Multivariate Analysis of Variance

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http://cw.felk.cvut.cz/wiki/courses/b4m36san/start

Agenda

- Bivariate statistical tests and their multivariate generalizations,
- relationship between continuous variables and a categorical variable
  - categorical variable = treatment, factor,
  - lots of methods, we will proceed from the most simple to most general,
- Review t-test for two groups
  - single continuous variable, binary factor/treatment,
  - non-parametric alternative,
  - multiple comparisons problem for more groups,
- Explain ANOVA
  - posthoc tests to find out which groups contributed most,
- Generalize towards MANOVA
  - two-way modification, non-parametric

Bivariate statistical models and tests

- assess strength of relationship between a pair of variables
  - independent (causal) and dependent (effect) variable,
  - rejection of null hypothesis does not imply causal relationship,
- all of them can be generalized towards multivariate statistics.

		dependent variable: categorical	dependent variable: continuous
independent variable	categorical	contingency table chi-square test	analysis of variance
	continuous	LDA logistic regression	correlation regression

Independence test for two categorical variables

- categorical variable
  - takes one of a limited (and fixed) number of possible values,
- contingency table
  - table showing observed (multivariate) joint frequency distribution,
  - for the moment concern two-way contingency tables only,
  - a pair of variables with r and c categories captured in a $r \times c$ table,
  - its elements represent frequency counts for the individual events,
  - an example: two binary variables $X_1 = gender$ and $X_2 = disease$

	X_{21}	$\dots$	X_{2c}	Σ
X_{11}	N_{11}		N_{1c}	$N_{1\bullet}$
$\dots$				
X_{1r}	N_{r1}		N_{rc}	$N_{r\bullet}$
Σ	$N_{\bullet 1}$		$N_{\bullet 2}$	N

	healthy	diseased	total
women	216	72	288
men	279	342	621
total	495	414	909

Independence test for two categorical variables

- independence assumption
  - H_0: two categorical variables are independent,
  - H_a: they have an association or relationship (of an unknown structure),
  - the frequency distribution does not change with the table rows,
- compare the observed frequencies with the expected ones
  - the expectations are derived from the marginal frequencies under the independence assumption, MLE approach is taken,
  - $E_{ij} = N\bar{p}_{i\bullet}\bar{p}_{\bullet j} = N\frac{N_{i\bullet}}{N}\frac{N_{\bullet j}}{N} = \frac{N_{i\bullet}N_{\bullet j}}{N}$.

O_{ij}	healthy	diseased	total
women	216	72	288
men	279	342	621
total	495	414	909

E_{ij}	healthy	diseased	total
women	157	131	288
men	338	283	621
total	495	414	909

Independence test for two categorical variables

- let us measure the discrepancy between the observed counts and the estimated expected counts under the null,
- Pearson's χ^2 is one of the options

$$X^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

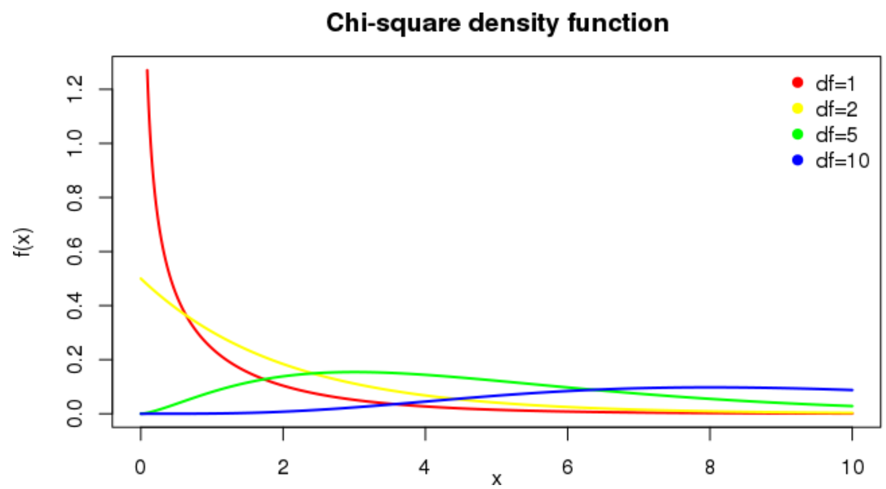
- a cumulative test statistic,
- it asymptotically approaches a χ^2 distribution
  - with $(r-1)(c-1)$ degrees of freedom,
- assumptions
  - non-parametric test, robust wrt distribution of the data,
  - one observation per subject, sufficient sample size ($E_{ij} \geq 5$).

