – optimized preprocessed version of **JPS** with goal bounding

    https://github.com/SteveRabin/JPSPlusWithGoalBounding

    http://www.gdcvault.com/play/1022094/JPS-Over-100x-Faster-than

Theta* – Any-Angle Path Planning Algorithm

- **Any-angle path planning algorithms** simplify the path during the search

- **Theta*** is an extension of A* with `LineOfSight()`

  Nash, A., Daniel, K, Koenig, S. and Felner, A. (2007): Theta*: Any-Angle Path Planning on Grids. AAAI.

Algorithm 2: Theta* Any-Angle Planning

if *LineOfSight(parent(s), s')* **then**
     /* Path 2 – any-angle path */
     if *g(parent(s))+ c(parent(s), s') < g(s')* **then**
         parent(s') := parent(s);
         g(s') := g(parent(s)) + c(parent(s), s');
else
     /* Path 1 – A* path */
     if *g(s) + c(s,s') < g(s')* **then**
         parent(s'):= s;
         g(s') := g(s) + c(s,s');

- Path 2: considers path from start to parent(s) and from parent(s) to s' if s' has line-of-sight to parent(s)

http://aigamedev.com/open/tutorials/theta-star-any-angle-paths/
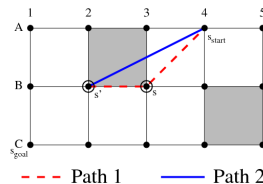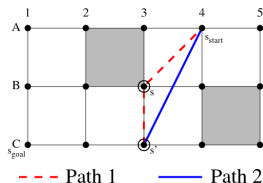
Theta* Any-Angle Path Planning Examples

- Example of found paths by the Theta* algorithm for the same problems as for the DT-based examples on Slide 16

Both algorithms implemented in C++
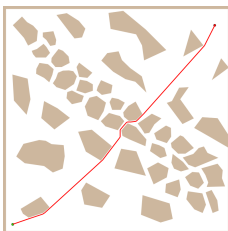
$\delta = 10$ cm, $L = 26.3$ m

$\delta = 30$ cm, $L = 40.3$ m

The same path planning problems solved by DT (without path smoothing) have $L_{\delta=10} = 27.2$ m and $L_{\delta=30} = 42.8$ m, while DT seems to be faster

- **Lazy Theta*** – reduces the number of line-of-sight checks

  Nash, A., Koenig, S. and Tovey, C. (2010): Lazy Theta*: Any-Angle Path Planning and Path Length Analysis in 3D. AAAI.

  http://aigamedev.com/open/tutorial/lazy-theta-star/

A* Variants – Online Search

- The state space (map) may not be known exactly in advance
    - Environment can **dynamically** change
    - True travel costs are **experienced** during the path execution
- Repeated A* searches can be computationally demanding
- **Incremental heuristic search**
    - Repeated planning of the path from the current state to the goal
    - Planning under the **free-space** assumption
    - **Reuse** information from the previous searches (**closed list** entries):
        - Focused Dynamic A* (**D***) – h^* is based on **traversability**, it has been used, e.g., for the Mars rover "Opportunity"

            Stentz, A. (1995): The Focussed D* Algorithm for Real-Time Replanning. IJCAI.

        - **D* Lite** – similar to D*

            Koenig, S. and Likhachev, M. (2005): Fast Replanning for Navigation in Unknown Terrain. T-RO.

- **Real-Time Heuristic Search**
    - Repeated planning with limited **look-ahead** – suboptimal but fast
        - Learning Real-Time A* (**LRTA***)

            Korf, E. (1990): Real-time heuristic search. JAI.

        - Real-Time Adaptive A* (**RTAA***)

            Koenig, S. and Likhachev, M. (2006): Real-time adaptive A*. AAMAS.

Real-Time Adaptive A* (RTAA*)

- Execute A* with limited **look-ahead**

- Learns better informed heuristic from the experience, initially $h(s)$, e.g., Euclidean distance

- Look-ahead defines **trade-off** between optimality and computational cost

    - `astar(lookahead)`

    A* expansion as far as "look-ahead" nodes and it terminates with the state s'

---

while $(s_{curr} \notin \text{GOAL})$ **do**
    astar(lookahead);
    if $s' = FAILURE$ **then**
        return FAILURE;
    for *all* $s \in CLOSED$ **do**
        H(s) := g(s') + h(s') - g(s);
    execute(plan); // perform one step
return SUCCESS;

