Graph Algorithms

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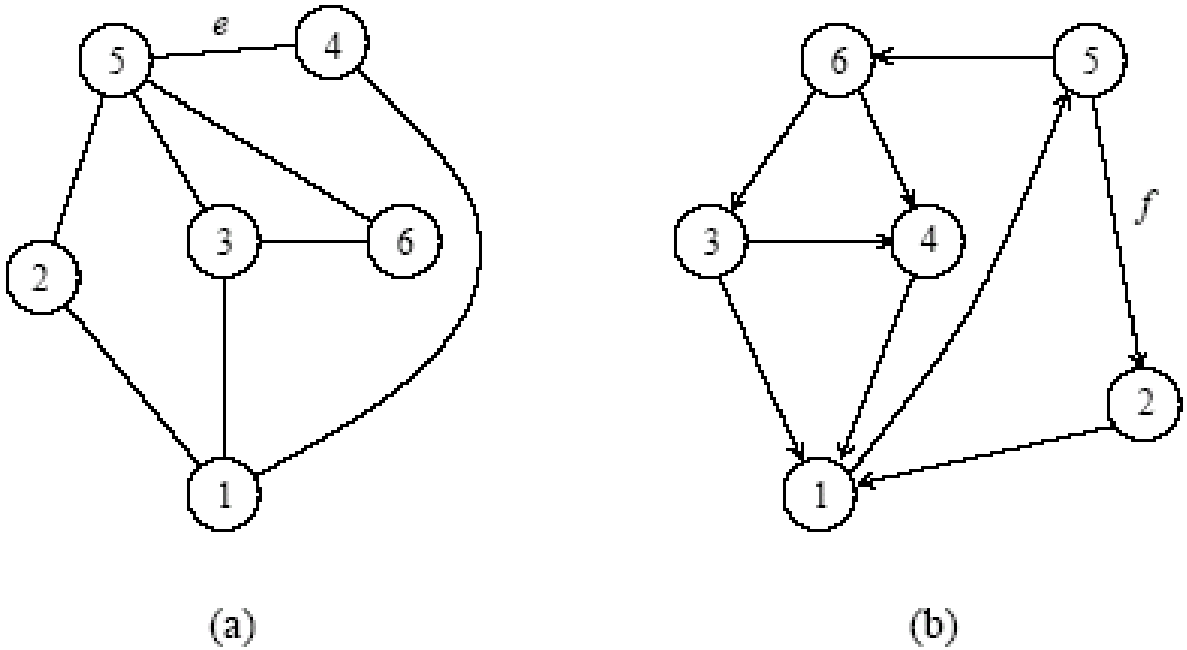
Topic Overview

- Definitions and Representation
- Minimum Spanning Tree: Prim's Algorithm
- Single-Source Shortest Paths: Dijkstra's Algorithm
- All-Pairs Shortest Paths
- Connected Components
- Algorithms for Sparse Graphs

Definitions and Representation

- An ***undirected graph*** *G* is a pair *(V,E)*, where *V* is a finite set of points called *vertices* and *E* is a finite set of *edges*.
- An edge $e \in E$ is an unordered pair *(u,v)*, where $u,v \in V$.
- In a **directed graph**, the edge *e* is an ordered pair *(u,v)*. An edge *(u,v)* is *incident from* vertex *u* and is *incident to* vertex *v*.
- A ***path*** from a vertex *v* to a vertex *u* is a sequence $<v_0,v_1,v_2,\ldots,v_k>$ of vertices where $v_0 = v$, $v_k = u$, and $(v_i, v_{i+1}) \in E$ for $i = 0, 1, \ldots, k-1$.
- The **length of a path** is defined as the number of edges in the path.

Definitions and Representation



a) An undirected graph and (b) a directed graph.

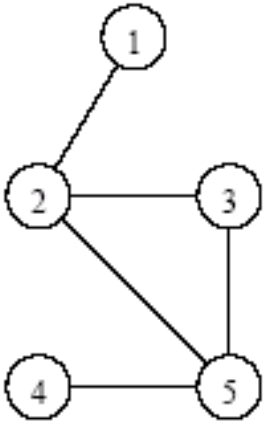
Definitions and Representation

- An undirected graph is ***connected*** if every pair of vertices is connected by a path.
- A ***forest*** is an acyclic graph, and a ***tree*** is a connected acyclic graph.
- A graph that has weights associated with each edge is called a ***weighted graph***.

Definitions and Representation

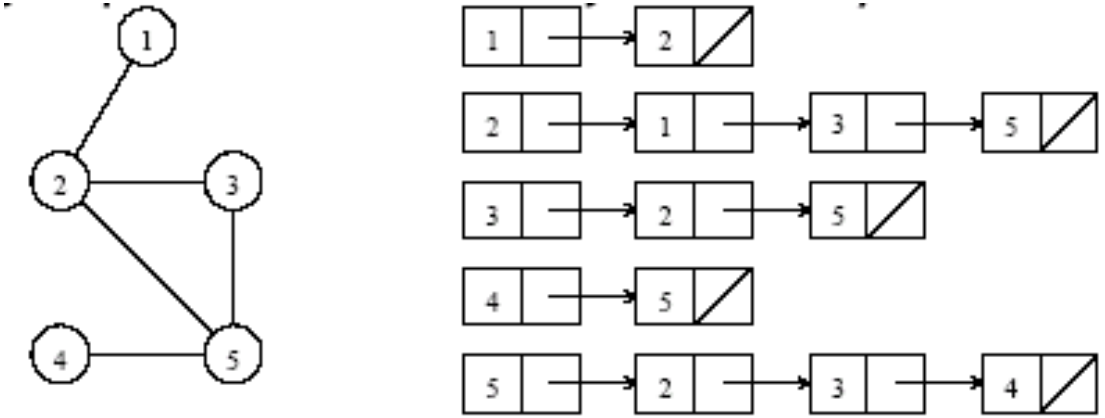
- Graphs can be represented by their **adjacency matrix** or an **edge** (or vertex) **list**.
- Adjacency matrices have a value $a_{i,j} = 1$ if nodes *i* and *j* share an edge; 0 otherwise. In case of a weighted graph, $a_{i,j} = w_{i,j}$, the weight of the edge.
- The ***adjacency list*** representation of a graph $G = (V,E)$ consists of an array $Adj[1..|V|]$ of lists. Each list $Adj[v]$ is a list of all vertices adjacent to *v*.
- For a graph with *n* nodes, adjacency matrices take $\Theta(n^2)$ space and adjacency list takes $\Theta(|E|)$ **space**.