Independence test for two categorical variables

- for the $gender$ and $disease$ relationship

$$X^2 = \frac{(216-157)^2}{157} + \frac{(72-131)^2}{131} + \frac{(279-338)^2}{338} + \frac{(342-283)^2}{283} = 71.3$$

- choose a significance level $\alpha = 0.01$ (type I error control),
- compare with the table value $\chi^2_{\alpha=0.01,df=1}$=6.635,
- since $X^2 > \chi^2_{df=1}$ reject H_0,
- the exact p-value: $p = 1 - F_{\chi^2(1)}(71.3) = 1.09e - 17$.



Independence test for two categorical variables

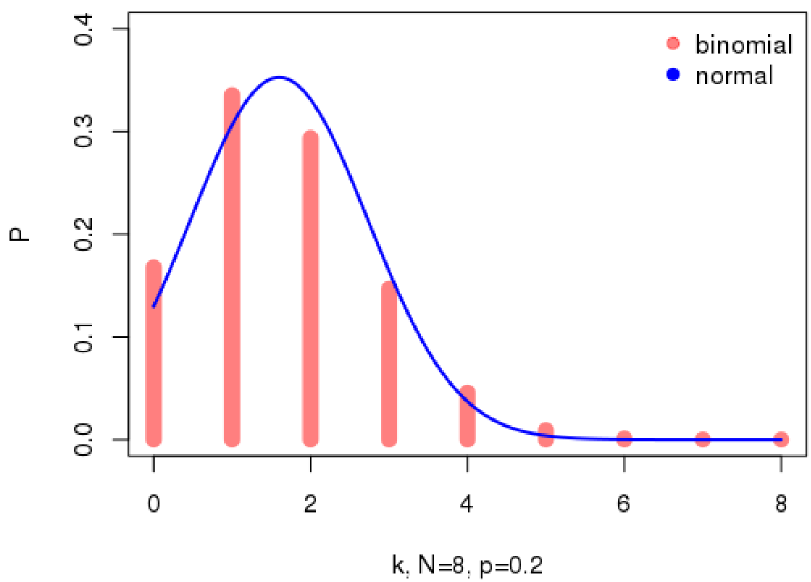
- clarification why the Pearson's test statistic follows χ^2 distribution,
- for simplicity, concern a simple goodness of fit test with only 2 categories
  - N trials, X observations in cat 1, $N - X$ observations in cat 2,
  - $p_1 = p = \frac{X}{N}$, $p_2 = 1 - p = \frac{N-X}{N}$
  - $H_0 : p = p_0$ (compare to a statistical model, a single number only here),
  - X follows binomial distribution

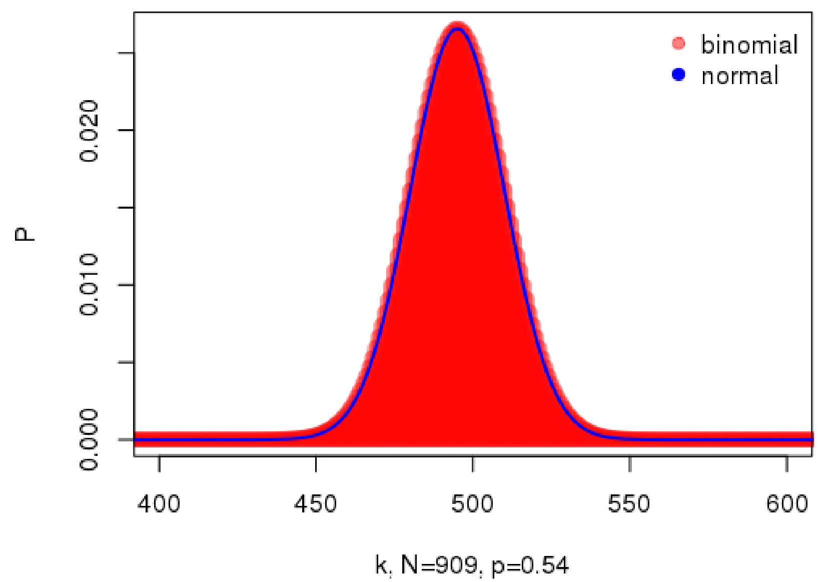
$$Pr(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$$

