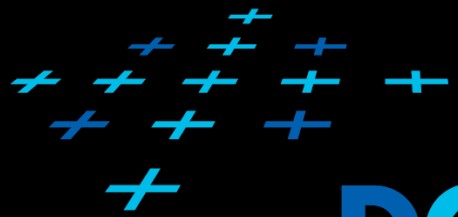

DCGI

KATEDRA POČÍTAČOVÉ GRAFIKY A INTERAKCE

DUALITY AND APPLICATIONS OF ARRANGEMENTS

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Based on [Berg], [Mount], and [Goswami]

Version from 7.1.2016

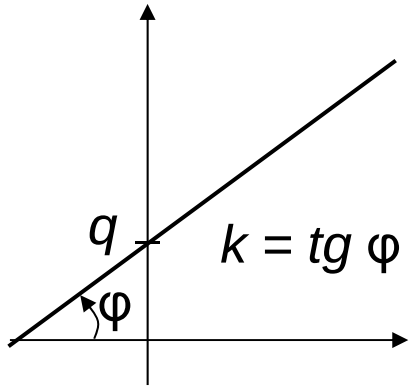
Talk overview

- Duality
  1. Points and lines
  2. Line segments
  3. Polar duality (different points and lines)
  4. Convex hull using duality
- Applications of duality and arrangements

1. Duality of lines and points in the plane

- Points and lines - both have 2 parameters:
  - Points – coords *x* and *y*
  - Lines – slope *k* and y-intercept *q*

    $y = kx + q$



- We can simply map points and lines 1:1
- Many mappings exist – it depends on the context

Why to use duality?

Some reasons why to use duality:

- Transforming a problem to dual plane may give a new view on the problem
- Looking from a different angle may give the insight needed to solve it
- Solution in dual space may be even simpler

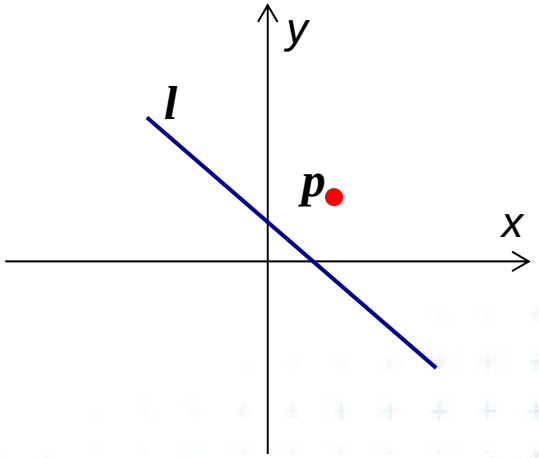
Definition of duality transformation *D*

Let *D* be the duality transform:

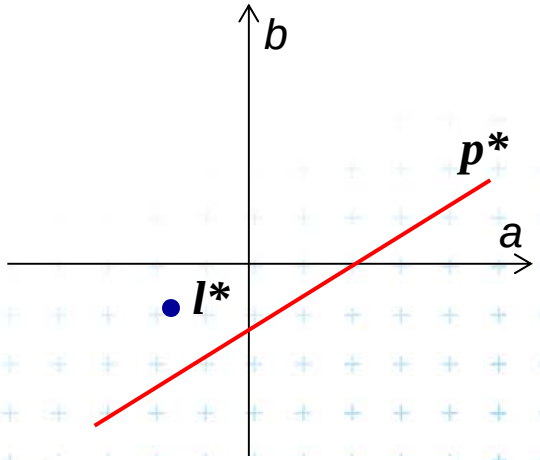
- Point $p = [\, p_x, p_y \,]$ is transformed to line $D_p = p^* := (b = p_x a - p_y)$
- Line $l : (y = ax - b)$ is transformed to point $D_l = l^* := [\, a, b \,]$

variables

constants



Primal plane (*xy*)



Dual plane (*ab*)

Example and more about duality *D*

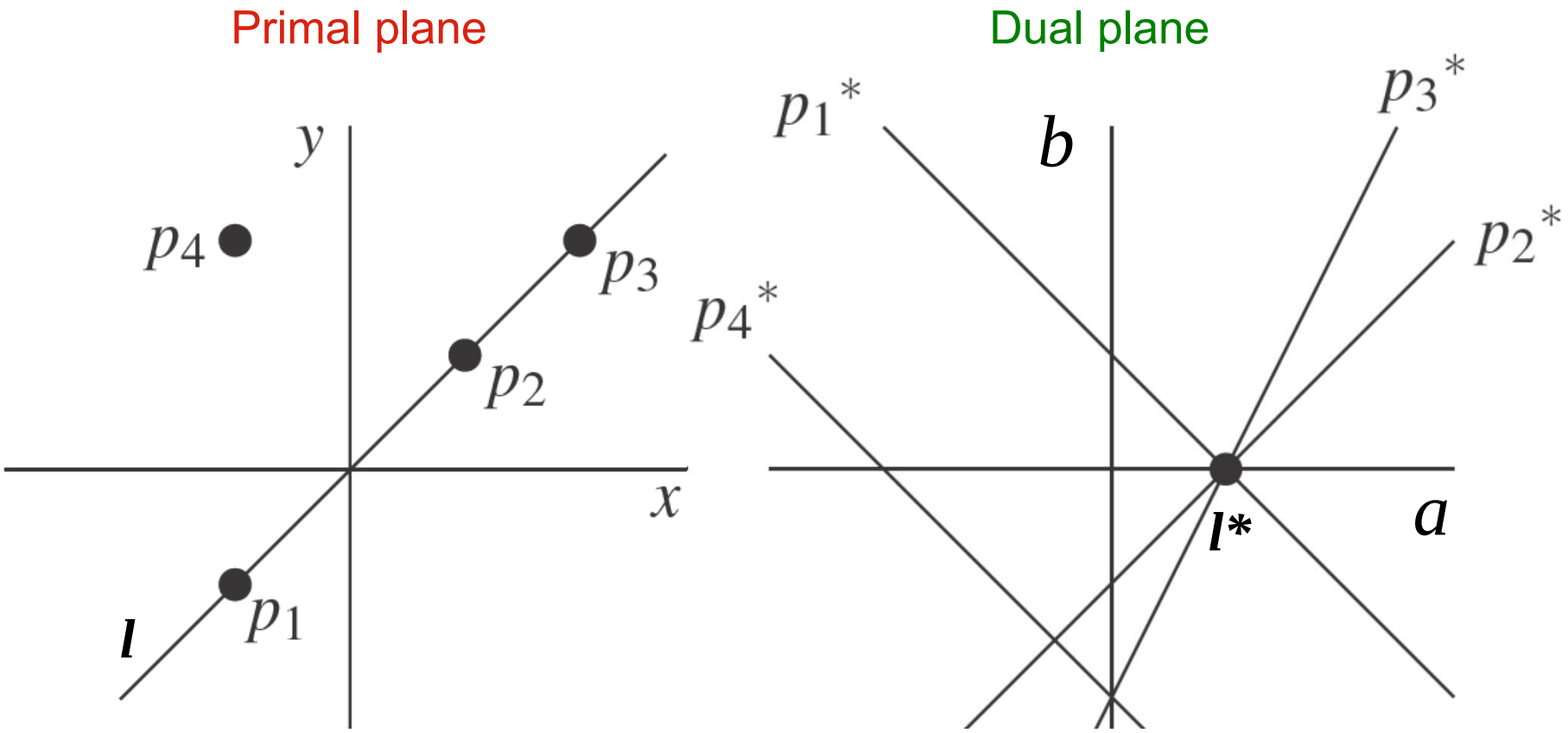
- Example:
  line $y = 5x - 3$
  can be represented as point $y^* = [5, 3]$

  See the [applet]

- Duality *D*
  - is its own inverse $DD_p = p$, $DD_l = l$
  - cannot represent vertical lines
    =>Take vertical lines as special cases, use lexicographic order, or rotate the problem space slightly.

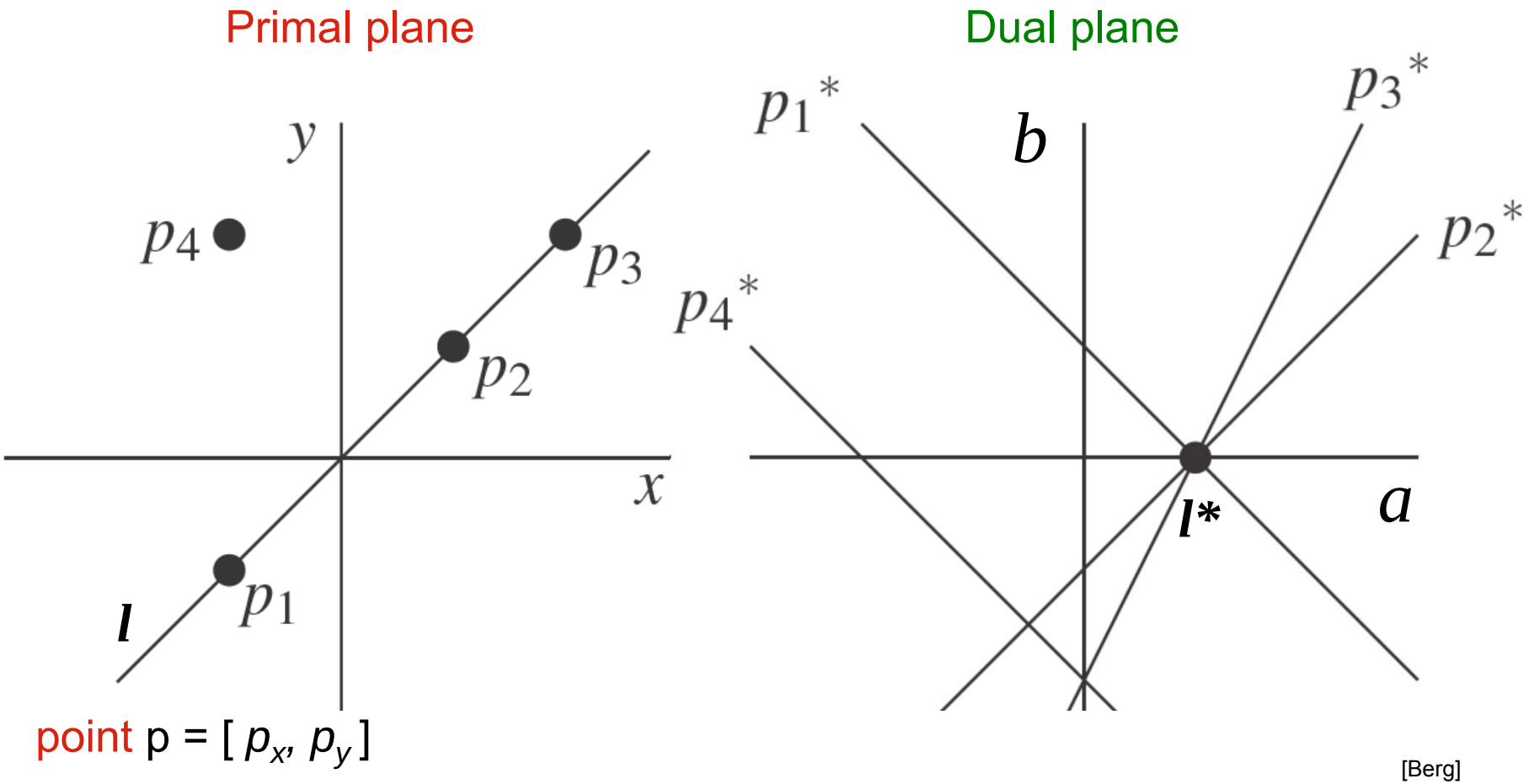
  - Primal plane – plane with coordinates *x, y*
  - Dual plane* – plane with coordinates *a, b*

Duality of lines and points in the plane



[Berg]

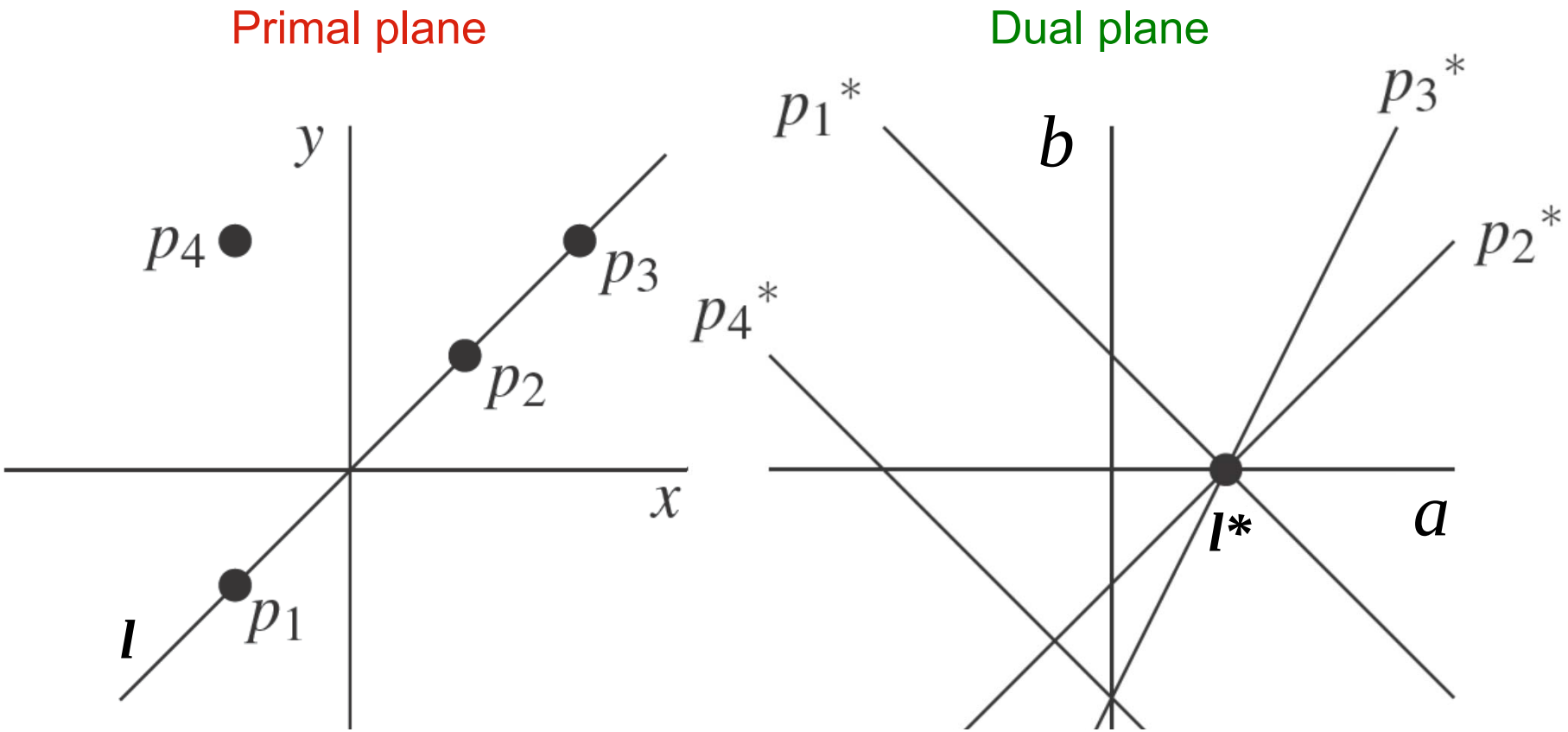
Duality of lines and points in the plane



point p = [p_x, p_y]