Definitions and Representation



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

An undirected graph and its **adjacency matrix** representation.

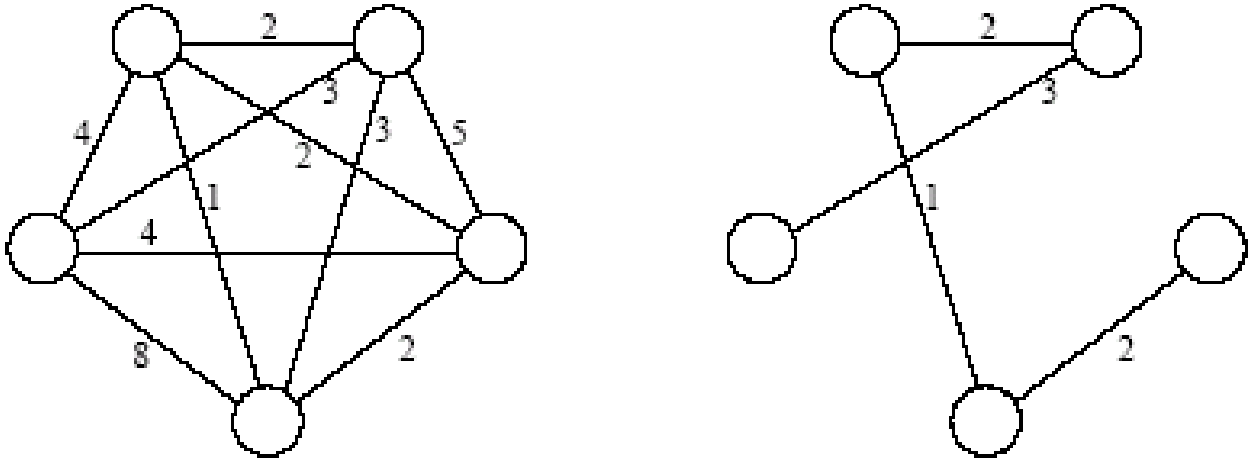


An undirected graph and its **adjacency list** representation.

Minimum Spanning Tree

- A ***spanning tree*** of an undirected graph *G* is a subgraph of *G* that is a tree containing all the vertices of *G*.
- In a weighted graph, the weight of a subgraph is the sum of the weights of the edges in the subgraph.
- A ***minimum spanning tree*** (MST) for a weighted undirected graph is a **spanning tree with minimum weight**.

Minimum Spanning Tree

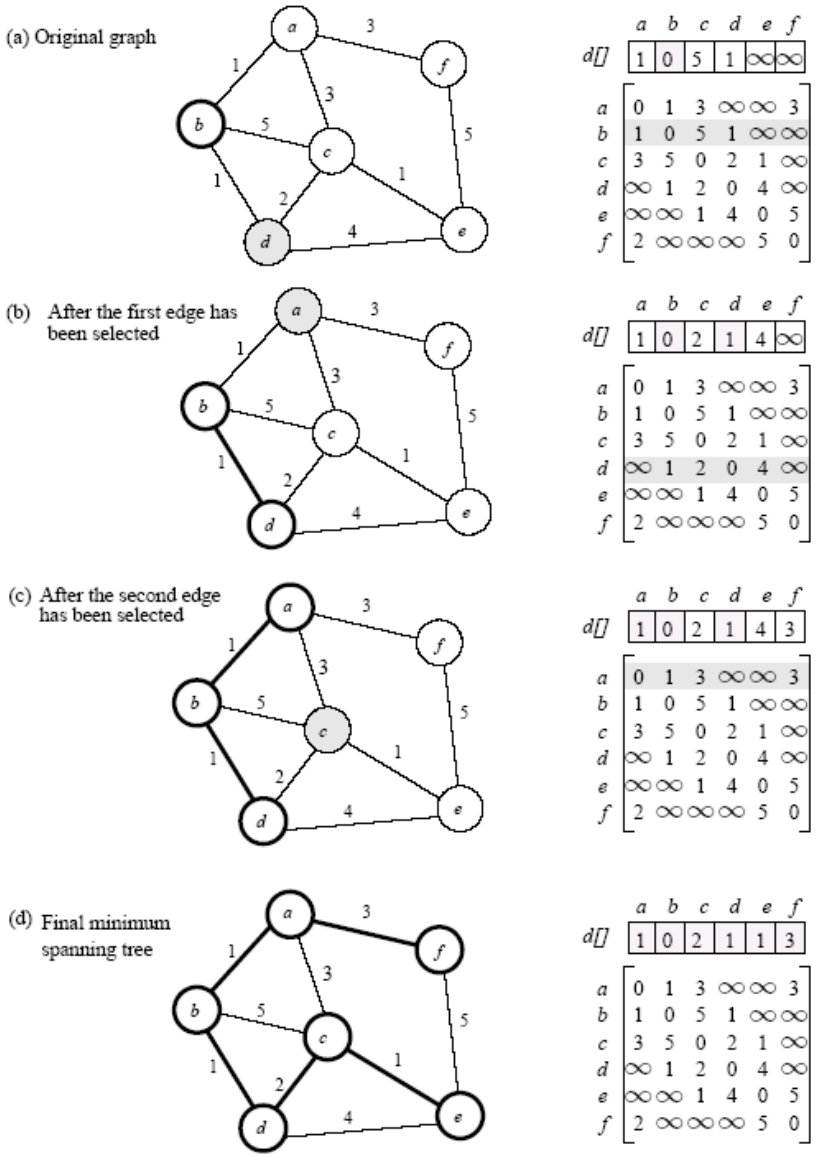


An undirected graph and its minimum spanning tree.

Minimum Spanning Tree: Prim's Algorithm

- **Prim's algorithm** for finding an MST is a **greedy algorithm**.
- **Start by selecting an arbitrary vertex**, include it into the current MST.
- **Grow the current MST** by inserting into it the vertex **closest** to one of the vertices already in current MST.

Minimum Spanning Tree: Prim's Algorithm



Prim's minimum spanning tree algorithm.