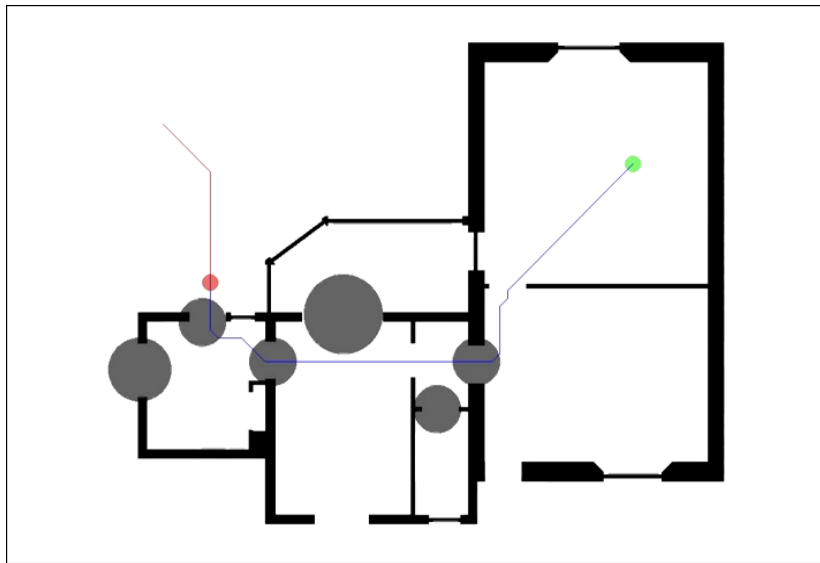s' is the last state expanded during the previous A* search

Outline

- Grid-based Planning

- DT for Path Planning

- Graph Search Algorithms

- D* Lite

D* Lite – Demo

D* Lite Overview

- It is similar to D*, but it is based on **Lifelong Planning A***

  Koenig, S. and Likhachev, M. (2002): D* Lite. AAAI.

- It searches from the goal node to the start node, i.e., g-values estimate the goal distance
- Store pending nodes in a priority queue
- Process nodes in order of increasing objective function value
- Incrementally repair solution paths when changes occur
- Maintains two estimates of costs per node
  - g – the objective function value – based on what we know
  - rhs – one-step lookahead of the objective function value – based on what we know
- **Consistency**
  - Consistent – $g = rhs$
  - Inconsistent – $g \neq rhs$
- Inconsistent nodes are stored in the priority queue (open list) for processing

D* Lite: Cost Estimates

- *rhs* of the node u is computed based on g of its successors in the graph and the transition costs of the edge to those successors

$$rhs(u) = \min_{s' \in Succ(u)} (g(s') + c(u, s'))$$

- The key/priority of a node s in the open list is the minimum of $g(s)$ and $rhs(s)$ plus a focusing heuristic h

$$[\min(g(s), rhs(s)) + h(s_{start}, s); \min(g(s), rhs(s))]$$

- The first term is used as the primary key
- The second term is used as the secondary key for tie-breaking

D* Lite Algorithm

- **Main** – repeat until the robot reaches the goal $\left(\text{or } g(s_{start}) = \infty \text{ there is no path}\right)$

```
Initialize();
ComputeShortestPath();
while (sstart ≠ sgoal) do
      sstart = argmins′∈Succ(sstart)(c(sstart, s′) + g(s′));

Move to sstart;
Scan the graph for changed edge costs;
if any edge cost changed perform then
      foreach directed edges (u, v) with changed edge costs do
            Update the edge cost c(u, v);
            UpdateVertex(u);

      foreach s ∈ U do
            U.Update(s, CalculateKey(s));

      ComputeShortestPath();
```

Procedure Initialize

```
U = 0;
foreach s ∈ S do
      rhs(s) := g(s) := ∞;

rhs(sgoal) := 0;
U.Insert(sgoal, CalculateKey(sgoal));
```

D* Lite Algorithm – ComputeShortestPath()

Procedure ComputeShortestPath

while $U.TopKey() < CalculateKey(s_{start})$ OR $rhs(s_{start}) \neq g(s_{start})$ **do**
    $u := U.Pop()$;
    if $g(u) > rhs(u)$ **then**
        $g(u) := rhs(u)$;
        foreach $s \in Pred(u)$ **do** UpdateVertex(s);
    else
        $g(u) := \infty$;
        foreach $s \in Pred(u) \bigcup \{u\}$ **do** UpdateVertex(s);

Procedure UpdateVertex

if $u \neq s_{goal}$ **then** $rhs(u) := \min_{s' \in Succ(u)}(c(u, s') + g(s'))$;
if $u \in U$ **then** U.Remove(u);
if $g(u) \neq rhs(u)$ **then** U.Insert(u, CalculateKey(u));

Procedure CalculateKey

return $[\min(g(s), rhs(s)) + h(s_{start}, s); \min(g(s), rhs(s))]$

Summary of the Lecture