[Berg]

Duality of lines and points in the plane



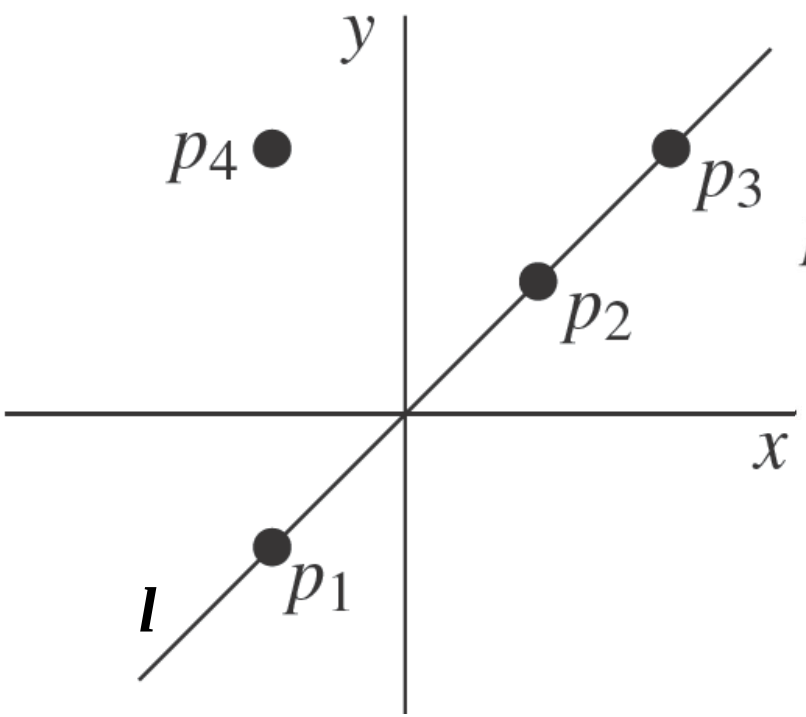
point p = [p_x, p_y]

line p* := (b = $p_x a - p_y$)

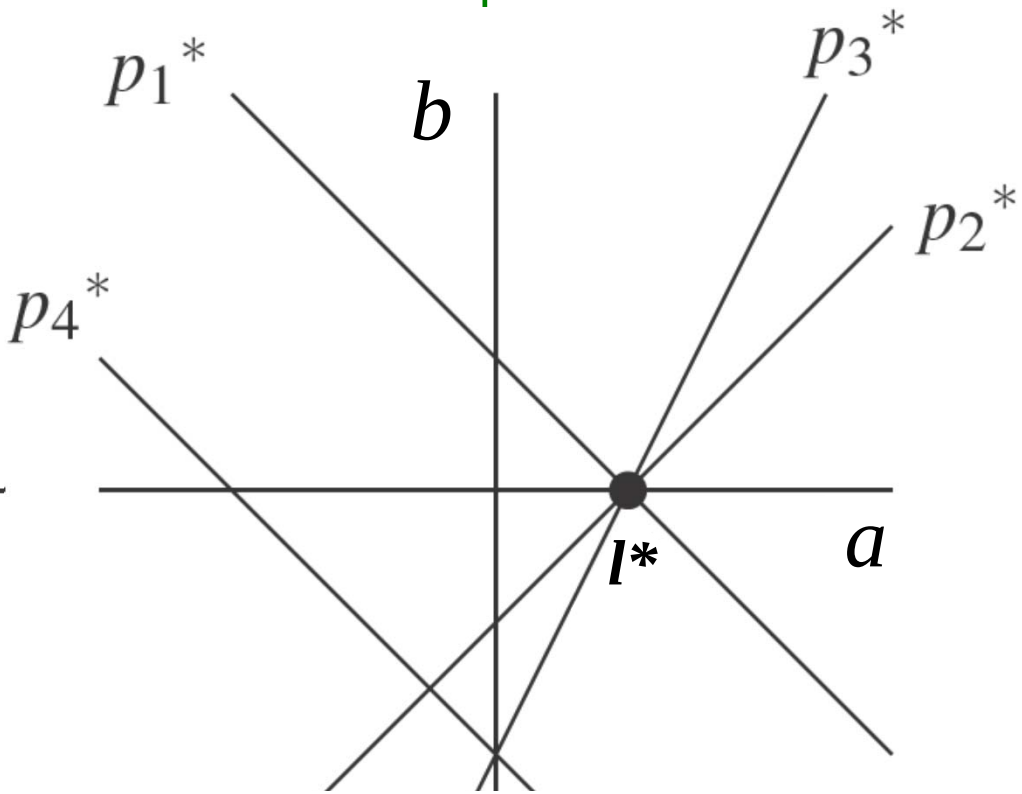
[Berg]

Duality of lines and points in the plane

Primal plane



Dual plane



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[Berg]

Duality of lines and points in the plane

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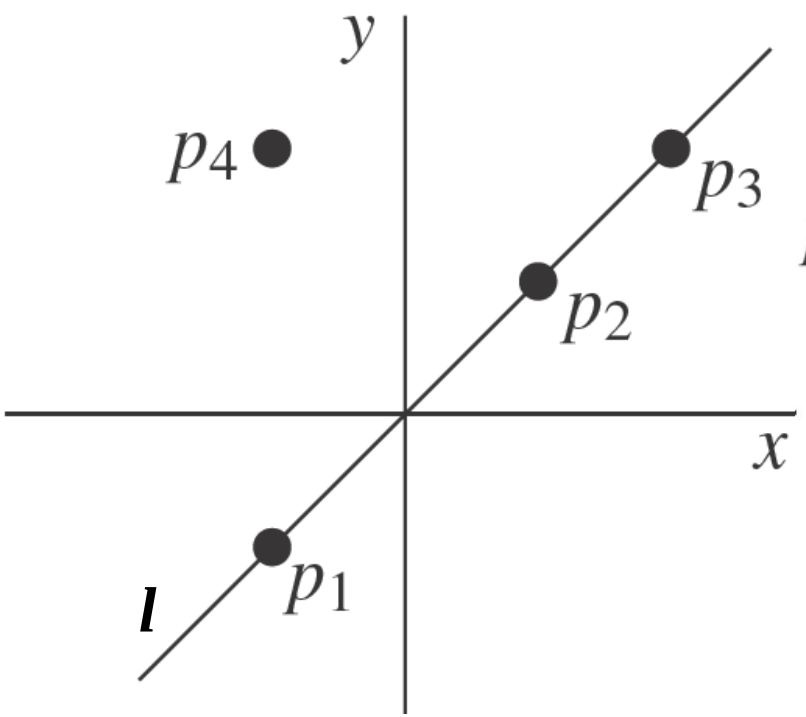

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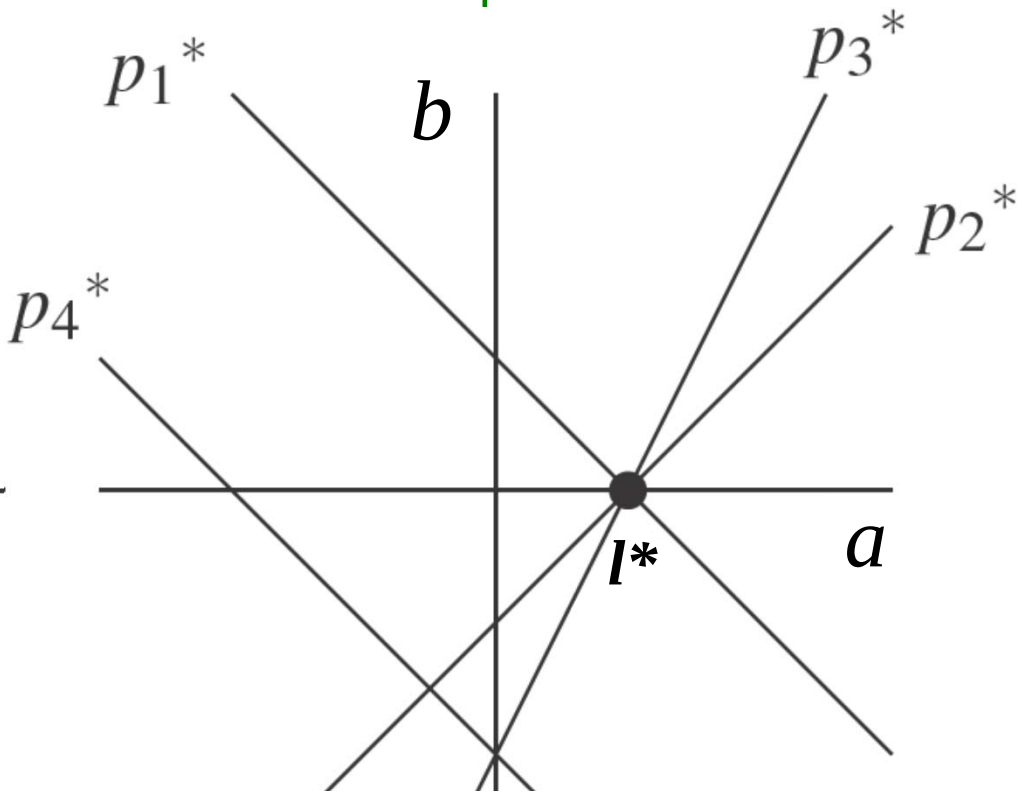


point p = [p_x, p_y]

line l := (y = ax + b)

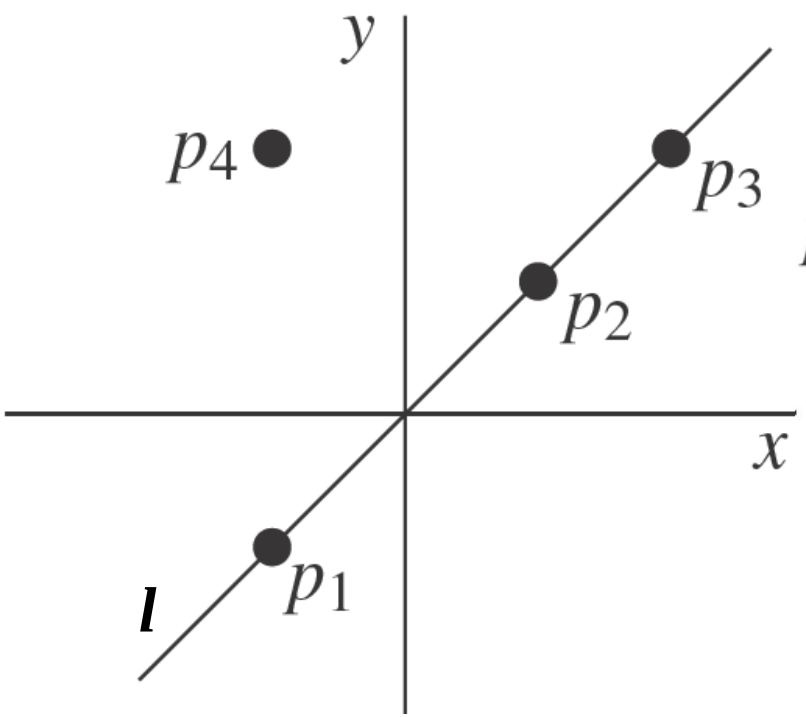
line p* := (b = $p_x a - p_y$)

Point l^* = [a, – b]

[Berg]

Duality of lines and points in the plane

Primal plane



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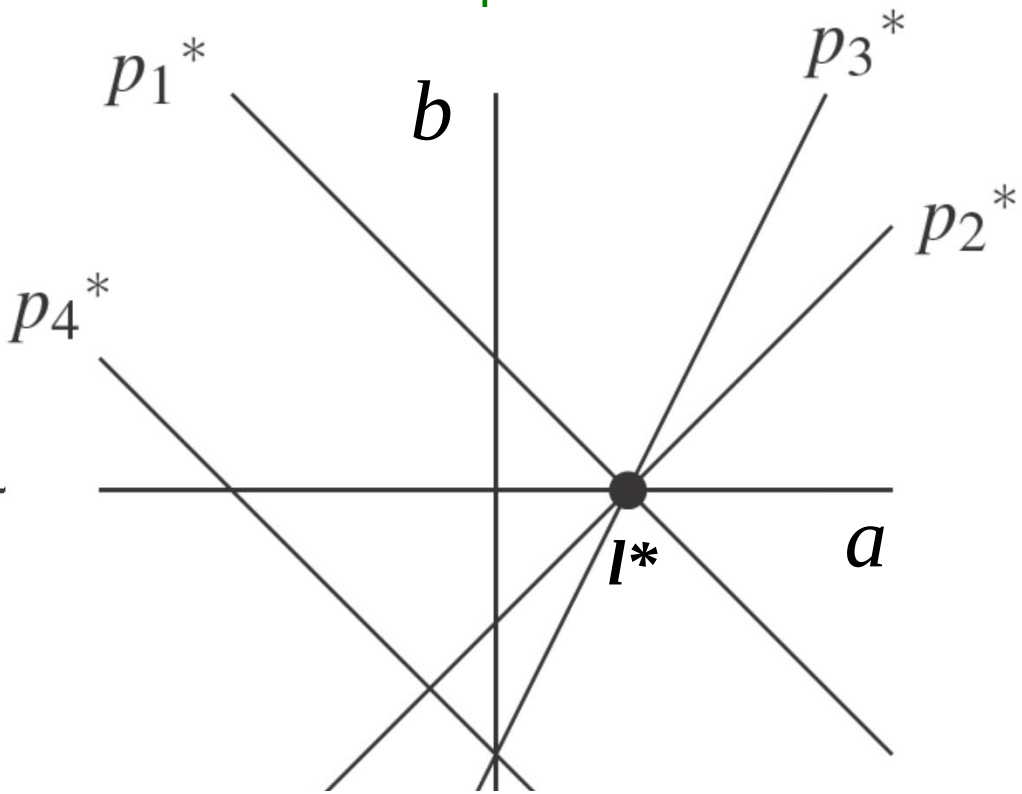

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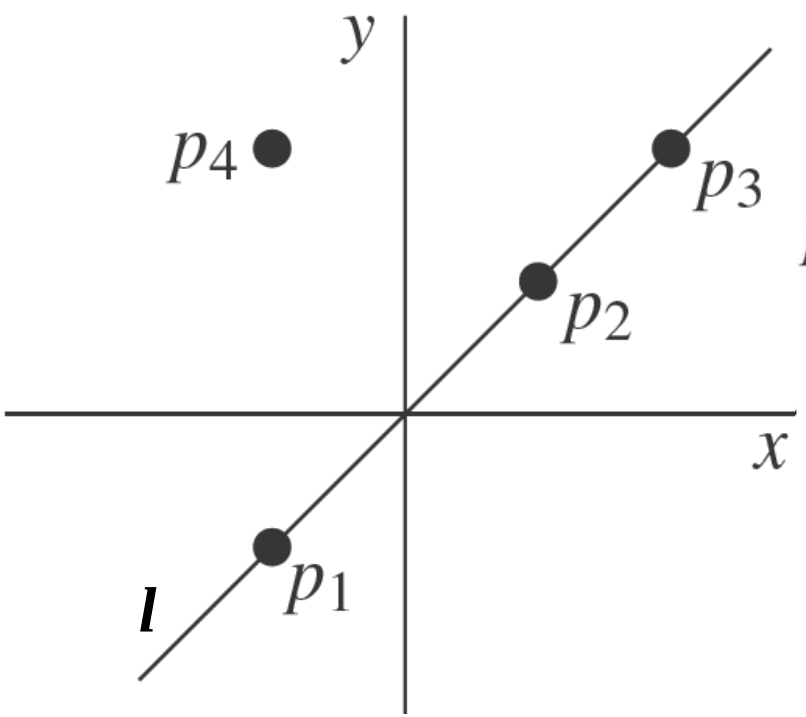
[Berg]