  - the probability of getting exactly k successes in N trials, each trial successful with probability p,
  - for large Ns can be approximated by $\mathrm{N}(Np, Np(1-p))$,
  - we can standardize X as $z = \frac{X-Np}{\sqrt{Np(1-p)}}$.

Independence test for two categorical variables

- compare binomial distribution and its approximation with normal distribution
  - **left**: small N, p $\ll$ 0.5, significant approximation error,
  - **right**: $disease$ variable from our smoking example, 495 healthy and 414 diseased individuals, N=909, p=0.54, negligible approximation error.





Independence test for two categorical variables

- χ^2_k – chi-square distribution with k degrees of freedom is the distribution of a sum of the squares of k independent standard normal random variables,
- a simple goodness of fit test with 2 categories can simply test whether

$$z^2 = \frac{(X-Np)^2}{Np(1-p)} \textit{ approximately} \sim \chi^2_1$$

- it can be shown that it is identical with Pearson's statistic

$$\begin{aligned}
\sum_{i=1}^{2} \frac{(O_i - E_i)^2}{E_i} &= \frac{(X-Np)^2}{Np} + \frac{[(N-X)-(N-Np)]^2}{N(1-p)} \\
&= \frac{(X-Np)^2}{Np} + \frac{(X-Np)^2}{N(1-p)} \\
&= (X-Np)^2\left(\frac{1}{Np} + \frac{1}{N(1-p)}\right)
\end{aligned}$$

Independence test for two categorical variables

- it further holds that

$$\frac{1}{Np} + \frac{1}{N(1-p)} = \frac{Np + N(1-p)}{NpN(1-p)} = \frac{1}{Np(1-p)}$$

- and consequently

$$\sum_{i=1}^{2} \frac{(O_i - E_i)^2}{E_i} = \frac{(X - Np)^2}{Np(1-p)} \quad \textit{approximately} \sim \chi^2_1$$

- the dependence between the two cells is compensated by diving by E_i instead of $E_i(1-p_i)$,
- this generalizes to multinomial distributions (larger contingency tables)
- the Pearson statistics has a distribution that asymptotically follows χ^2_{k-1},
- likelihood-ratio statistics $G = 2\sum_{ij} O_{ij} \ln \frac{O_{ij}}{E_{ij}}$ is actually preferred.