Minimum Spanning Tree: Prim's Algorithm

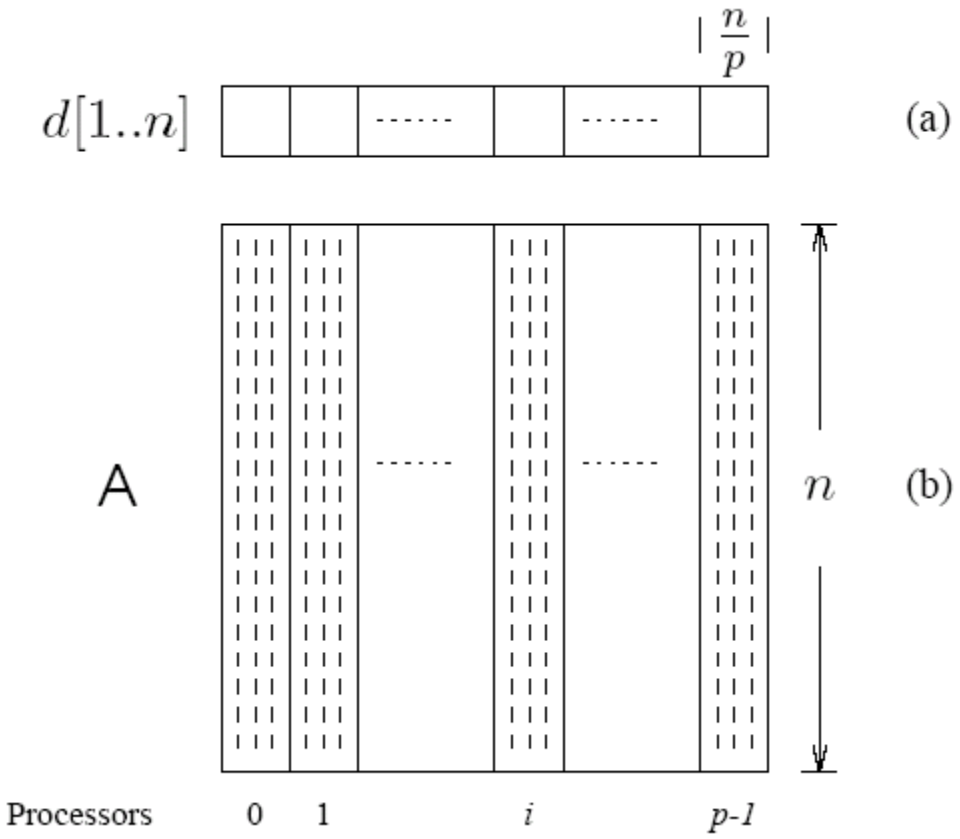
```
1.   procedure PRIM_MST(V, E, w, r)
2.   begin
3.       V_T := {r};
4.       d[r] := 0;
5.       for all v ∈ (V − V_T) do
6.           if edge (r, v) exists set d[v] := w(r, v);
7.           else set d[v] := ∞;
8.       while V_T ≠ V do
9.       begin
10.          find a vertex u such that d[u] := min{d[v]|v ∈ (V − V_T)};
11.          V_T := V_T ∪ {u};
12.          for all v ∈ (V − V_T) do
13.              d[v] := min{d[v], w(u, v)};
14.      endwhile
15.  end PRIM_MST
```

Prim's sequential minimum spanning tree algorithm.

Prim's Algorithm: Parallel Formulation

- The algorithm works in ***n* outer iterations** - it is hard to execute these iterations concurrently.
- The inner loop is relatively easy to parallelize. Let ***p* be the number of processes**, and let ***n* be the number of vertices**.
- The **adjacency matrix is partitioned in a 1-D** block fashion, with distance vector *d* partitioned accordingly.
- In each step, a **processor selects the locally closest node**, followed by a **global reduction to select globally closest node**.
- This node is inserted into MST, and **the choice broadcast to all processors**.
- Each **processor updates** its part of the *d* vector locally.

Prim's Algorithm: Parallel Formulation



The partitioning of the distance array *d* and the adjacency matrix *A* among *p* processes.

Prim's Algorithm: Parallel Formulation

- The cost to **select the minimum entry** is $O(n/p + \log p)$.
- The cost of a **broadcast** is $O(\log p)$.
- The cost of **local updation** of the *d* vector is $O(n/p)$.

- The **parallel time per iteration** is $O(n/p + \log p)$.
- The **total parallel time** is given by $O(n^2/p + n \log p)$.

- The corresponding **isoefficiency** is $O(p^2 \log^2 p)$.

Single-Source Shortest Paths

- For a weighted graph *G = (V,E,w)*, the *single-source shortest paths* problem is to **find the shortest paths from a vertex *v* ∈ *V* to all other vertices** in *V*.
- **Dijkstra's algorithm is similar to Prim's algorithm**. It maintains a set of nodes for which the shortest paths are known.
- It **grows this set** based on the **node closest to source** using one of the nodes in the current shortest path set.

Single-Source Shortest Paths: Dijkstra's Algorithm

```
1.   procedure DIJKSTRA_SINGLE_SOURCE_SP(V, E, w, s)
2.   begin
3.       V_T := {s};
4.       for all v ∈ (V − V_T) do
5.           if (s, v) exists set l[v] := w(s, v);
6.           else set l[v] := ∞;
7.       while V_T ≠ V do
8.       begin
9.           find a vertex u such that l[u] := min{l[v]|v ∈ (V − V_T)};
10.          V_T := V_T ∪ {u};
11.          for all v ∈ (V − V_T) do
12.              l[v] := min{l[v], l[u] + w(u, v)};
13.      endwhile
14.  end DIJKSTRA_SINGLE_SOURCE_SP
```

Dijkstra's sequential single-source shortest paths algorithm.

Dijkstra's Algorithm: Parallel Formulation

- Very **similar** to the parallel formulation of **Prim's** algorithm for minimum spanning trees.
- The weighted adjacency matrix is partitioned using the **1-D block mapping**.
- Each process selects, locally, the **node closest to the source**, followed by a **global reduction** to select next node.
- The node is **broadcast** to all processors and the *l*-vector **updated**.
- The **parallel performance** of Dijkstra's algorithm is **identical** to that of Prim's algorithm.