Duality of lines and points in the plane

Primal plane



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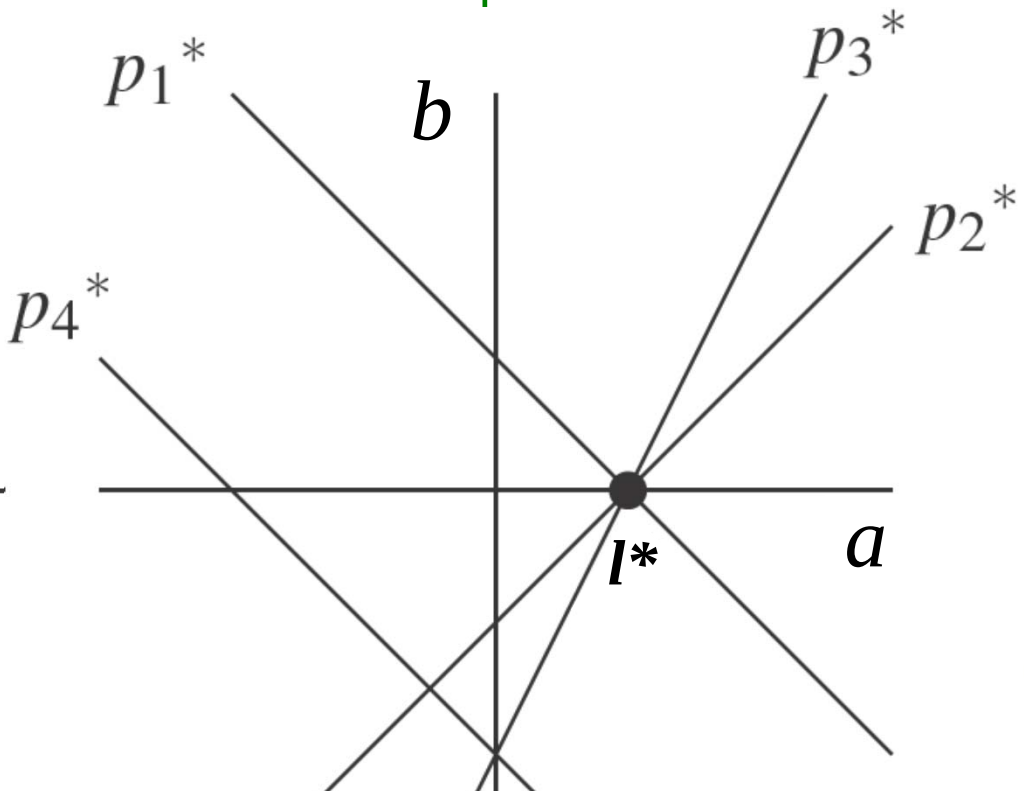

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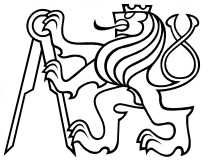
~~Point l^* = [a, – b]~~

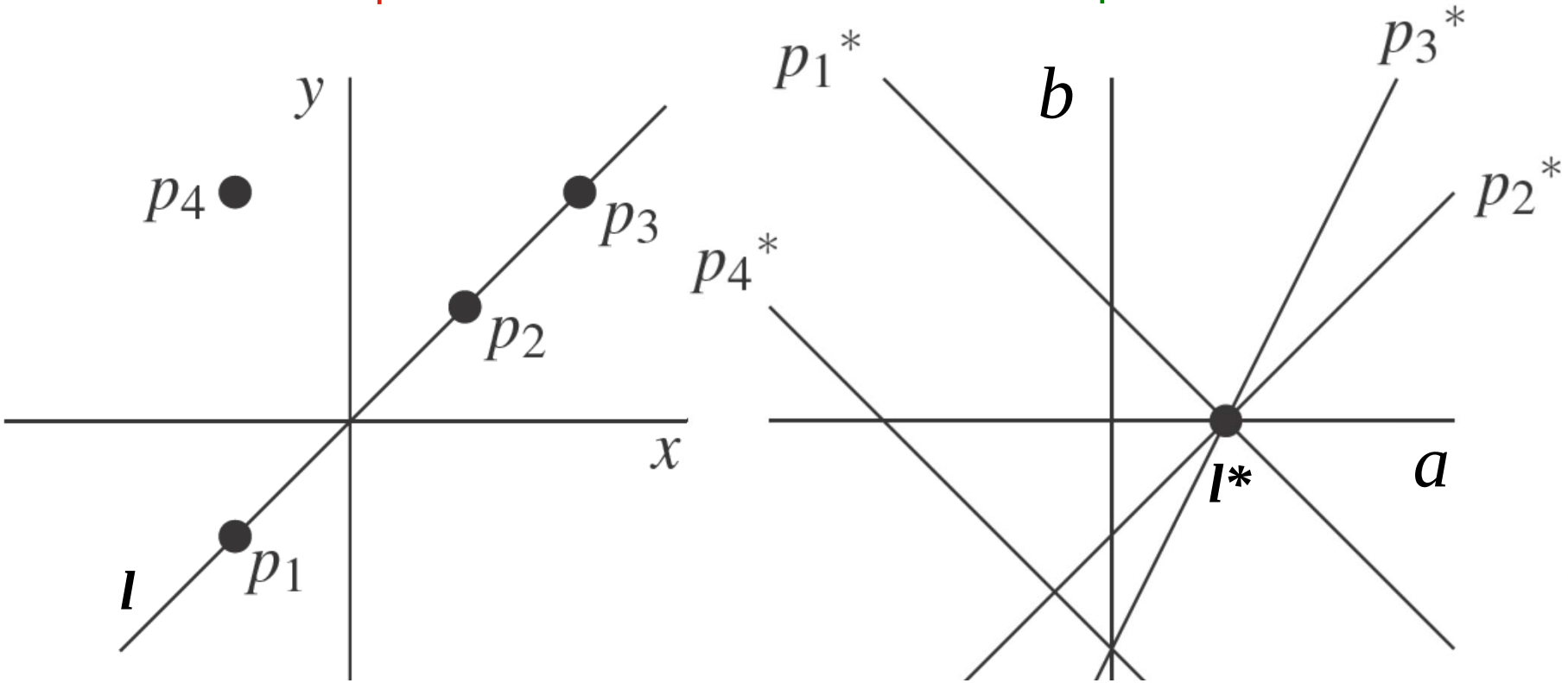
[Berg]



Duality of lines and points in the plane

Primal plane

Dual plane



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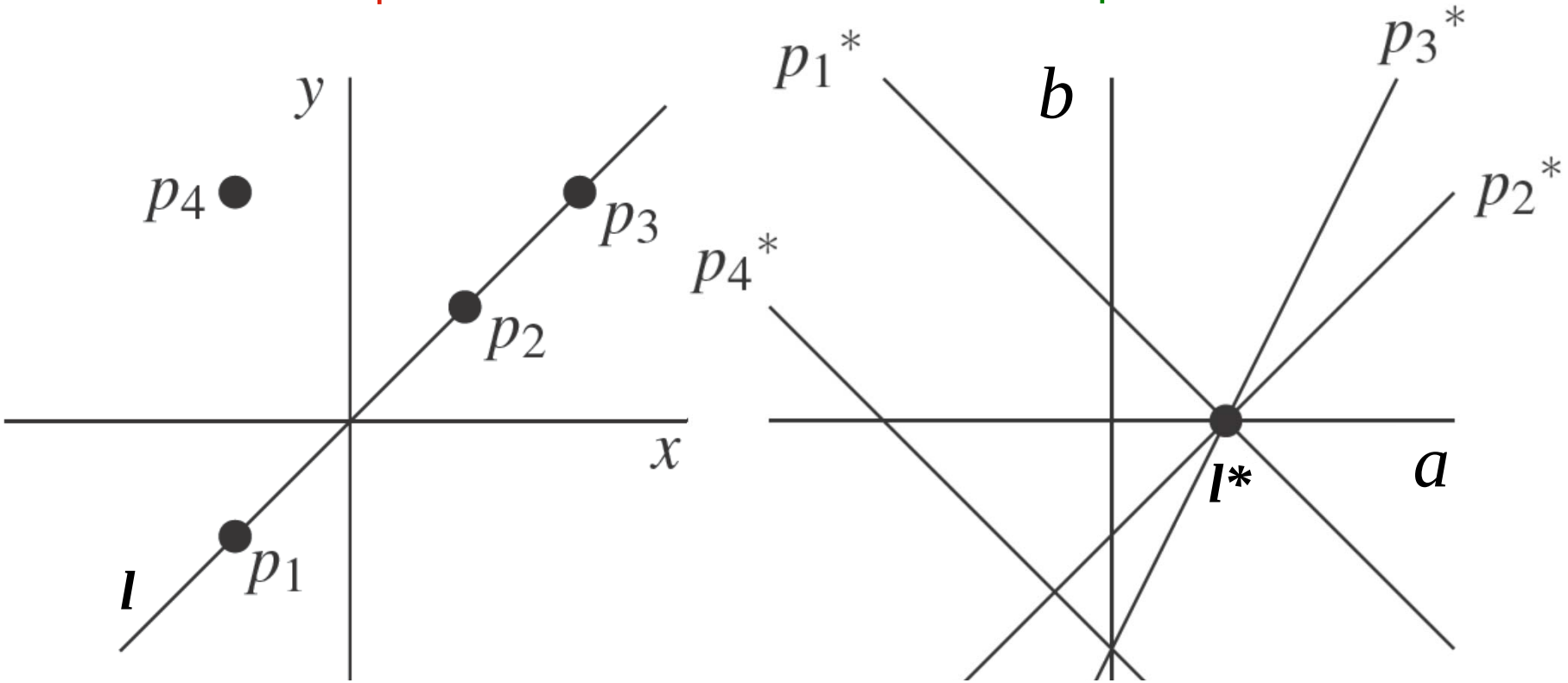
[Berg]

Duality of lines and points in the plane

Primal plane

Dual plane



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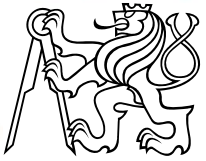
Point l^* = [a, b]

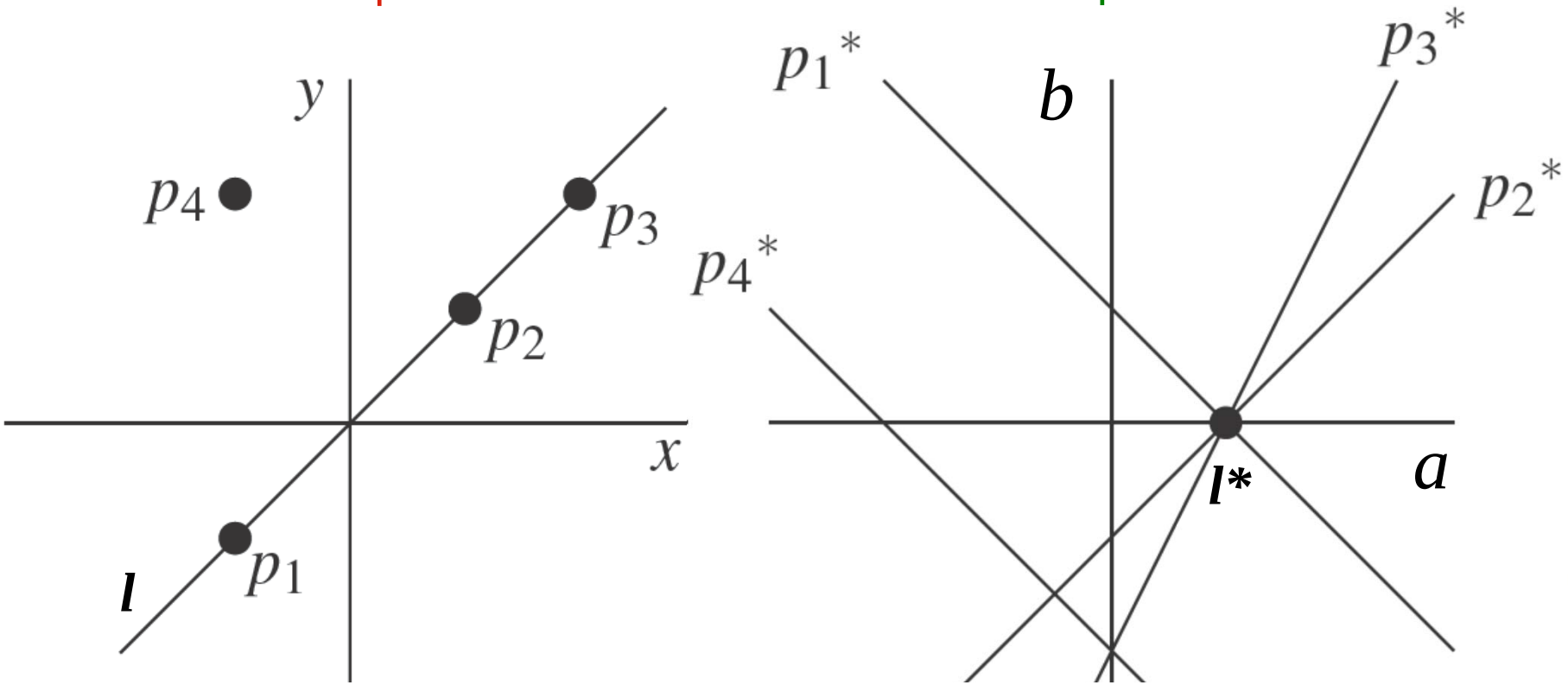
[Berg]



Duality of lines and points in the plane

Primal plane

Dual plane



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Point l^* = [a, b]