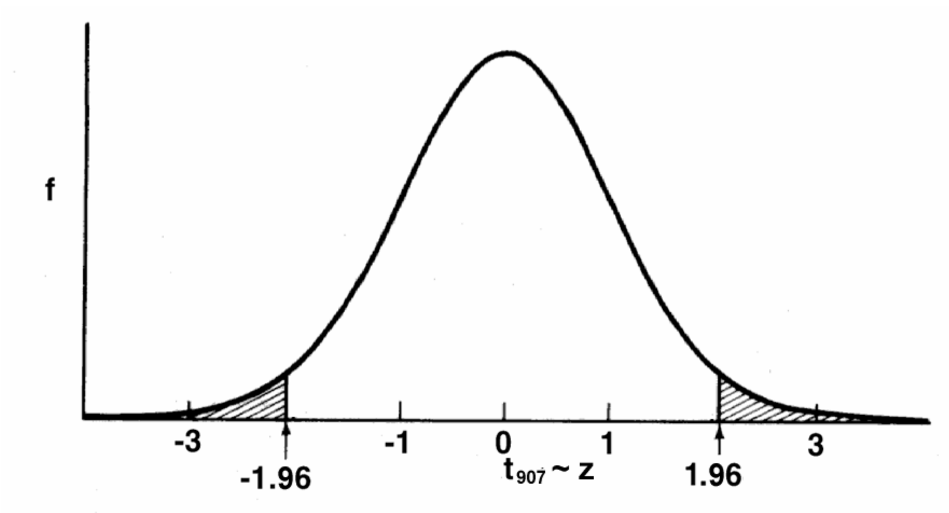
Review t-test for two groups

- a test in which the test statistic follows a Student's t-distribution ...
  - under the null hypothesis,
- consider a two sample t-test, $H_0 : \mu_1 = \mu_2$, $H_a : \mu_1 \neq \mu_2$
  - the two populations should follow a normal distribution,
  - variances of the two populations assumed equal $\rightarrow$ Student's t-tests,
  - variances can differ $\rightarrow$ **Welch's test** (see below),

$$t_{obs} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad \sim \quad t_{df}$$

  - $\bar{X}_i$, s_i^2 and n_i... sample means, variances and sizes,
  - $df \leq n_1 + n_2 - 2$, the exact formula complicated,
  - reject H_0 if $|t_{obs}| \geq t_{df,1-\alpha/2}$.

t-distribution



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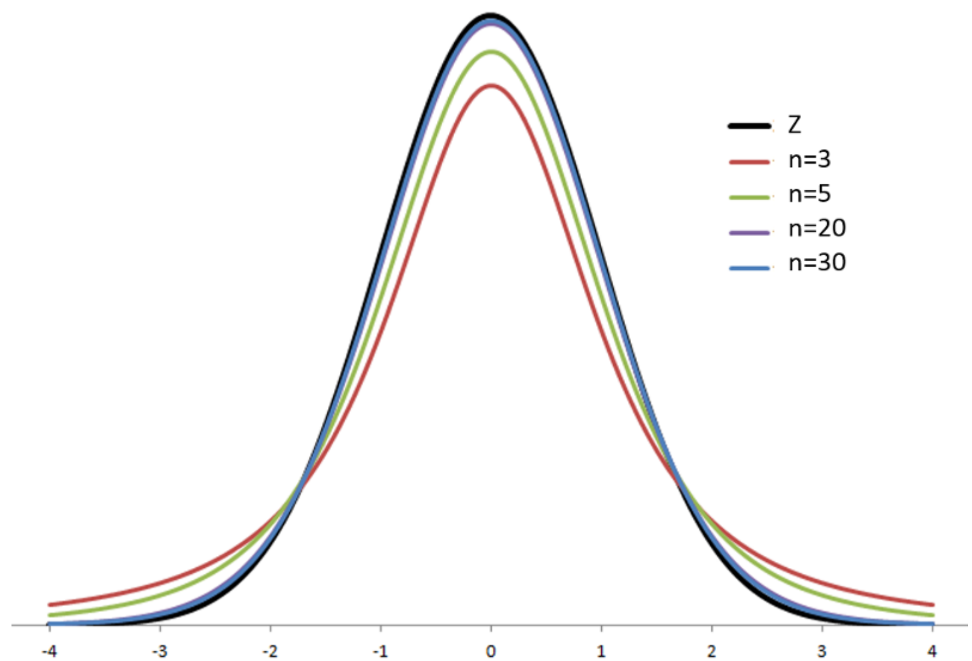

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T-test for multiple groups

- Concern a categorical variable with many levels $\rightarrow$ multiple groups,
- conduct a two-sample t-test for a difference in means for each pair of groups
  - the number of comparisons grows quadratically with the number of groups/levels,
- for $\alpha = 0.05$ for each comparison
  - there is a 5% chance that each comparison will falsely be called significant,
  - the overall probability of Type I error is elevated above 5%,
  - we falsely reject at least one of the partial null hypothesis with probability

$$1 - (1 - \alpha)^{\binom{g}{2}}$$

  - e.g., for 4 levels it makes $0.26 \gg \alpha$,
- **multiple comparisons** must be corrected.