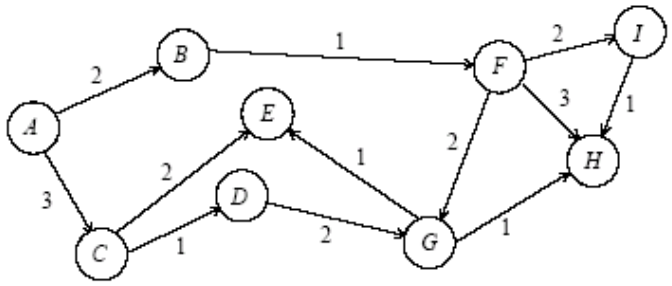
All-Pairs Shortest Paths

- Given a weighted graph *G(V,E,w)*, the *all-pairs shortest paths* problem is to find the **shortest paths between all pairs** of vertices v_i, $v_j \in V$.
- A number of algorithms are known for solving this problem.

All-Pairs Shortest Paths: Matrix-Multiplication Based Algorithm

- Consider the **multiplication of the weighted adjacency matrix with itself** - except, in this case, we **replace** the **multiplication** operation in matrix multiplication by **addition**, and the **addition** operation by **minimization**.
- Notice that the product of weighted adjacency matrix with itself returns a matrix that contains **shortest paths of length 2** between any pair of nodes.
- It follows from this argument that $\boldsymbol{A^n}$ **contains all shortest paths**.

Matrix-Multiplication Based Algorithm



$$A^1 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & 0 & 1 & 2 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & 2 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & \infty & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^8 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

Matrix-Multiplication Based Algorithm

- A^n is computed by **doubling powers -** i.e., as A, A^2, A^4, A^8, and so on.
- We need ***log n* matrix multiplications**, each taking time $O(n^3)$.
- The **serial complexity** of this procedure is $\boldsymbol{O(n^3 \log n)}$.
- This algorithm is **not optimal**, since the best known algorithms have complexity $O(n^3)$.

Matrix-Multiplication Based Algorithm: Parallel Formulation

- Each of the ***log n*** **matrix multiplications** can be performed **in parallel**.
- We can **use** $n^3/\log n$ **processors** to compute each matrix-matrix product **in time** ***log n***.
- The entire process takes $O(\log^2 n)$ time.

Dijkstra's Algorithm

- Execute ***n* instances** of the **single-source shortest** path problem, one for each of the *n* source vertices.
- Complexity is $O(n^3)$.

Dijkstra's Algorithm: Parallel Formulation

- **Two parallelization strategies** - execute each of the *n* shortest path problems on a different processor (**source partitioned**), or use a parallel formulation of the shortest path problem to increase concurrency (**source parallel**).

Dijkstra's Algorithm: Source Partitioned Formulation

- Use ***n* processors**, each processor P_i **finds the shortest paths from vertex** v_i to all other vertices by executing Dijkstra's sequential single-source shortest paths algorithm.
- It requires **no interprocess communication** (provided that the adjacency matrix is replicated at all processes).
- The **parallel run time** of this formulation is: $\Theta(n^2)$.
- While the algorithm is cost optimal, it **can only use *n* processors**. Therefore, the **isoefficiency** due to concurrency is p^3.

Dijkstra's Algorithm: Source Parallel Formulation

- In this case, **each of the shortest path problems is further executed in parallel**. We can therefore use up to n^2 **processors**.
- Given p processors ($p > n$), **each single source shortest path problem** is executed by p/n **processors**.
- Using previous results, this takes time:

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n \log p).}^{\text{communication}}$$

- For cost optimality, we have $p = O(n^2/\log n)$ and the **isoefficiency** is $\Theta((p \log p)^{1.5})$.

Floyd's Algorithm

- For any pair of vertices v_i, $v_j \in V$, consider all paths from v_i to v_j whose **intermediate vertices belong to the set $\{v_1, v_2, \ldots, v_k\}$**. Let $p_{i,j}^{(k)}$ (of weight $d_{i,j}^{(k)}$ be the minimum-weight path among them.
- **If vertex v_k is not in** the shortest path from v_i to v_j, then $p_{i,j}^{(k)}$ is the same as $p_{i,j}^{(k-1)}$.
- **If v_k is in $p_{i,j}^{(k)}$**, then we can **break $p_{i,j}^{(k)}$ into two paths** - one from v_i to v_k and one from v_k to v_j . Each of these paths uses vertices from $\{v_1, v_2, \ldots, v_{k-1}\}$.

Floyd's Algorithm

From our observations, the following **recurrence relation** follows:

$$d_{i,j}^{(k)} = \begin{cases} w(v_i, v_j) & \text{if } k = 0 \\ \min\left\{d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}\right\} & \text{if } k \geq 1 \end{cases}$$

This equation **must be computed for each pair of nodes and for *k = [1, n]***. The serial complexity is ***O(n³)***.