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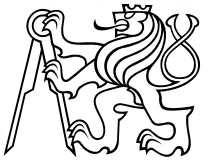
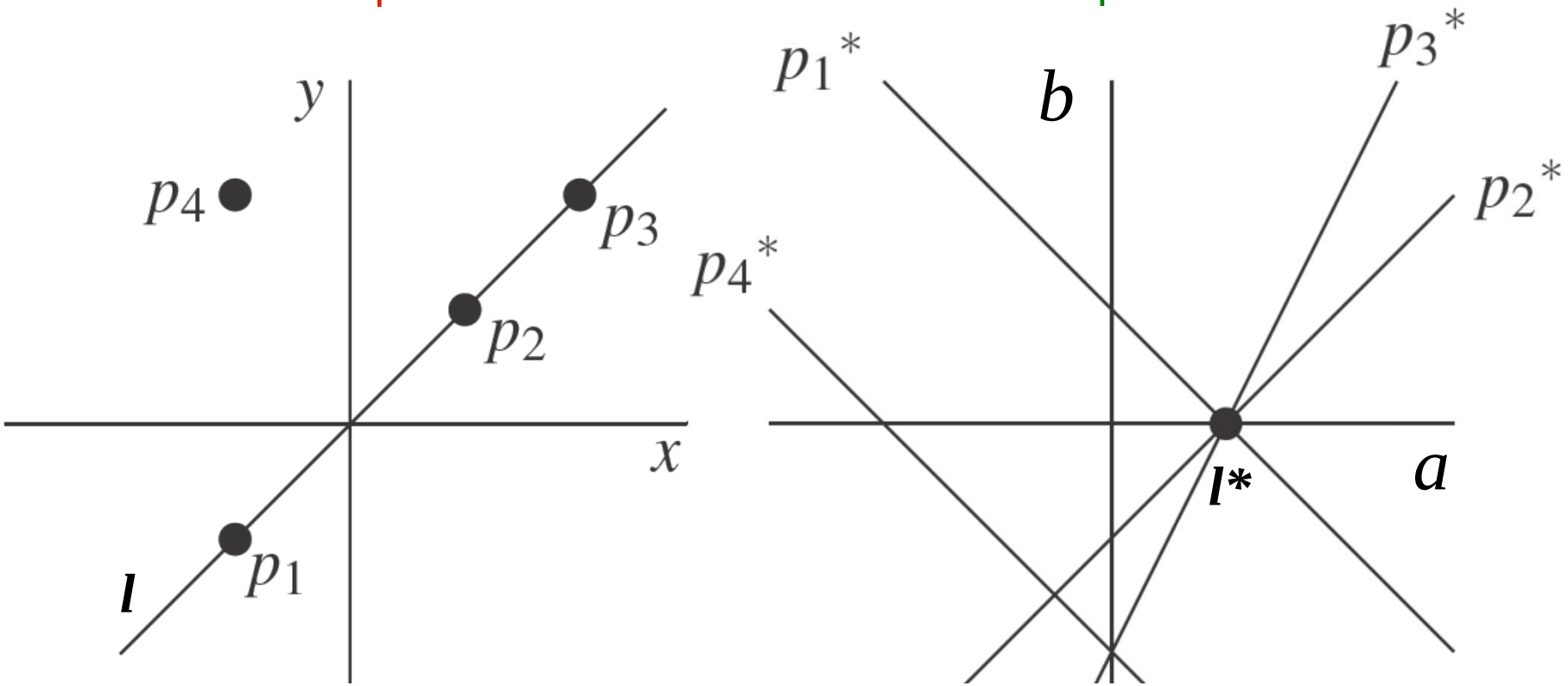


Duality of lines and points in the plane

Primal plane

Dual plane



point p = [p_x, p_y]

~~line l := (y = ax + b)~~

line l := (y = ax – b)

line p* := (b = $p_x a$ – p_y)

~~Point l^* = [a, – b]~~

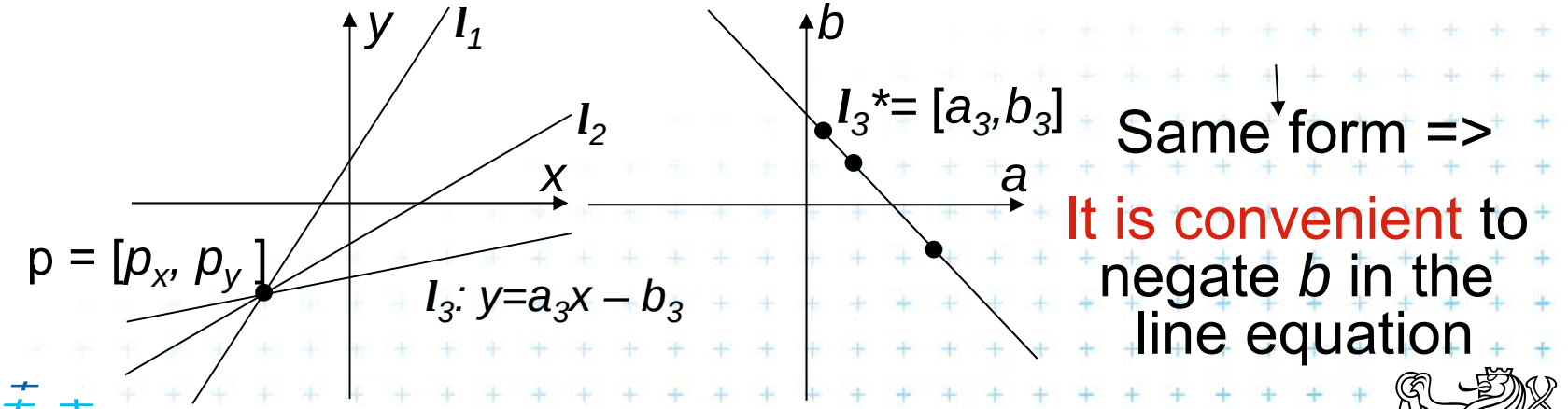
Point l^* = [a, b]

[Berg]

Same form => It is convenient to negate b in the line equation

Why is *b* negated in the line equation?

- In primal plane, consider
  - point $p = [\, p_x, p_y \,]$ and
  - set of non-vertical lines $l_i : y = a_i x - b_i$ passing through p satisfy the equation $p_y = a_i p_x - b_i$ (each line with different constants a_i, b_i)
- In dual plane, these lines transform to collinear points $\{\, l_i^* = [a_i, b_i] : b_i = p_x a_i - p_y \,\}$



If *b not* negated in the line equation…

Lines l_i have equartion $l_i : y = a_i x - b_i$ OR $y = a_i x + b_i$

Passing through point p = [p_x, p_y] :

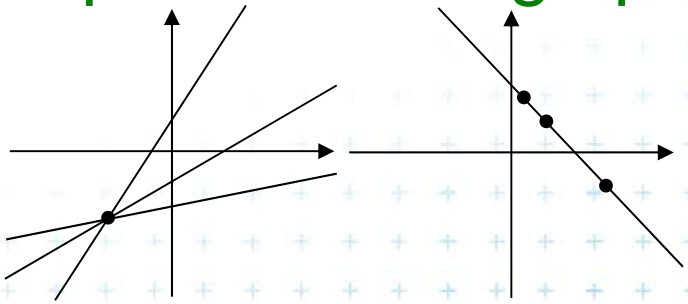
- With minus
  - equation l_i: $p_y = a_i p_x - b_i$
    dual points $\{l_i^* = [a_i, b_i] : b_i = p_x a_i - p_y \}$ … same form
- With plus
  - equation l_i: $p_y = a_i p_x + b_i$
    dual $\{l_i^* = [a_i, b_i] : b_i = -p_x a_i + p_y \}$ … different form

Properties of points and lines duality

Incidence is preserved

- A point *p* is incident to the line *l* in primal plane
  iff
  point *l** is incident to the line *p** in the dual plane.

- Lines l_1, l_2 intersects at point *p*
  iff
  line *p** passes through points l_1*, l_2*.

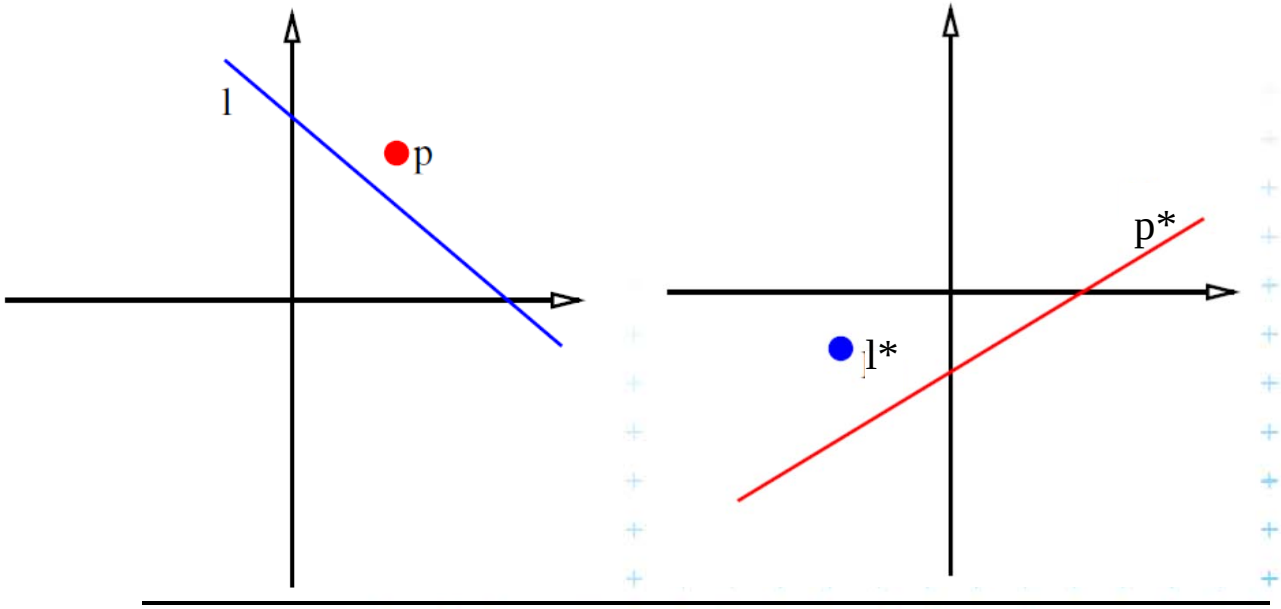


Properties of points and lines duality

But order is reversed

- Point *p* lies above (below) line *l* in the primal plane **iff**
  line *p** passes below (above) point *l** in the dual plane

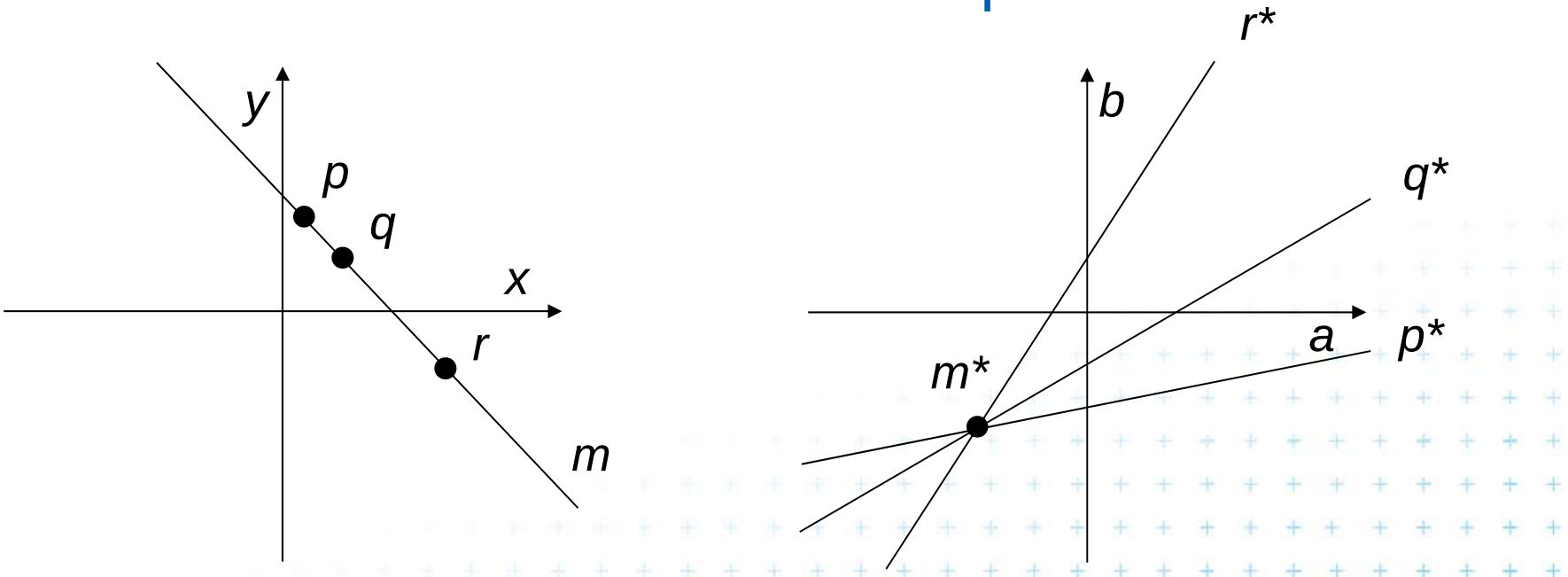
Or said order is preserved: … **iff** Point *l** lies above (below) line *p**



Properties of points and lines duality

Collinearity

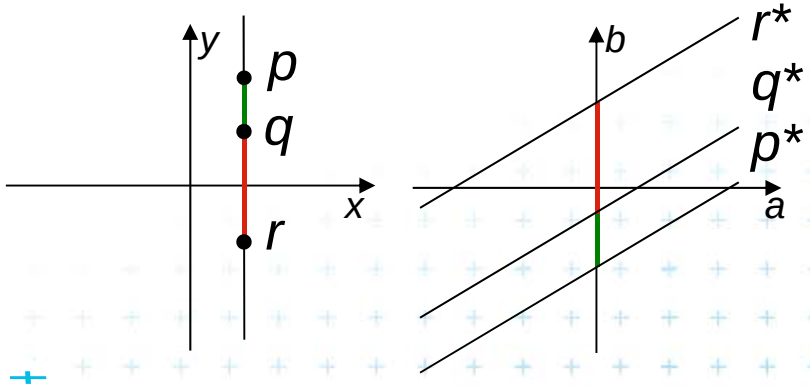
- Points are collinear in the primal plane **iff** their dual lines intersect in a common point



- This does not hold for points on vertical line

Handling of vertical lines

- Dual transform is undefined for vertical lines
  - Points with same *x* coordinate dualize to lines with the same slope (parallel lines) and therefore
  - These dual lines do not intersect (as should for collinear points)
  - Vertical line through these points does not dualize to an intersection point
  - For detection of vertically collinear points use other method - O(*n*) vertical lines -> O(n^2) brute force 3|| lines s.
    -> O(*n*) after O(*n* log *n*) sorting by *x*