Multiple comparisons

- **multiple comparisons** must be corrected.
  - the most simple is the **Bonferroni correction**,
  - test each hypothesis at level $\alpha_{indiv} = \alpha_{overall}/m$,
    * m stands for the number of individual pair tests,
    * follows from Boferroni inequality for independent tests

$$\alpha_{overall} = 1 - (1 - \alpha)^m \leq m\alpha_{indiv}$$

    * e.g., $0.26 = 1 - 0.95^6 < 0.05 * 6 = 0.3$,
  - however, this adjustment may be too conservative.

Analysis of variance (ANOVA)

- compares means for multiple (usually $g \geq 3$) independent populations
  - parametric and unpaired, one-way,
  - relationship between a categorical factor F and a continuous outcome Y,
  - extends a two sample t-test to multiple groups,

Subject	F	Y
1	f_1	y_1
2	f_2	y_2
...		
N	f_N	y_N

		1	...	g
Subject	1	y_{11}	...	y_{g1}
	2	y_{12}	...	y_{g2}
	...	...	...	...
	n_i	y_{1n_1}	...	y_{gn_g}

- y_{ij} ...observation for subject j in group i,
- n_i ...number of subjects in group i,
- $N = n_1 + n_2 + ... + n_g$ ...total sample size.

Analysis of variance (ANOVA)

- assumptions
  - the subjects are **independently sampled**
    - employ repeated measures ANOVA otherwise,
  - the data are **normally distributed** in each group
    - $E(Y_{i.}) = \mu_i$, e.g., no group sub-populations with different means,
    - residuals of the model below show the normal distribution
      $y_{ij} = \mu + \alpha_i + \epsilon_{ij} = \mu_i + \epsilon_{ij}$
    - employ non-parametric Kruskal-Wallis test otherwise,
  - the data are **homoscedastic**
    - the variability in the data does not depend on group membership,
    - there is a common variance $var(Y_{ij}) = \sigma^2$,
  - multiple comparisons problem for more groups,
- the hypotheses of interest
  - $H_0 : \mu_1 = \mu_2 = \cdots = \mu_g$,
  - $H_a : \mu_i \neq \mu_j$ for at least one $i \neq j$.

Analysis of variance (ANOVA)

- method
  - partition SS_{total}, the total variation in a response variable,
  - distinguish **within groups variability** SS_{error},
  - and **between groups variability** SS_{treat},

$$SS_{total} = \sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{..})^2 =$$
$$= \sum_{i=1}^{g}\sum_{j=1}^{n_i}\left((y_{ij}-\bar{y}_{i.})+(\bar{y}_{i.}-\bar{y}_{..})\right)^2 =$$
$$= \underbrace{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i.})^2}_{SS_{error}} + \underbrace{\sum_{i=1}^{g} n_i(\bar{y}_{i.}-\bar{y}_{..})^2}_{SS_{treat}}$$

  - $\bar{y}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$ . . . group i sample mean,
  - $\bar{y}_{..} = \frac{1}{N}\sum_{i=1}^{g}\sum_{j=1}^{n_i} y_{ij}$ . . . grand mean.

Analysis of variance (ANOVA)

- method
  - in a similar manner, partition the number of degrees of freedom that stand behind the observed sums of the squared deviations

$$DF_{total} = N - 1 = DF_{error} + DF_{treat} = (N - g) + (g - 1) = N - 1$$

  - decide whether group averages differ more than based on random variability observed in the dependent variable under the null hypothesis,
  - employ **mean square** variability, both within groups and between groups

$$MS_{error} = \frac{SS_{error}}{DF_{error}} = \frac{SS_{error}}{N - g} \qquad MS_{treat} = \frac{SS_{treat}}{DF_{treat}} = \frac{SS_{treat}}{g - 1}$$

Analysis of variance (ANOVA)

- method
  - compare the variance between the groups and within the groups,

$$F_{obs} = \frac{MS_{treat}}{MS_{error}} \sim F_{g-1,N-g}$$

  - if F_{obs} is small (close to 1), then variability between groups is negligible compared to variation within groups and the grouping does not explain much variation in the data,
  - if F_{obs} is large, then variability between groups is large compared to variation within groups and the grouping explains a lot of the variation in the data
- decision rule based on F_{obs}
  - reject H_0 if $F_{obs} \geq F_{\alpha,g-1,N-g}$,
  - fail to reject H_0 if $F_{obs} < F_{\alpha,g-1,N-g}$.