Floyd's Algorithm

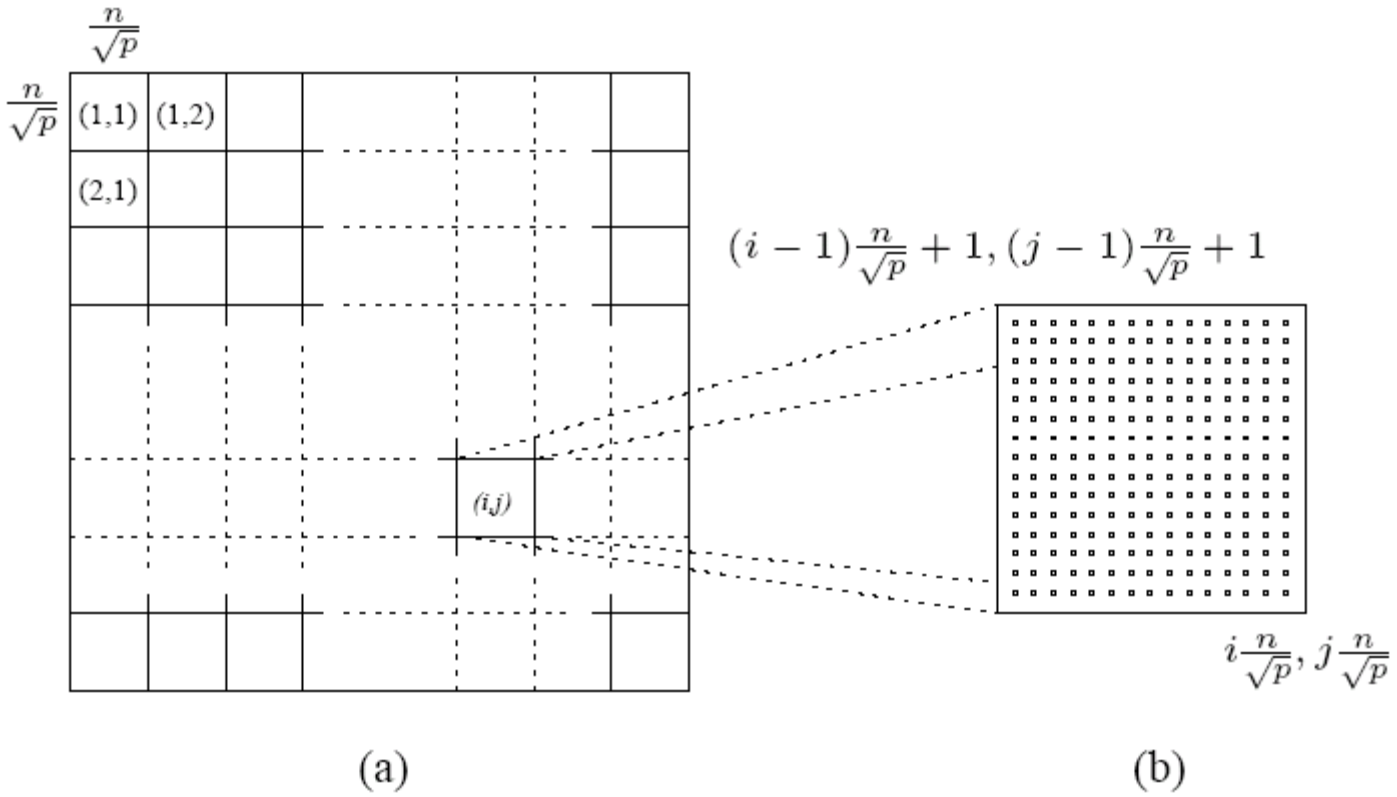
```
1.  procedure FLOYD_ALL_PAIRS_SP(A)
2.  begin
3.      D^(0) = A;
4.      for k := 1 to n do
5.          for i := 1 to n do
6.              for j := 1 to n do
7.                  d_{i,j}^(k) := min( d_{i,j}^(k-1), d_{i,k}^(k-1) + d_{k,j}^(k-1) );
8.  end FLOYD_ALL_PAIRS_SP
```

Floyd's all-pairs shortest paths algorithm. This program computes the all-pairs shortest paths of the graph *G* = *(V,E)* with adjacency matrix *A*.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

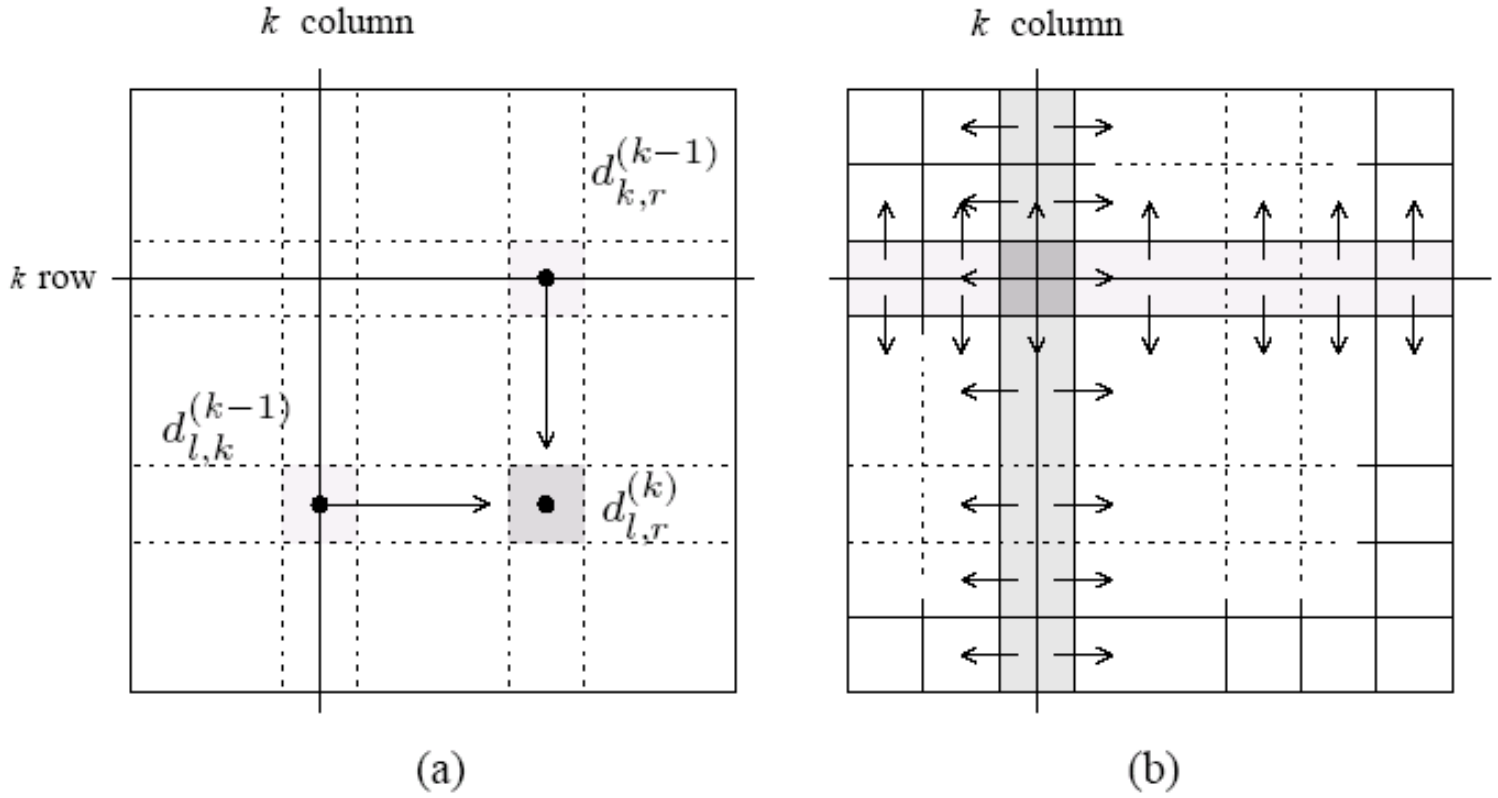
- Matrix $D^{(k)}$ **is divided into** p **blocks** of size $(n / \sqrt{p}) \times (n / \sqrt{p})$.
- **Each processor updates its part of the matrix** during each iteration.
- To compute $d_{l,k}^{(k-1)}$ processor $P_{i,j}$ **must get** $d_{l,k}^{(k-1)}$ **and** $d_{k,r}^{(k-1)}$.
- In general, during the k^{th} iteration, each of the $\sqrt{p}$ processes containing part of the k^{th} row send it to the $\sqrt{p} - 1$ processes in the same column.
- Similarly, each of the $\sqrt{p}$ processes containing part of the k^{th} column sends it to the $\sqrt{p} - 1$ processes in the same row.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping