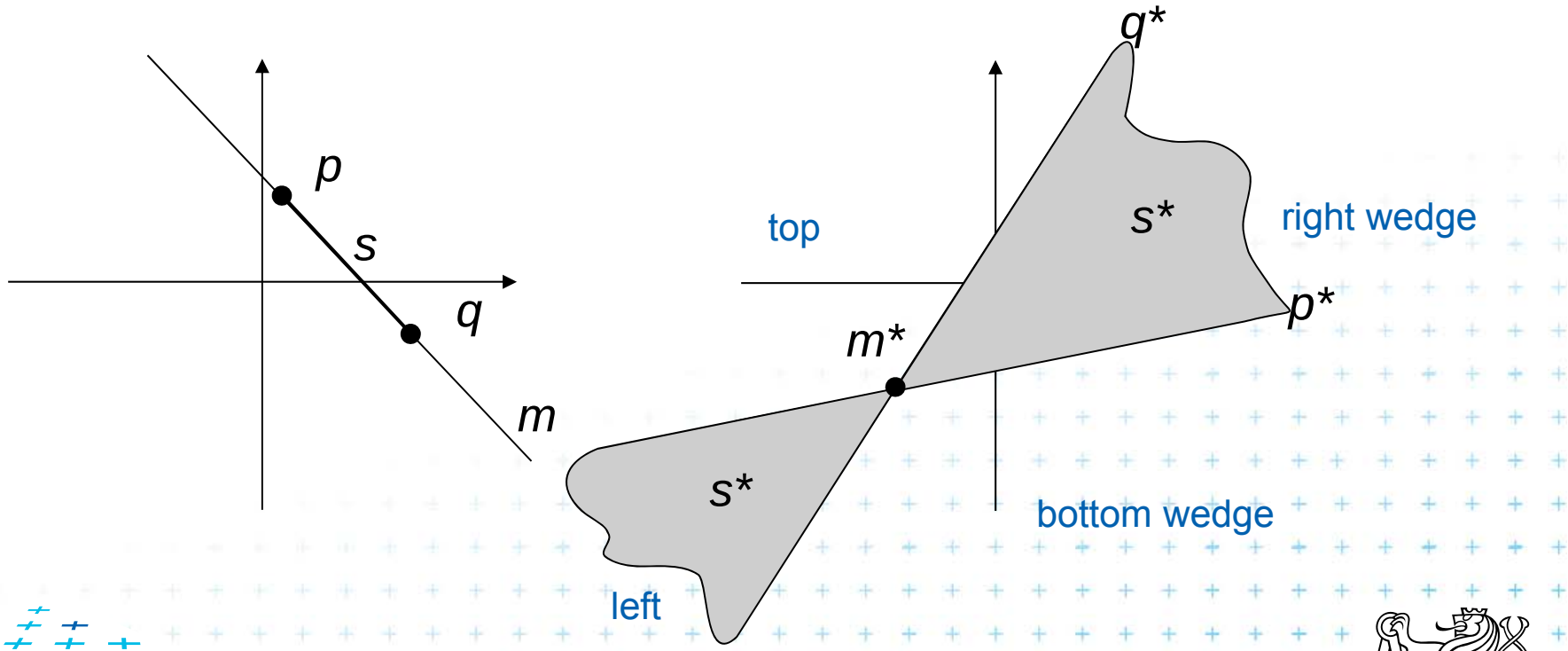


Vertical distances of such duals are "preserved". For $p_x = q_x$

$vertDist(q^*_b, p^*_b) = p_y - q_y$

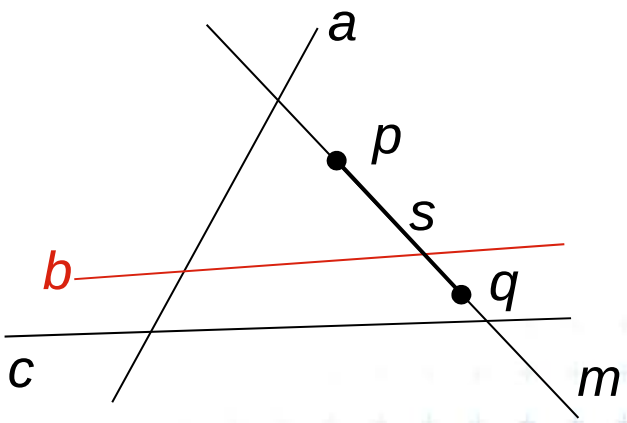
2. Duality of line segments

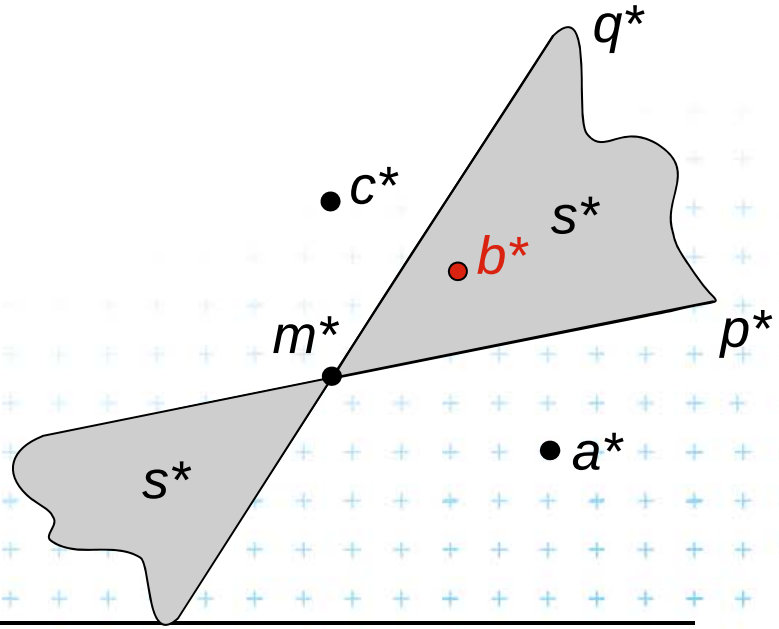
- Line segment *s*
  - = set of collinear points —dual—> set of lines passing one point
  - – union of these lines is a (left-right) double wedge *s** (Dvojitý klín)



Intersection of line and line segment

- Line *b* intersects line segment *s*
  - if point *b** lays in the double wedge *s**,
    i.e., between the duals *p*,q** of segment endpoints *p,q*
  - point *p* lies above line *b* and *q* lies below line *b*
  - point b* lies above line *p** and *b** lies below line *q**



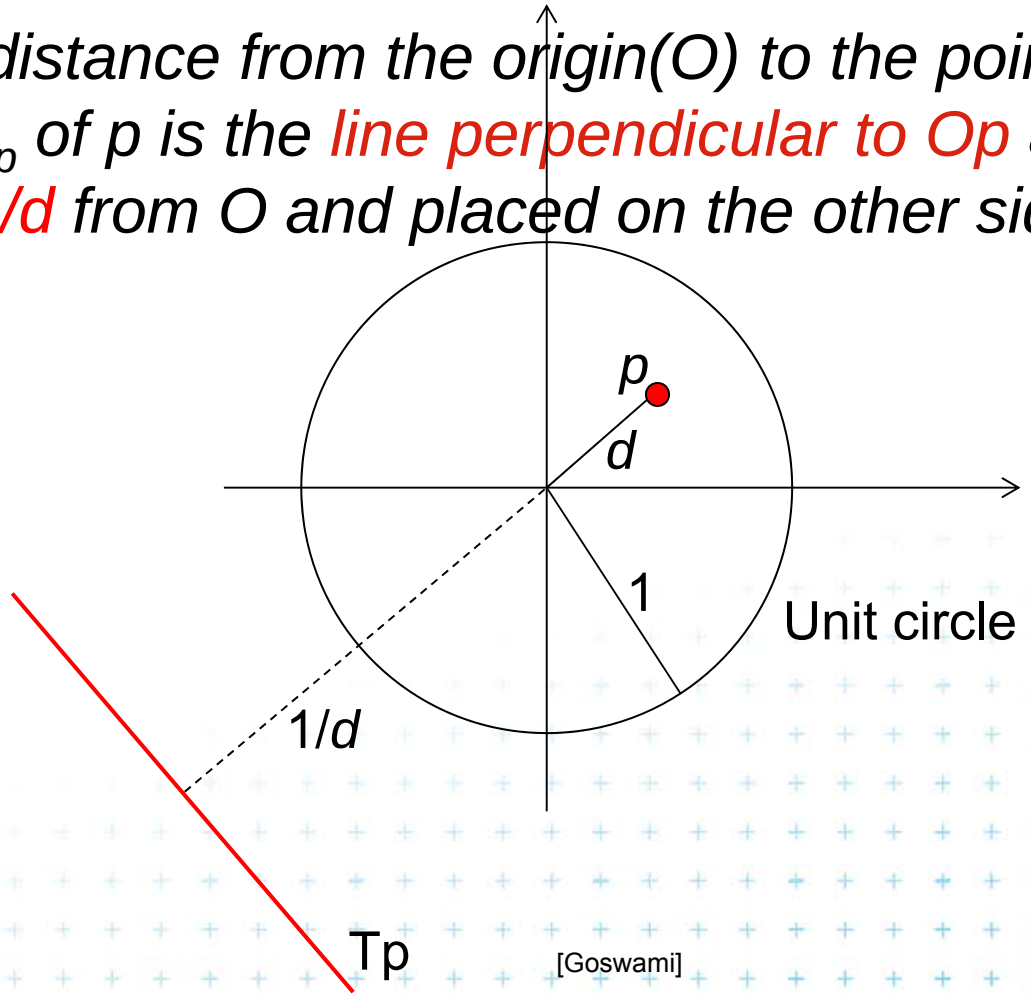


3. Polar duality (Polarity)

- Another example of point-line duality
- In 2D: Point p = (p_x, p_y) in the primal plane corresponds to a line T_p with equation $ax + by = 1$ in the dual plane and vice versa
- In dD: Point p is taken as a radius-vector (starts in origin O). The dot product (p . x) = 1 defines a polar hyperplane p* = { $x \in R^d$: (p . x) = 1 }
- Used in theory of polytopes

Polar duality (Polarity)

- *Geometrically in 2D, this means that*
  - *if d is the distance from the origin(O) to the point p, the dual T_p of p is the line perpendicular to Op at distance 1/d from O and placed on the other side of O.*



[Goswami]

4. Convex hull using duality – definitions

- An optimal algorithm
- Let *P* be the given set of *n* points in the plane.
- Let $p_a \in P$ be the point with smallest x-coordinate
- Let $p_d \in P$ be the point with largest x-coordinate

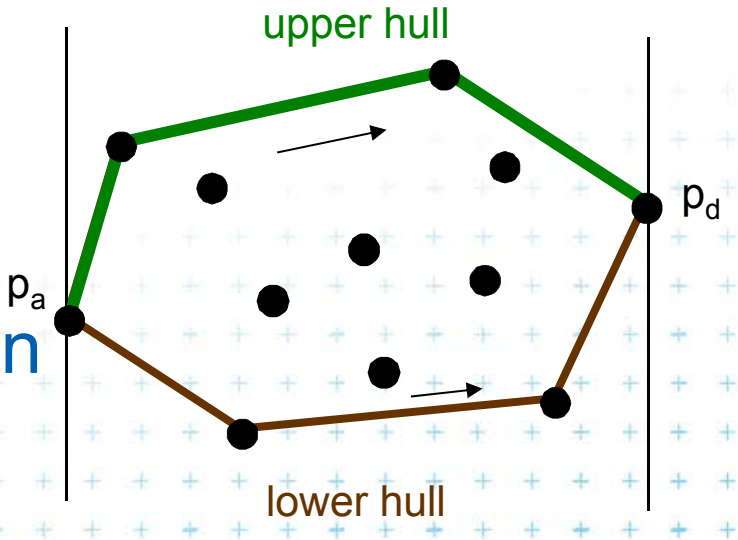
  Both p_a and $p_d \in CH(P)$

Upper hull = CW polygonal chain

$p_a, \ldots, p_d$ along the hull

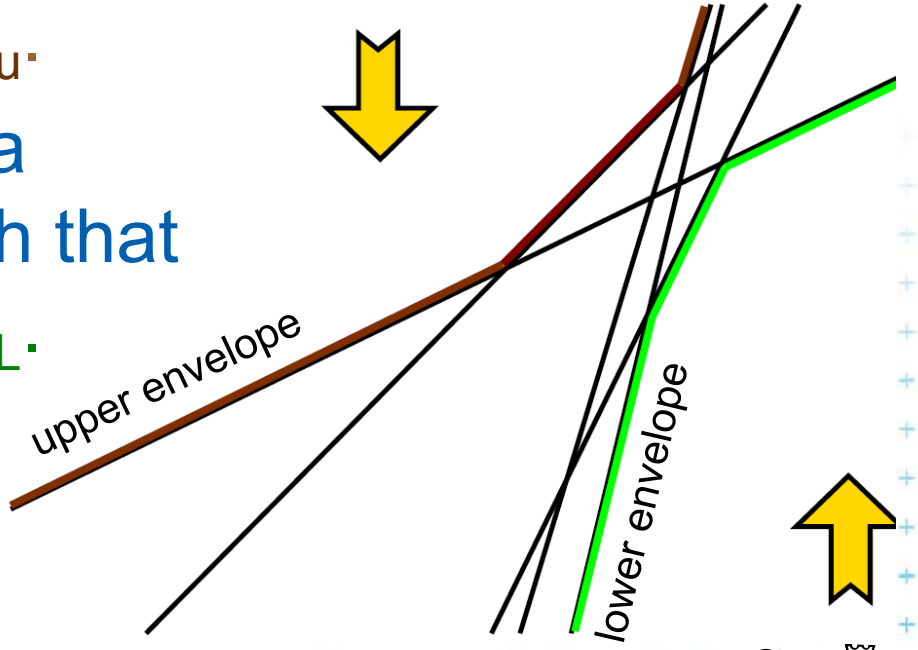
Lower hull = CCW polygonal chain

$p_a, \ldots, p_d$ along the hull



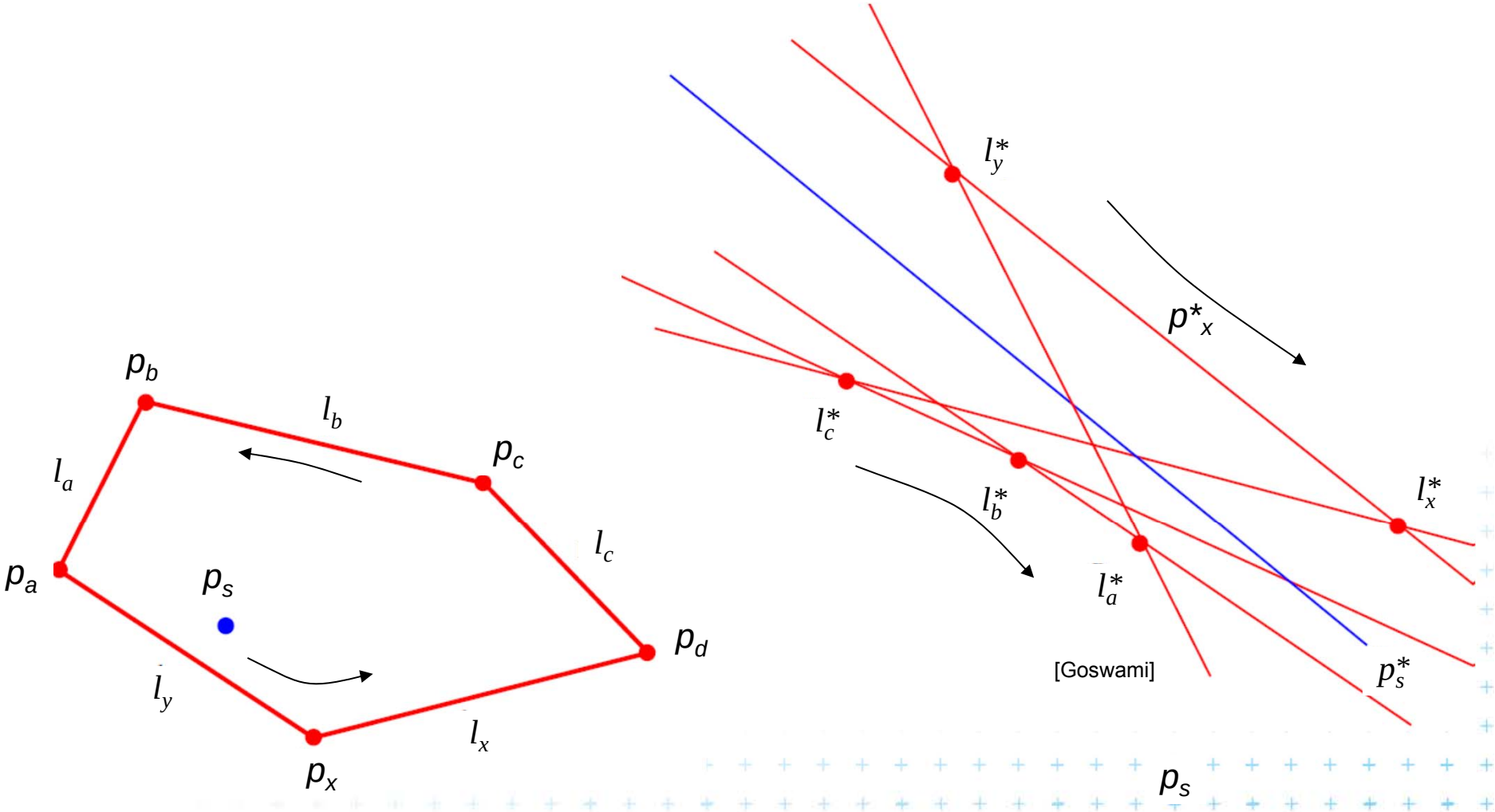
Definitions

- Let L be a se of lines in the plane
- The upper envelope is a polygonal chain E_u such that no line $l \in L$ is above E_u.
- The lower envelope is a polygonal chain E_L such that no line $\boldsymbol{l} \in \boldsymbol{L}$ is below E_L.