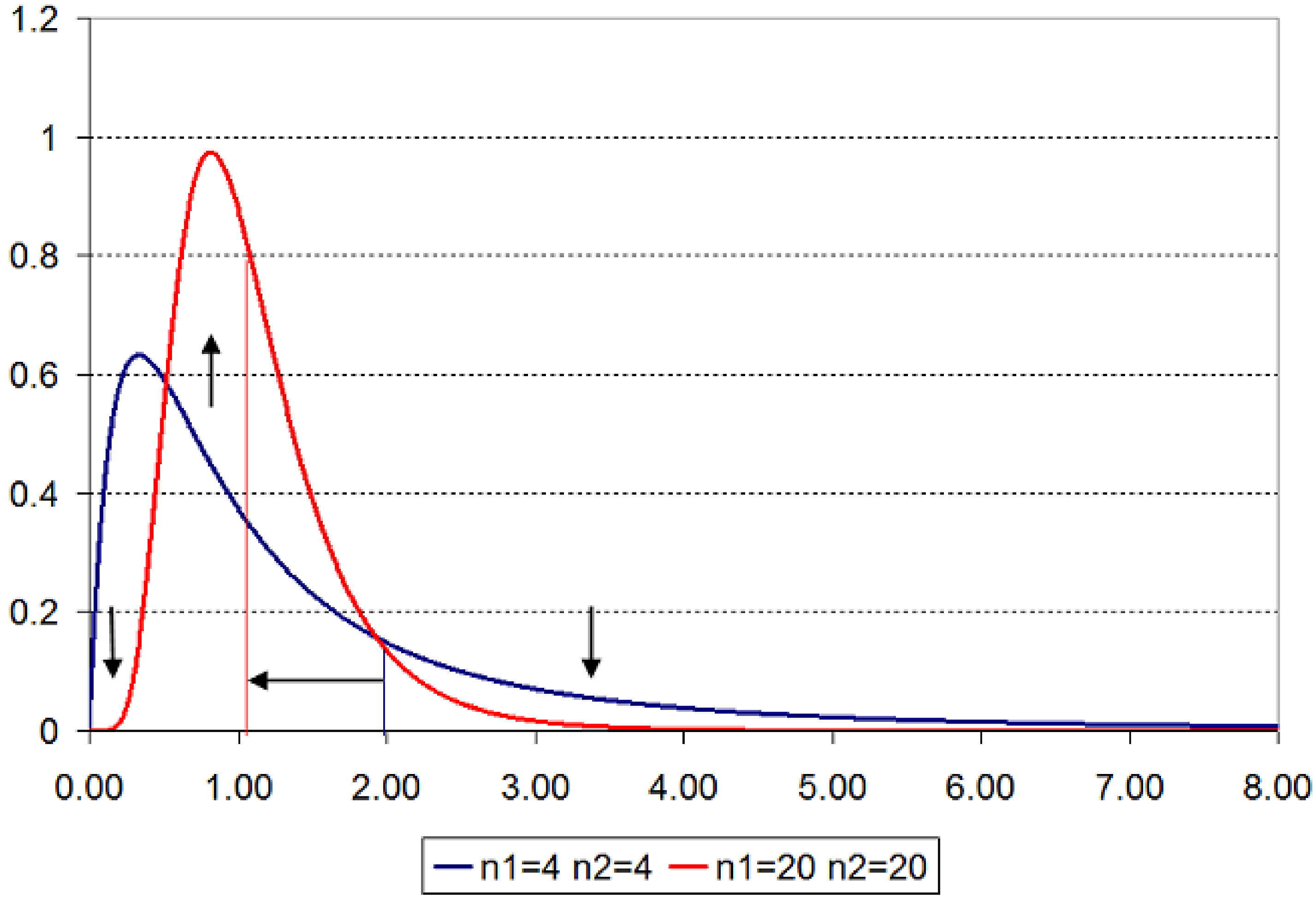
F-distribution

- F-distribution is any distribution obtained by taking the quotient of two χ^2 distributions divided by their respective degrees of freedom,
- consequently, any F-distribution has two parameters corresponding to the degrees of freedom for the two χ^2 distributions
- given $X_1 \sim \chi^2_{df_1}$ and $X_2 \sim \chi^2_{df_2}$

$$\frac{X_1/df_1}{X_2/df_2} \sim F_{df_1,df_2}$$

- F-distribution in R
  - find the value of $F_{\alpha,g-1,N-g}$:
    `qf(alpha, df1, df2, lower.tail = F)`,
  - find the ANOVA p-value when knowing F_{obs}:
    `pf(Fobs, df1, df2, lower.tail = F)`.