(a) Matrix $D^{(k)}$ distributed by 2-D block mapping into $\sqrt{p}$ x $\sqrt{p}$ subblocks, and (b) the subblock of $D^{(k)}$ assigned to process $P_{i,j}$.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping



(a) Communication patterns used in the 2-D block mapping. When computing $d_{i,j}^{(k)}$, information must be sent to the highlighted process from two other processes along the same row and column. (b) The row and column of $\sqrt{p}$ processes that contain the k^{th} row and column send them along process columns and rows.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

```
1.    procedure FLOYD_2DBLOCK(D^(0))
2.    begin
3.        for k := 1 to n do
4.        begin
5.            each process P_{i,j} that has a segment of the k^th row of D^(k-1);
                  broadcasts it to the P_{*,j} processes;
6.            each process P_{i,j} that has a segment of the k^th column of D^(k-1);
                  broadcasts it to the P_{i,*} processes;
7.            each process waits to receive the needed segments;
8.            each process P_{i,j} computes its part of the D^(k) matrix;
9.        end
10.   end FLOYD_2DBLOCK
```

Floyd's parallel formulation using the 2-D block mapping. $P_{*,j}$ denotes all the processes in the j^{th} column, and $P_{i,*}$ denotes all the processes in the i^{th} row. The matrix $D^{(0)}$ is the adjacency matrix.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

- During each iteration of the algorithm, the k^{th} row and k^{th} column of processors perform a **one-to-all broadcast along their rows/columns**.
- The size of this broadcast is $n/\sqrt{p}$ elements, taking time $\Theta((n \log p)/\sqrt{p})$.
- The **synchronization** step takes time $\Theta(\log p)$.
- The **computation time** is $\Theta(n^2/p)$.
- The parallel run time of the 2-D block mapping formulation of Floyd's algorithm is

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta\left(\frac{n^2}{\sqrt{p}}\log p\right)}^{\text{communication}}.$$

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

- The above formulation **can use $O(n^2 / \log^2 n)$ processors cost-optimally**.
- The isoefficiency of this formulation is $\Theta(p^{1.5} \log^3 p)$.
- This algorithm **can be further improved** by relaxing the strict synchronization after each iteration.

Floyd's Algorithm: Speeding Things Up by Pipelining

- The **synchronization step** in parallel Floyd's algorithm **can be removed** without affecting the correctness of the algorithm.
- **A process starts working on the k^{th} iteration as soon as it has computed the $(k-1)^{th}$ iteration and has the relevant parts of the $D^{(k-1)}$ matrix**.

Floyd's Algorithm: Speeding Things Up by Pipelining

- The overall **parallel run time** of this formulation is

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n).}^{\text{communication}}$$

- The pipelined formulation of Floyd's algorithm **uses up to $O(n^2)$ processes efficiently**.
- The corresponding **isoefficiency** is $\Theta(p^{1.5})$.

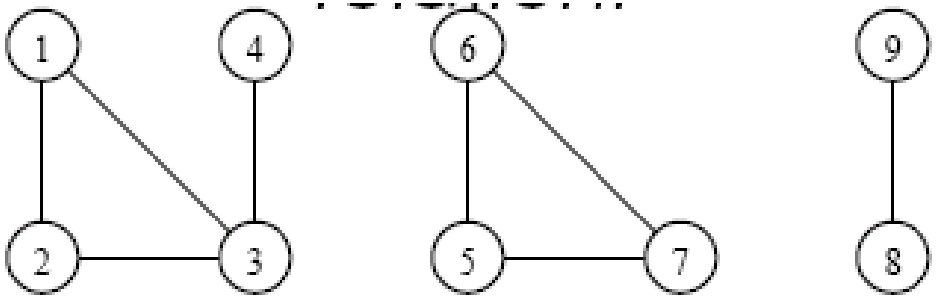
All-pairs Shortest Path: Comparison

- The performance and scalability of the all-pairs shortest paths algorithms on various architectures with bisection bandwidth. Similar run times apply to all cube architectures, provided that processes are properly mapped to the underlying processors.

	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Dijkstra source-partitioned	$\Theta(n)$	$\Theta(n^2)$	$\Theta(p^3)$
Dijkstra source-parallel	$\Theta(n^2/\log n)$	$\Theta(n \log n)$	$\Theta((p \log p)^{1.5})$
Floyd 2-D block	$\Theta(n^2/\log^2 n)$	$\Theta(n \log^2 n)$	$\Theta(p^{1.5} \log^3 p)$
Floyd pipelined 2-D block	$\Theta(n^2)$	$\Theta(n)$	$\Theta(p^{1.5})$

Connected Components

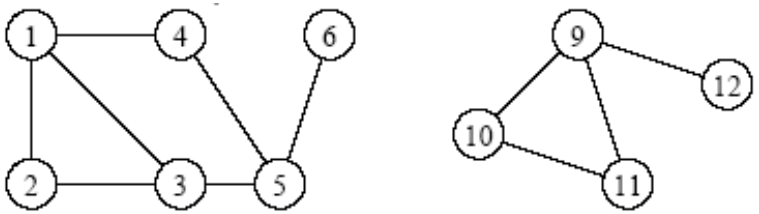
- The **connected components** of an **undirected graph** are the equivalence classes of vertices under the ``**is reachable from**'' relation.



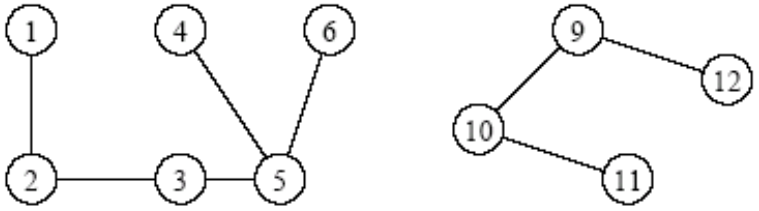
A graph with three connected components: *{1,2,3,4}*, *{5,6,7}*, and *{8,9}*.