[Goswami]

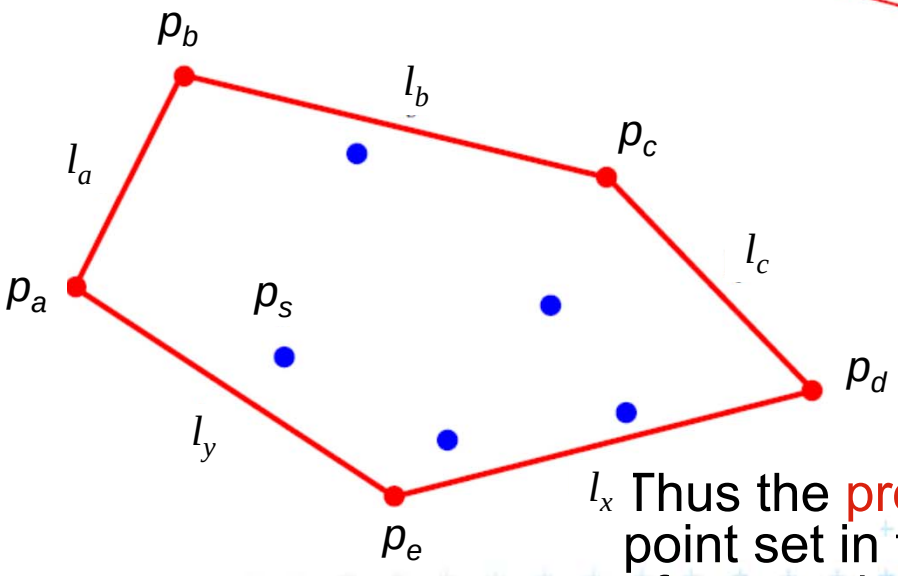
Connection between Hull and Envelope

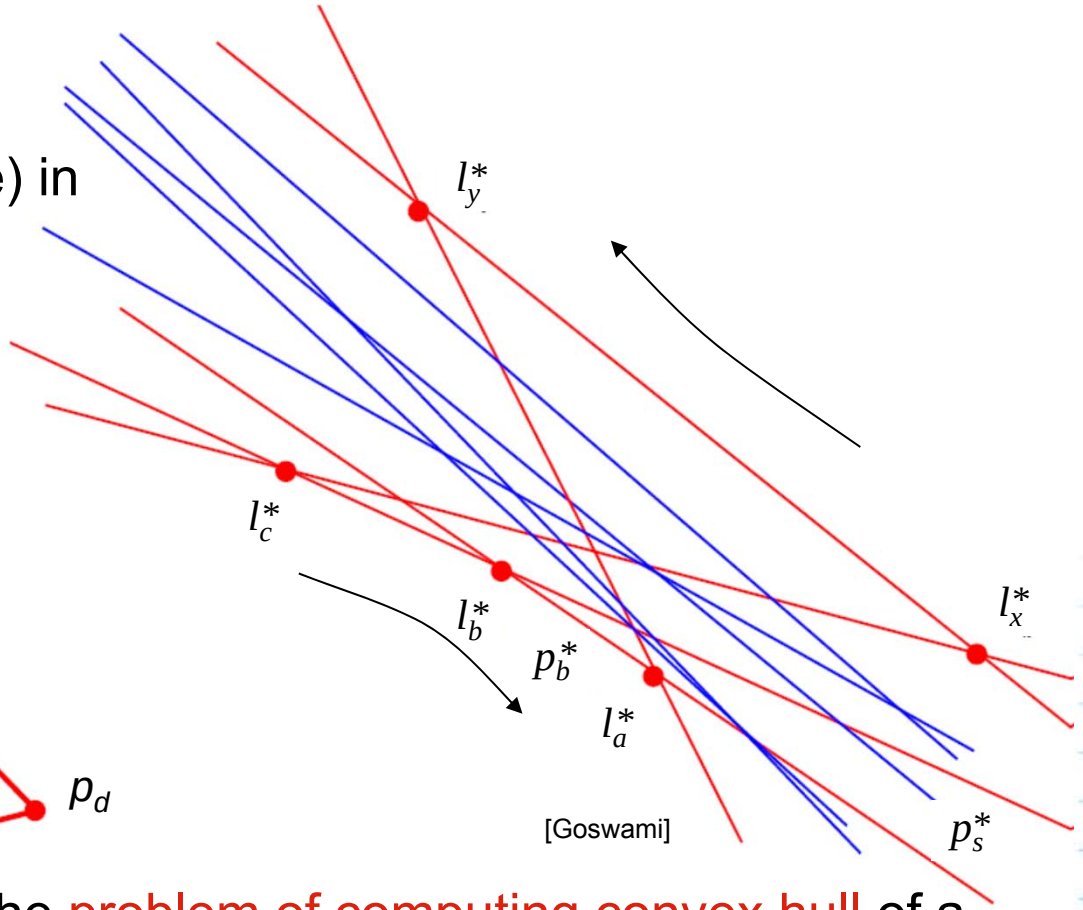


[Goswami]

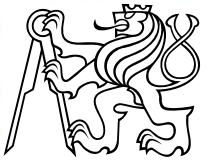
Connection between Hull and Envelope

Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.





l_x Thus the problem of computing convex hull of a point set in the primal plane reduces to the problem of computing upper and lower envelopes of the line set in the dual plane.



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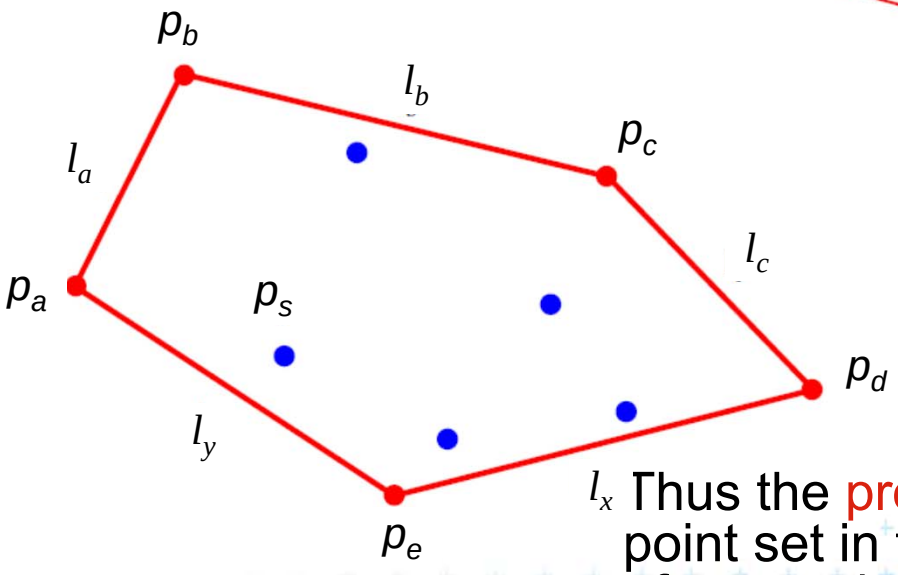

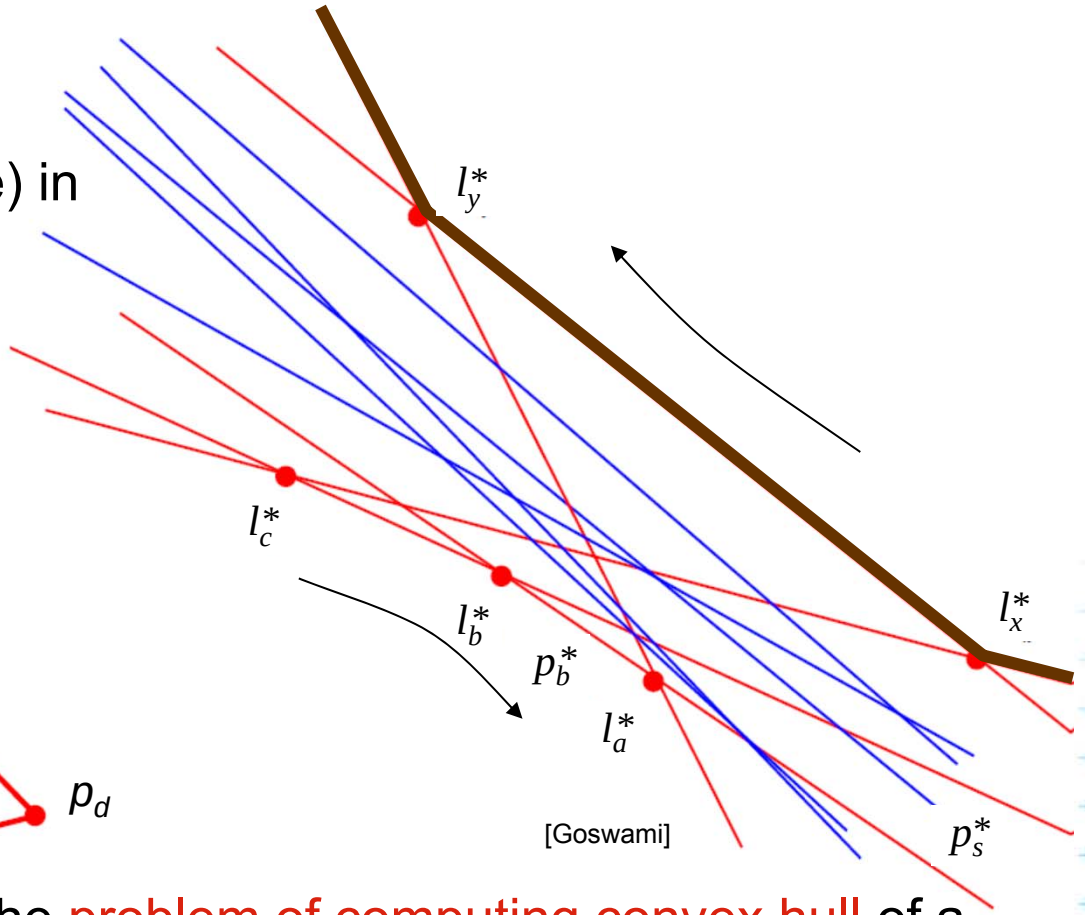

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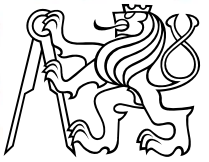
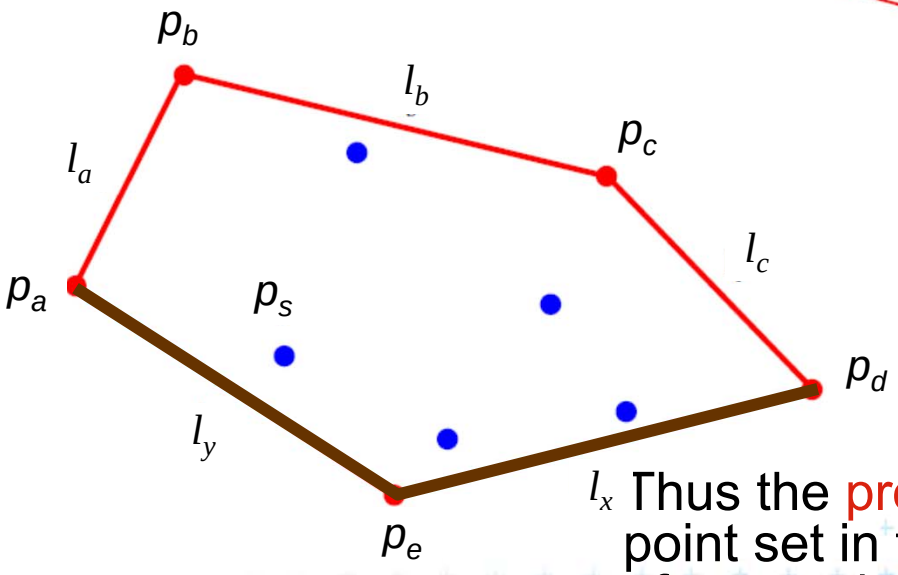
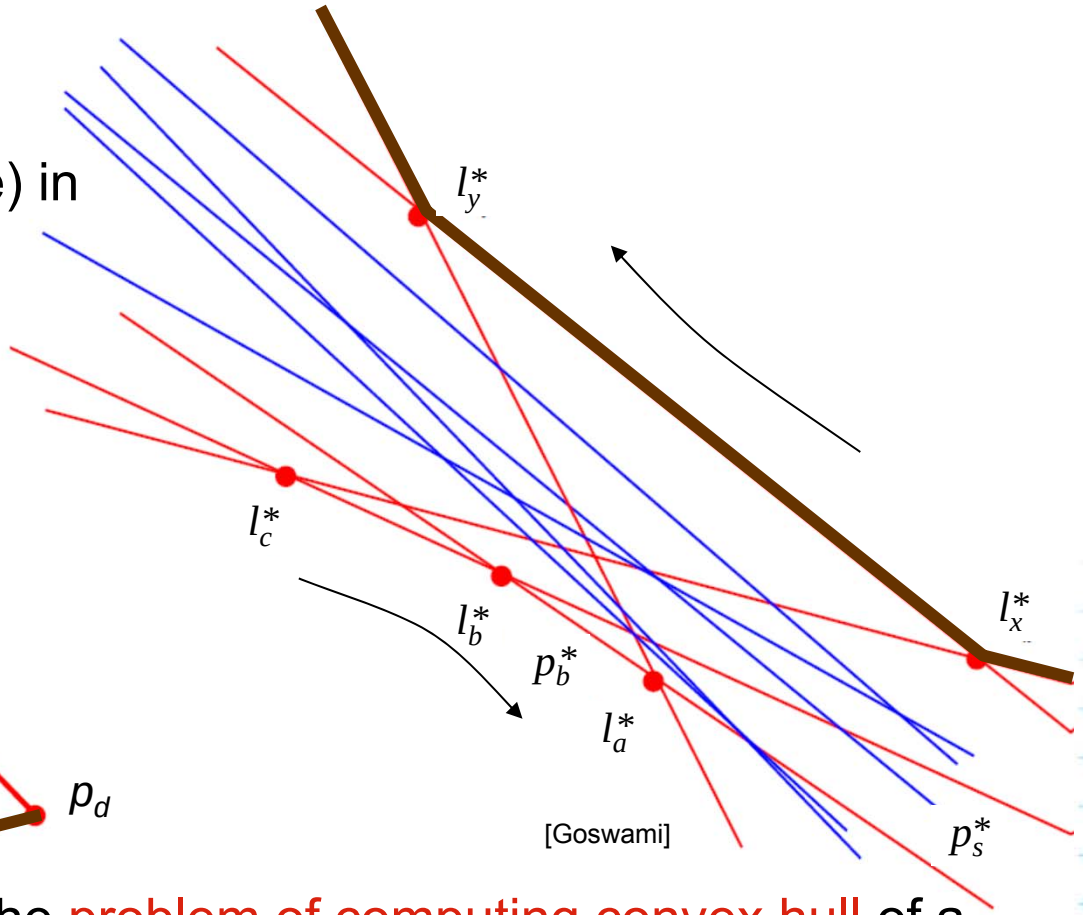


Connection between Hull and Envelope

Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.