F-distribution



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Post-hoc ANOVA tests

- after performing ANOVA (and rejecting the null hypothesis)
  - we only assume that there is some difference in group means,
- a post-hoc test identifies which particular groups stand behind the test outcome,
- Tukey's HSD (honest significant difference) test
  - a t-test that controls for family-wise arror rate (FWER),
  - compares all pairs of group means,
  - identifies all pairs whose difference is larger than expected standard error,
  - observed test statistics related to the studentized range distribution,

$$q_{obs} = \frac{\bar{y}_{i.} - \bar{y}_{j.}}{\sqrt{\frac{MS_{error}}{n^*}}} \quad \sim \quad q_{g,N-g}$$

  - n^* ... number of observations per group (their harmonic mean if not equal),
  - always positive, sort the means before its application.

ANOVA extensions/alternatives

- up to now we talked about ANOVA that
  - is parametric,
  - deals with independent measurements,
  - is one-way (with a single factor),
  - concerns a single target variable only,
- other options
  - non-parametric analysis (Wilcoxon test $\rightarrow$ Kruskal-Wallis analysis),
  - compares all possible group means (repeated measures ANOVA, Friedman test if non-parametric too),
  - main effects ANOVA and factorial ANOVA,
  - multivariate ANOVA (MANOVA).

Multivariate analysis of variance (MANOVA)

- p variables measured on each subject, objects categorized into g disjoint groups.
- y_{ijk} ...an observation for variable k from subject j in group i,
- $\mathbf{y_{ij}}$ ...a vector of dependent variables for subject i in group i,
- assumptions
  - the subjects are **independently sampled**,
  - the data are **multivariate normally distributed** in each group,
  - the data from all groups have common covariance matrix Σ,
  - the data from group i has common mean vector $\mu_{\mathbf{i}}$ of length p,
- the hypotheses of interest
  - $H_0 : \mu_{\mathbf{1}} = \mu_{\mathbf{2}} = \cdots = \mu_{\mathbf{g}}$,
  - $H_a : \mu_{\mathbf{ik}} \neq \mu_{\mathbf{jk}}$ for at least one $i \neq j$ and at least one variable k.

Analysis of variance (ANOVA)

- method
  - the analogy of SS_{total} in ANOVA is a $p \times p$ **cross products matrix** $\mathbf{T}$,
  - similarly to ANOVA, it can be decomposed into the **Error Sum of Squares and Cross Products** $\mathbf{E}$, and the **Hypothesis Sum of Squares and Cross Products** $\mathbf{H}$.

$$\mathbf{T} = \sum_{\mathbf{i=1}}^{\mathbf{g}} \sum_{\mathbf{j=1}}^{\mathbf{n_i}} (\mathbf{y_{ij}} - \bar{\mathbf{y}}_{..})(\mathbf{y_{ij}} - \bar{\mathbf{y}}_{..})' =$$

$$= \sum_{\mathbf{i=1}}^{\mathbf{g}} \sum_{\mathbf{j=1}}^{\mathbf{n_i}} \{(\mathbf{y_{ij}} - \bar{\mathbf{y}}_{\mathbf{i}}) + (\bar{\mathbf{y}}_{\mathbf{i}} - \bar{\mathbf{y}}_{..})\}\{(\mathbf{y_{ij}} - \bar{\mathbf{y}}_{\mathbf{i}}) + (\bar{\mathbf{y}}_{\mathbf{i}} - \bar{\mathbf{y}}_{..})\}' =$$