Connected Components: Depth-First Search Based Algorithm

- Perform **DFS on the graph to get a forest** - arc tree in the forest corresponds to a separate connected component.



(a)



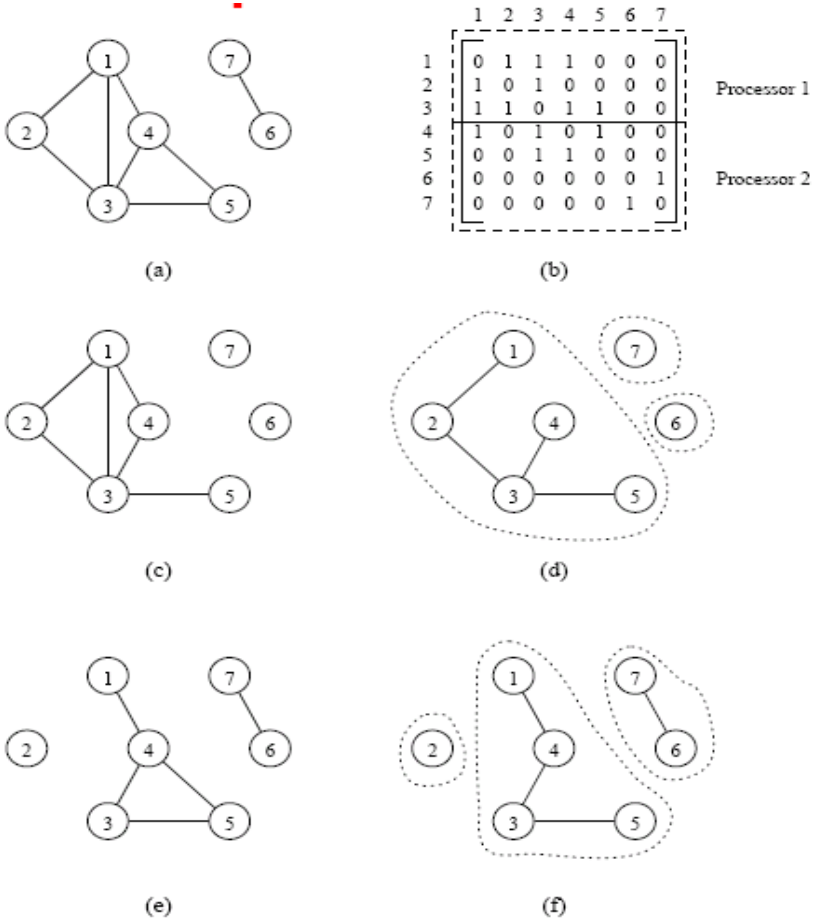
(b)

Part (b) is a **depth-first forest** obtained from depth-first traversal of the **graph in part (a)**. Each of these trees is a connected component of the graph in part (a).

Connected Components: Parallel Formulation

- **Partition the graph across processors** and **run independent connected component algorithms** on each processor. At this point, we have *p* spanning forests.
- In the second step, **spanning forests are merged pairwise** until only one spanning forest remains.

Connected Components: Parallel Formulation



Computing connected components in parallel. The adjacency matrix of the graph *G* in (a) is partitioned into two parts (b). Each process gets a subgraph of *G* ((c) and (e)). Each process then computes the spanning forest of the subgraph ((d) and (f)). Finally, the two spanning trees are merged to form the solution.

Connected Components: Parallel Formulation

- To merge pairs of spanning forests efficiently, the algorithm uses disjoint sets of edges.
- We **define the following operations** on the disjoint sets:
- ***find*(*x*)**
  - returns a pointer to the representative element of the set containing *x* . Each set has its own unique representative.
- ***union*(*x*, *y*)**
  - unites the sets containing the elements *x* and *y*. The two sets are assumed to be disjoint prior to the operation.

Connected Components: Parallel Formulation

- For merging forest *A* into forest *B*, **for each edge *(u,v)* of *A*, a *find* operation is performed to determine if the vertices are in the same tree of *B***.
- **If not**, then the **two trees (sets) of *B* containing *u* and *v* are united** by a *union* operation.
- Otherwise, no *union* operation is necessary.
- Hence, merging *A* and *B* requires at **most *2(n-1) find* operations** and ***(n-1) union* operations**.

Connected Components: Parallel 1-D Block Mapping

- The *n x n* adjacency matrix is **partitioned into *p* blocks** (1-D).
- Each processor can **compute its local spanning forest** in time ***Θ(n²/p)***.
- **Merging** is done by embedding a logical tree into the topology. There are ***log p* merging stages**, and **each takes time *Θ(n)***. Thus, the cost due to merging is *Θ(n log p)*.
- During each merging stage, spanning forests are sent between nearest neighbors. Recall that ***Θ(n)* edges of the spanning forest are transmitted**.

Connected Components: Parallel 1-D Block Mapping

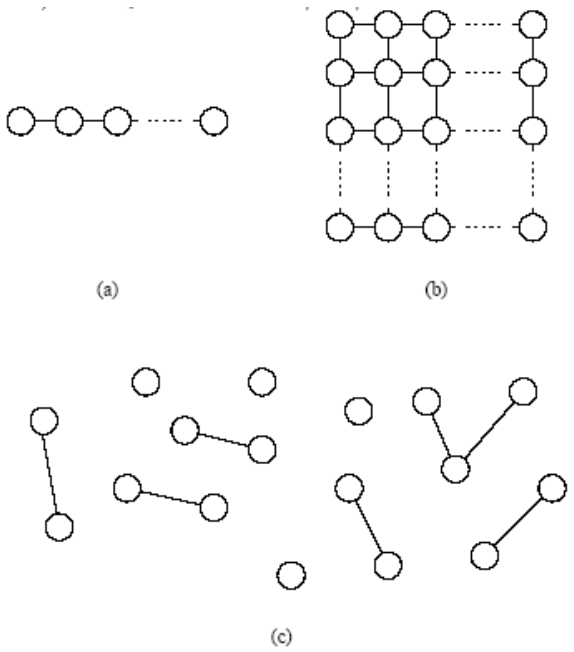
- The parallel run time of the connected-component algorithm is

$$T_P = \overbrace{\Theta\left(\frac{n^2}{p}\right)}^{\text{local computation}} + \overbrace{\Theta(n \log p).}^{\text{forest merging}}$$

- For a cost-optimal formulation $p = O(n / \log n)$. The corresponding **isoefficiency is $\Theta(p^2 \log^2 p)$**.