[Goswami]

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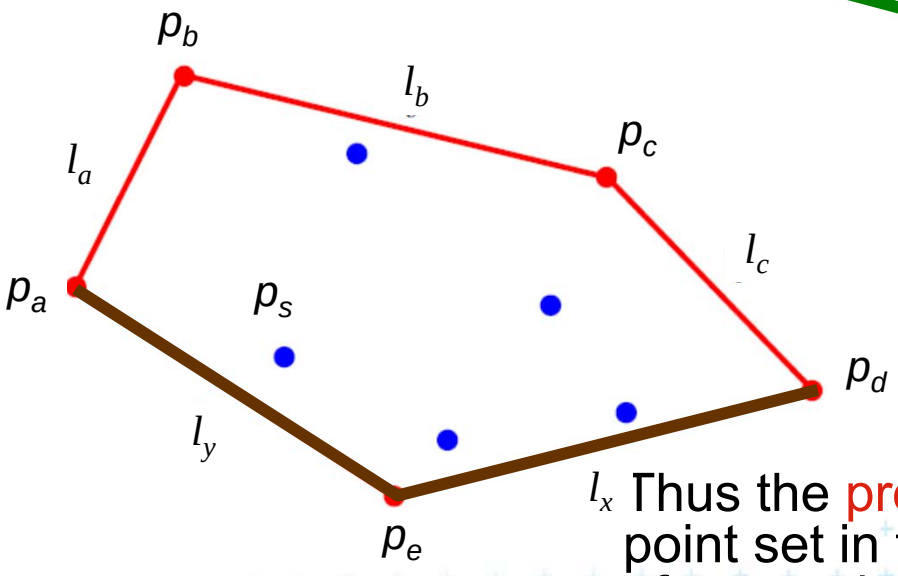

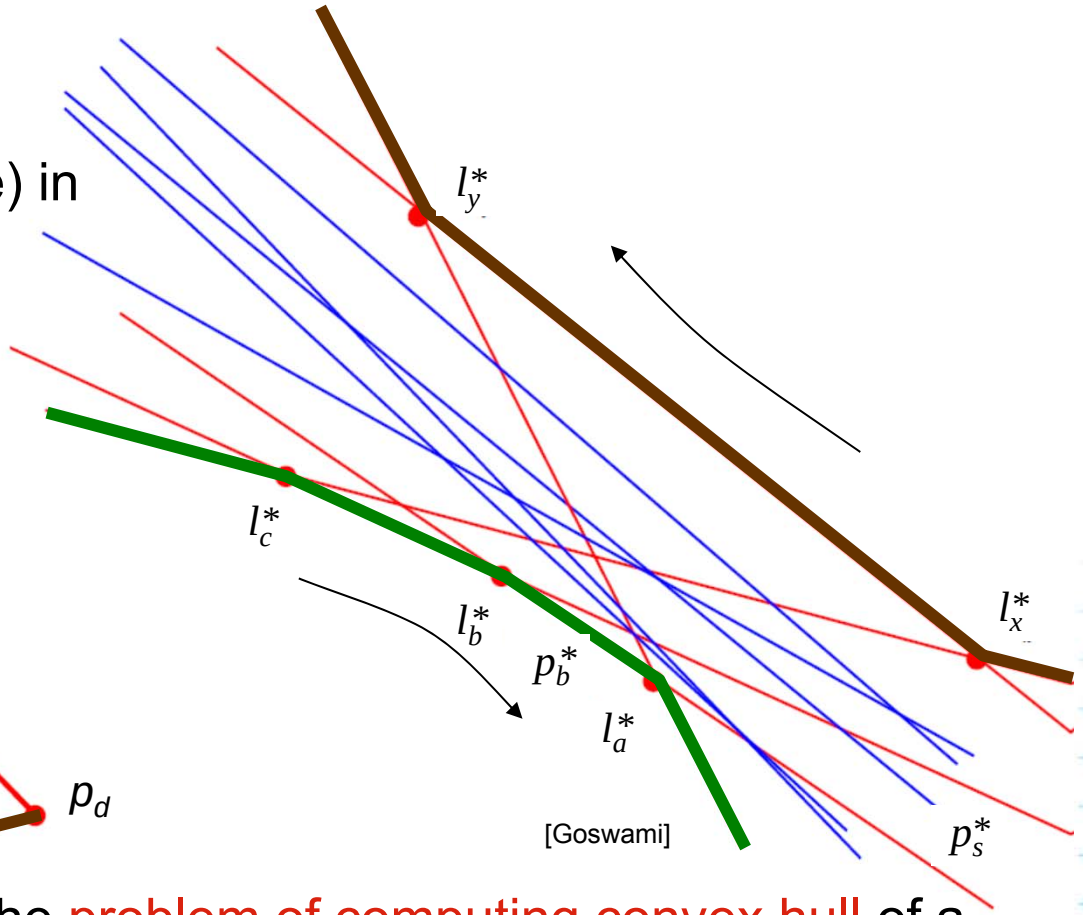

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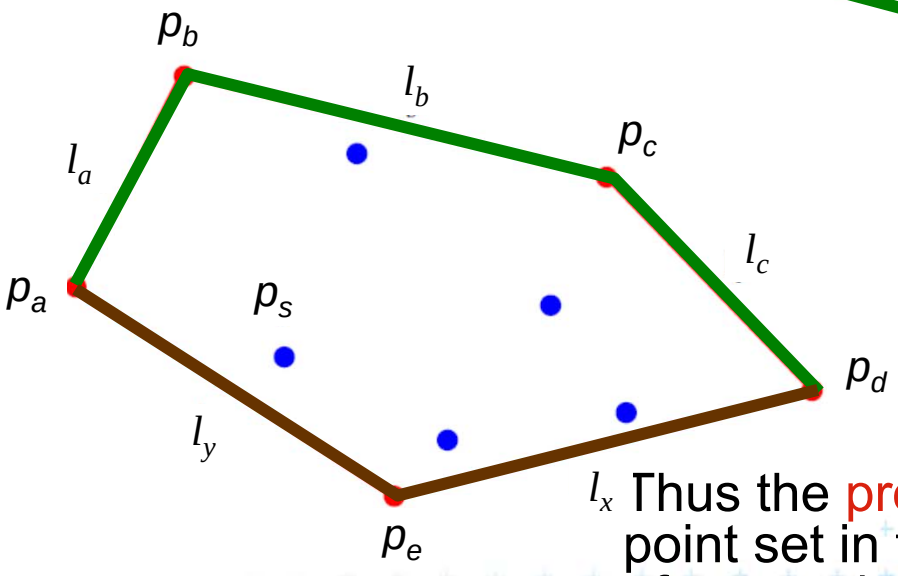

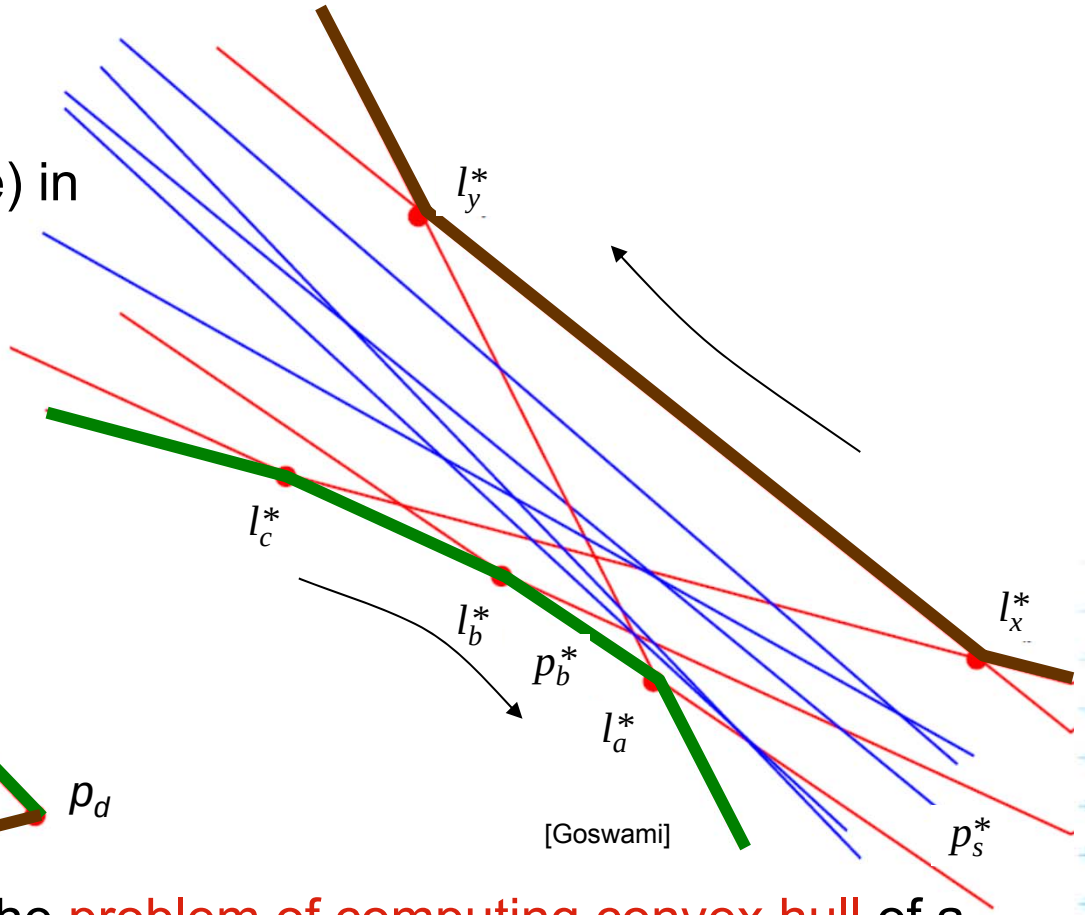

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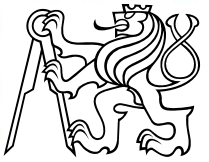

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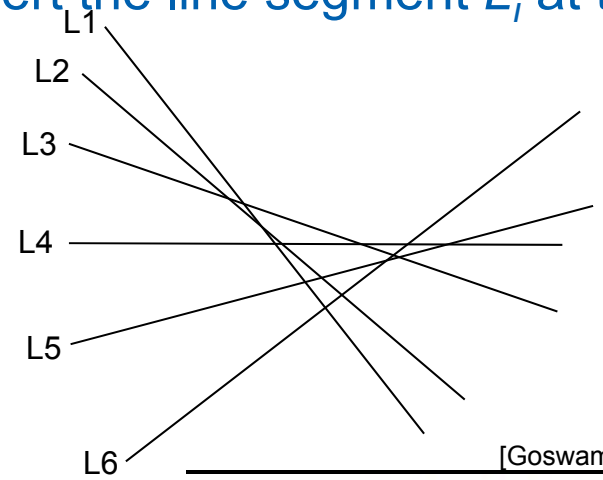

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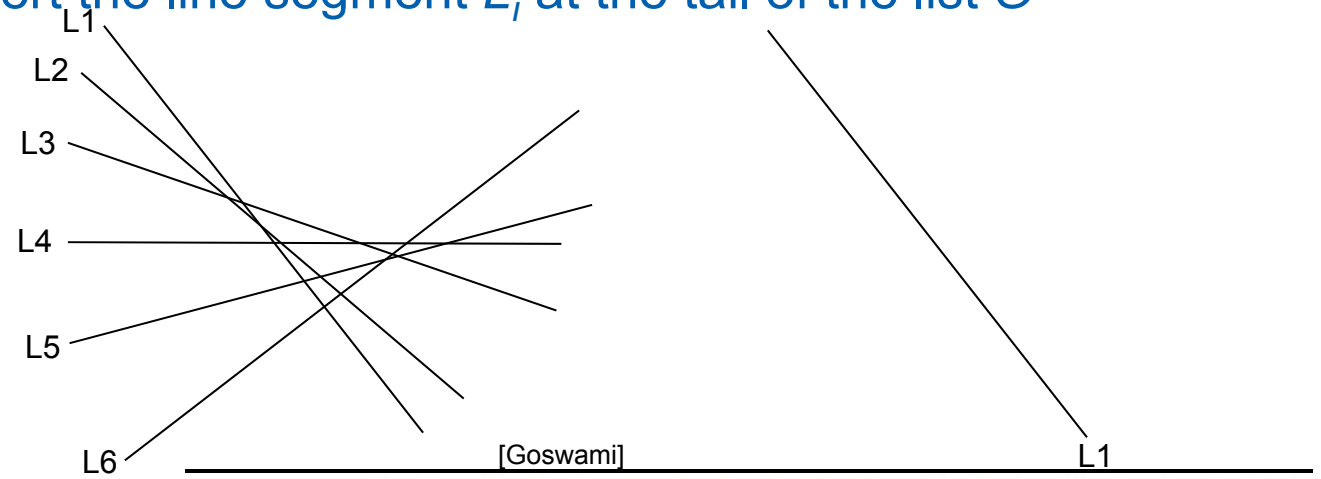

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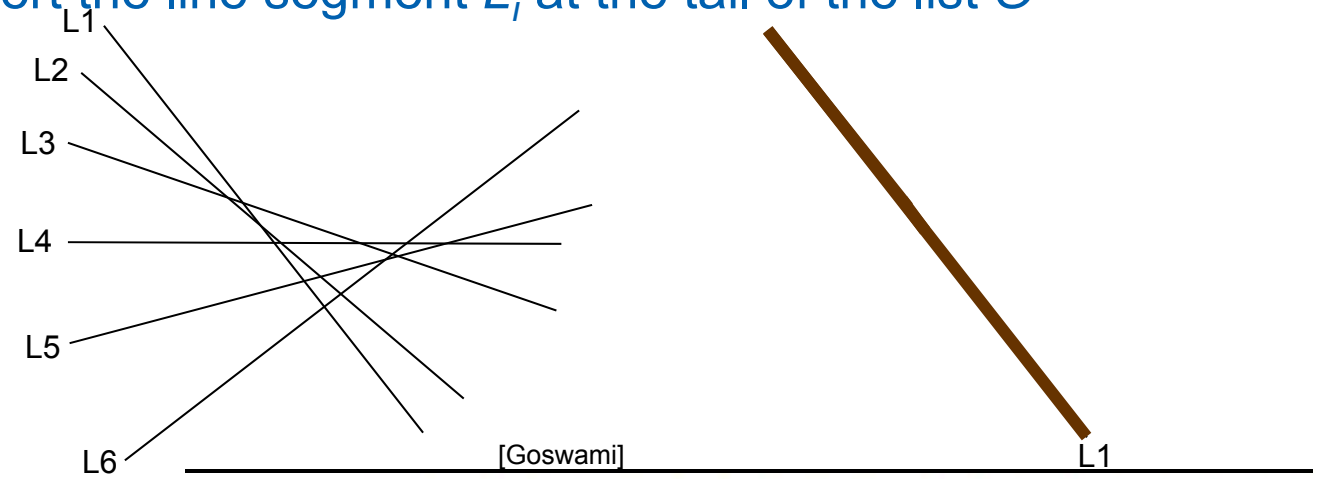