$$= \underbrace{\sum_{\mathbf{i=1}}^{\mathbf{g}} \sum_{\mathbf{j=1}}^{\mathbf{n_i}} (\mathbf{y_{ij}} - \bar{\mathbf{y}}_{\mathbf{i.}})(\mathbf{y_{ij}} - \bar{\mathbf{y}}_{\mathbf{i.}})'}_{\mathbf{E}} + \underbrace{\sum_{\mathbf{i=1}}^{\mathbf{g}} \mathbf{n_i}(\bar{\mathbf{y}}_{\mathbf{i.}} - \bar{\mathbf{y}}_{..})(\bar{\mathbf{y}}_{\mathbf{i.}} - \bar{\mathbf{y}}_{..})'}_{\mathbf{H}}$$

* $\bar{\mathbf{y}}_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{y}_{ij}$ ... sample mean vector for group i,
* $\bar{\mathbf{y}}_{..} = \frac{1}{N} \sum_{i=1}^{g} \sum_{j=1}^{n_i} \mathbf{y}_{ij}$ ... grand mean vector of length p.

Multivariate analysis of variance (MANOVA)

- explanation of the elements of $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{H}$
  - the element $\mathbf{t}_{k,l}$ is
$$\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ijk}-\bar{y}_{..k})(y_{ijl}-\bar{y}_{..l})$$
  - for $k = l$ it is the total sum of squares for variable k, and measures the total variation in the kth variable, for $k \neq l$, this measures the dependence between variables k and l across all of the observations,
  - the element $\mathbf{e}_{k,l}$ is
$$\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ijk}-\bar{y}_{i.k})(y_{ijl}-\bar{y}_{i.l})$$
  - for $k = l$ it is the error sum of squares for variable k, and measures the within treatment variation for the kth variable, for $k \neq l$ it measures the dependence between variables k and l after taking into account the treatment,

Multivariate analysis of variance (MANOVA)

- explanation of the elements of $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{H}$
  - the element $\mathbf{h}_{k,l}$ is

$$\sum_{i=1}^{g} n_i(\bar{y}_{i.k} - \bar{y}_{..k})(\bar{y}_{i.l} - \bar{y}_{..l})$$

  - for $k = l$ it is the treatment sum of squares for variable k, and measures the between treatment variation for the kth variable, for $k \neq l$, this measures dependence of variables k and l across treatments.
- consequently, if the hypothesis sum of squares and cross products $\mathbf{H}$ is large relative to the error sum of squares and cross products matrix $\mathbf{E}$ we wish to reject H_0.

Multivariate analysis of variance (MANOVA)

- Wilk's lambda test statistics for MANOVA (several other statistics exist too)
  - the determinant of the error matrix $\mathbf{E}$ is divided by the determinant of the total matrix $\mathbf{T} = \mathbf{H} + \mathbf{E}$, we will reject the null hypothesis if Wilk's lambda is small/close to zero as then $\mathbf{H}$ is large relative to $\mathbf{E}$ too.

$$\Lambda^* = \frac{|\mathbf{E}|}{|\mathbf{H} + \mathbf{E}|}$$

  - can also be computed using the eigenvalues $\hat{\lambda}$ of $\mathbf{E^{-1}B}$ ($s = min(p, g-1)$)

$$\Lambda^* = \prod_{i=1}^{s} \frac{1}{1 + \hat{\lambda}_i}$$

  - the distribution of Λ^* is not tractable, we can only have approximations,
  - e.g., Bartlett's approximation can be used if N is large

$$-\left(N - 1 - \frac{p+g}{2}\right) \ln \lambda^* > \chi^2_{p(g-1),\alpha}$$

The main references

:: Resources (slides, scripts, tasks) and reading

- STAT 505 course on Applied Multivariate Statistical Analysis, PennState University, `https://onlinecourses.science.psu.edu/stat505/`.
- G. James, D. Witten, T. Hastie and R. Tibshirani: **An Introduction to Statistical Learning with Applications in R.** Springer, 2014.
- A. C. Rencher, W. F. Christensen: **Methods of Multivariate Analysis.** 3rd Edition, Wiley, 2012.
- T. Hastie, R. Tibshirani and J. Friedman: **The Elements of Statistical Learning: Data Mining, Inference, and Prediction.** Springer, 2009.