Algorithms for Sparse Graphs

- A graph $G = (V,E)$ is sparse if $|E|$ is much smaller than $|V|^2$.



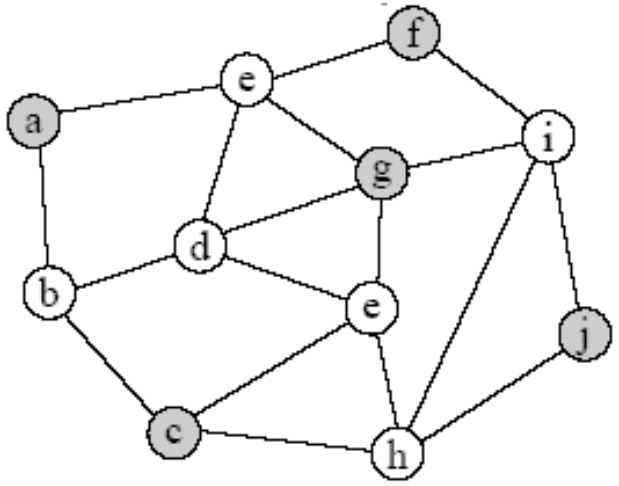
Examples of sparse graphs: (a) a linear graph, in which each vertex has two incident edges; (b) a grid graph, in which each vertex has four incident vertices; and (c) a random sparse graph.

Algorithms for Sparse Graphs

- **Dense algorithms can be improved significantly** if we make use of the sparseness. For example, the run time of **Prim's minimum spanning tree algorithm can be reduced from $\Theta(n^2)$ to $\Theta(|E| \log n)$**.
- Sparse algorithms **use adjacency list instead of an adjacency matrix**.
- **Partitioning adjacency lists is more difficult** for sparse graphs - do we balance number of vertices or edges?
- **Parallel algorithms typically make use of graph structure** or degree information for performance.

Finding a Maximal Independent Set

- A set of vertices $I \subset V$ is called *independent* **if no pair of vertices in *I* is connected** via an edge in *G*. An independent set is called ***maximal*** **if by including any other vertex not in *I*, the independence property is violated**.



{a, d, i, h} is an independent set

{a, c, j, f, g} is a maximal independent set

{a, d, h, f} is a maximal independent set

Examples of independent and maximal independent sets.

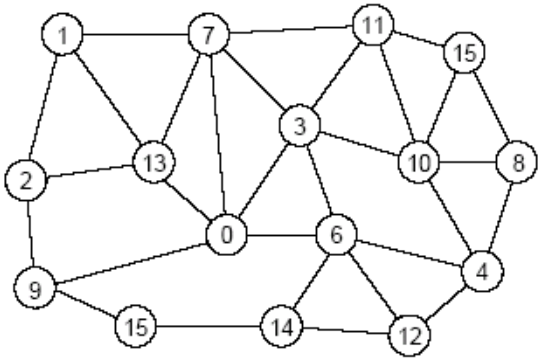
Finding a Maximal Independent Set (MIS)

- Simple algorithms start by MIS ***I* to be empty**, and assigning **all vertices to a candidate set *C***.
- Vertex ***v* from *C* is moved into *I*** and **all vertices adjacent to *v* are removed from *C***.
- This process is **repeated until *C* is empty**.
- This process is inherently serial!

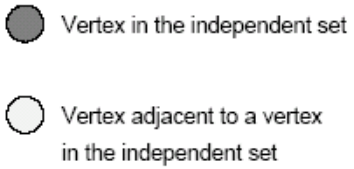
Finding a Maximal Independent Set (MIS)

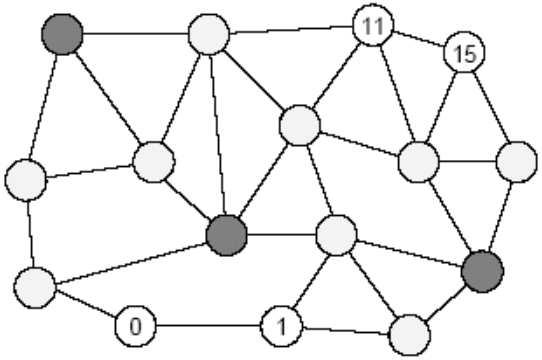
- Parallel MIS algorithms **use randimization** to gain concurrency (**Luby's algorithm** for graph coloring).
- Initially, each node is in the candidate set *C*. Each node **generates a (unique) random number** and **communicates it to its neighbors**.
- If a nodes number is **smalests out of all its neighbors, it joins set *I***. All of its **neighbors are removed from *C***.
- This process continues **until *C* is empty**.
- **On average**, this algorithm **converges after *O(log|V|)*** such steps.

Finding a Maximal Independent Set (MIS)

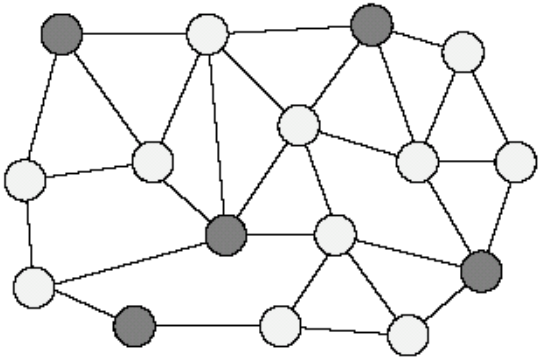


(a) After the 1st random number assignment





(b) After the 2nd random number assignment



(c) Final maximal independent set

The different augmentation steps of Luby's randomized maximal independent set algorithm. The numbers inside each vertex correspond to the random number assigned to the vertex.

Finding a Maximal Independent Set (MIS): Parallel Formulation

- We use **three arrays**, each of length *n*. Array ***I***, which stores nodes in MIS, ***C***, which stores the candidate set, and ***R***, the random numbers.
- Partition *C* across *p* processors. **Each processor generates the corresponding values in the *R* array**, and from this, computes which candidate vertices can enter MIS.
- The ***C* array is updated by deleting all the neighbors** of vertices that entered *MIS*.
- The performance of this algorithm is dependent on the structure of the graph.