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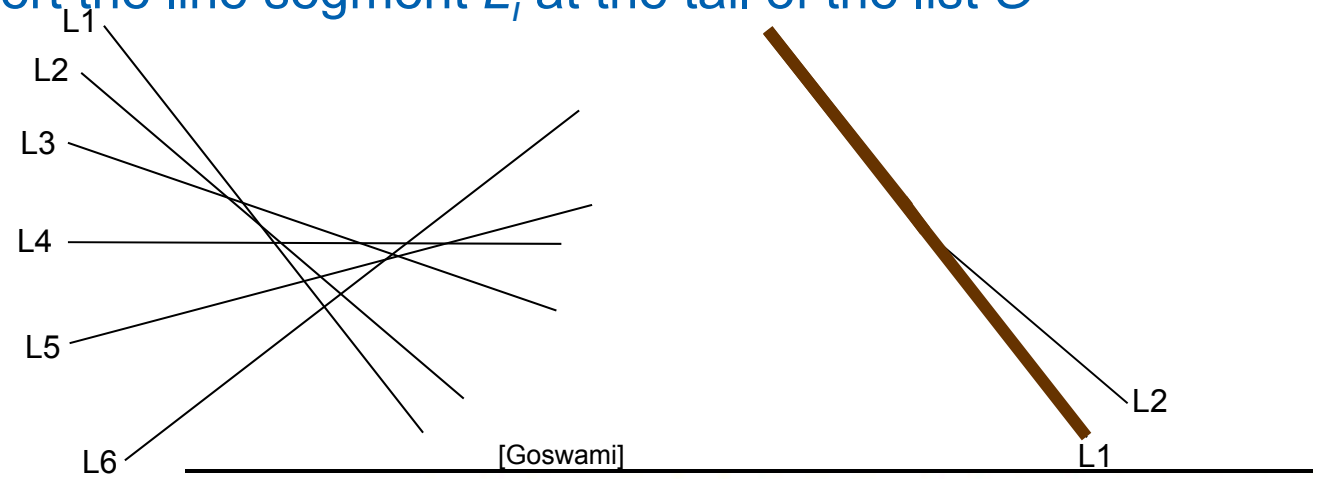

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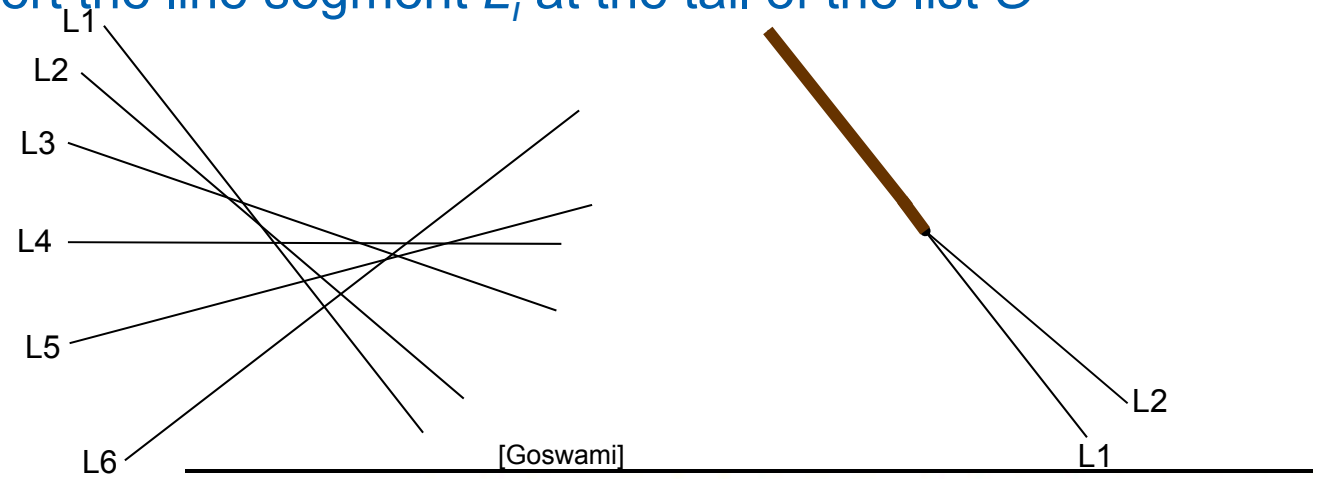

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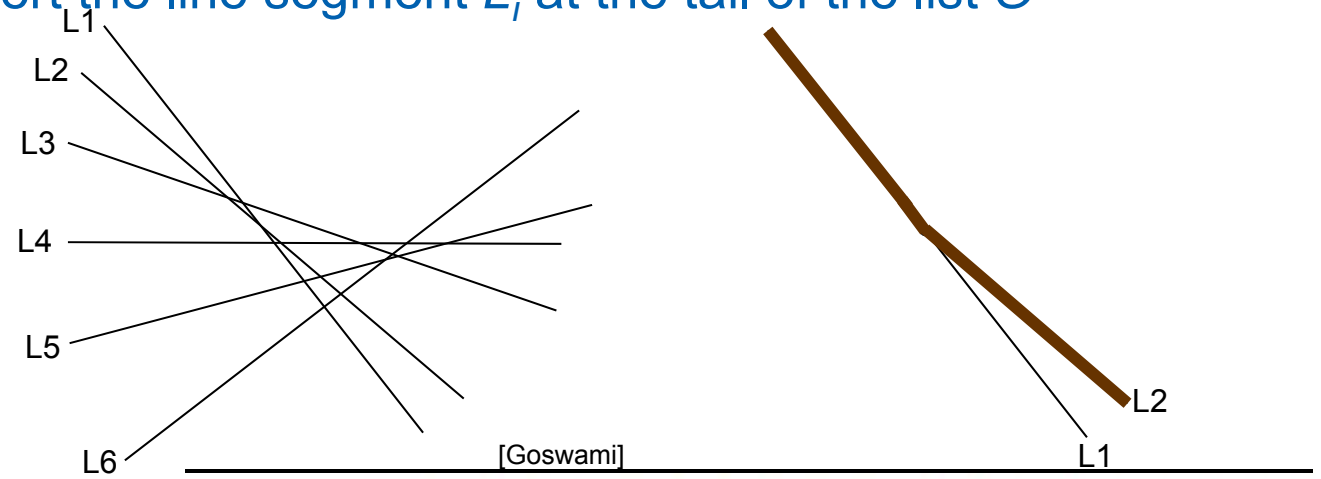

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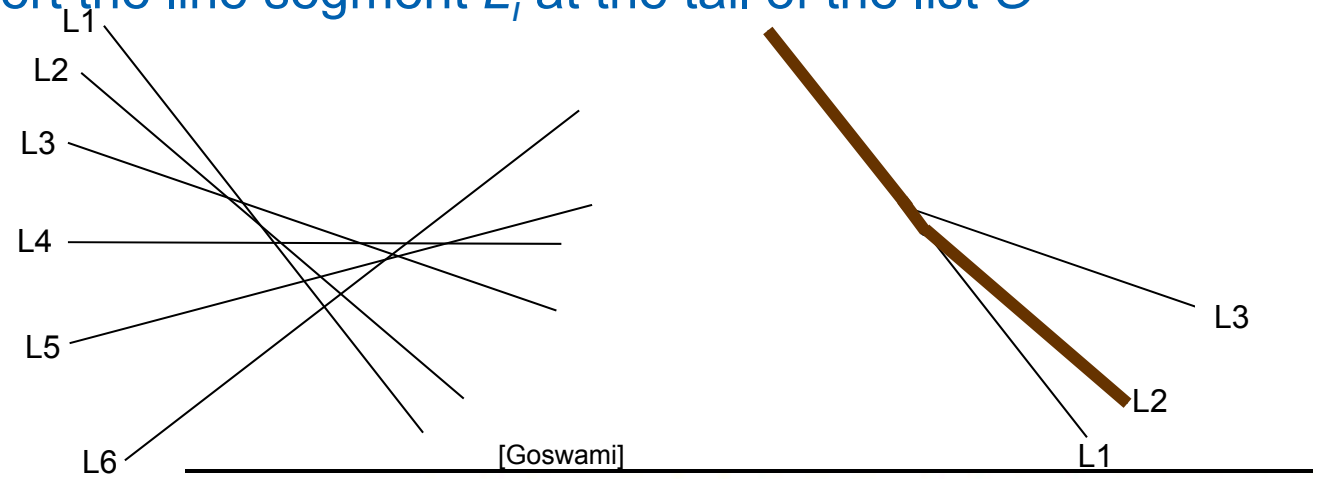

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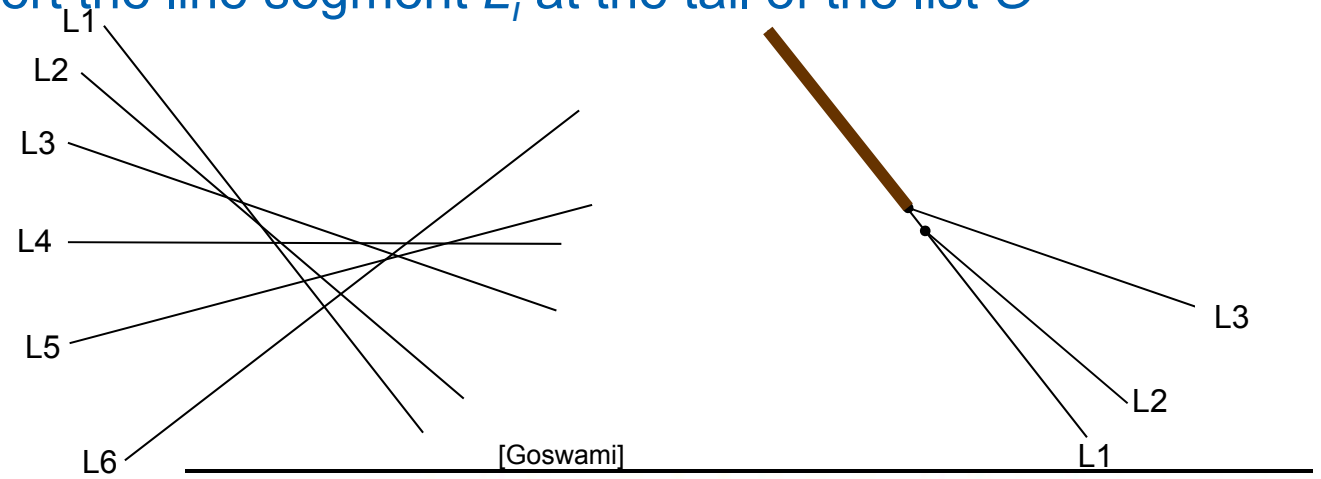

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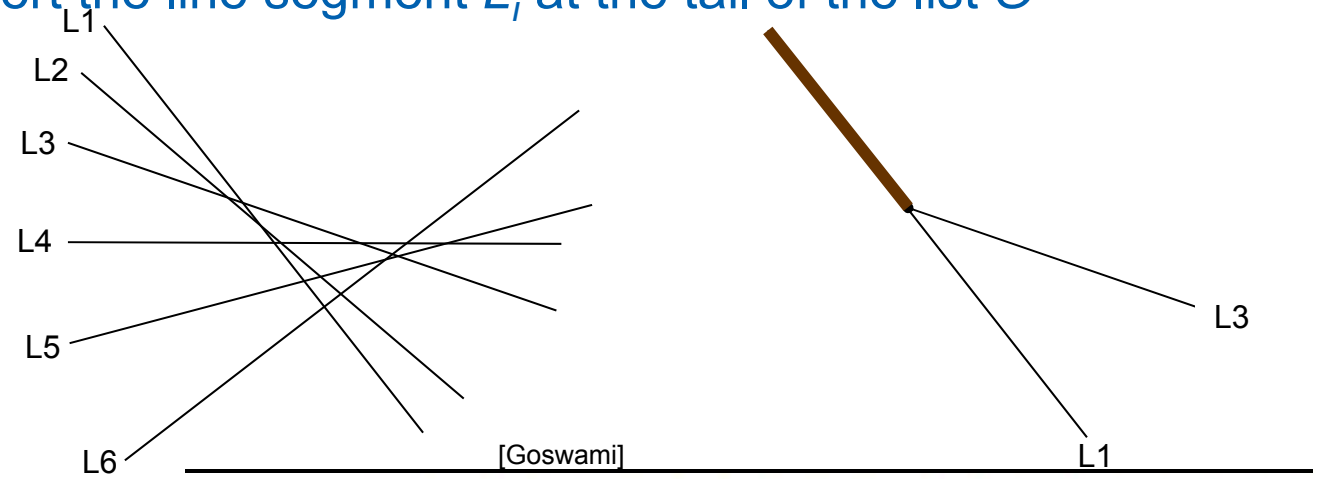

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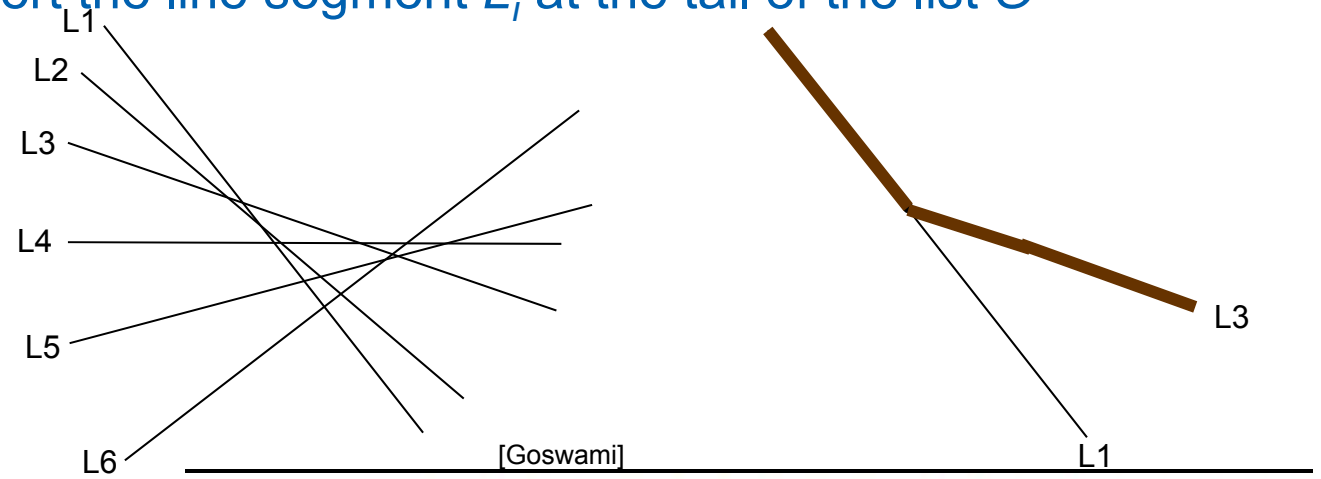

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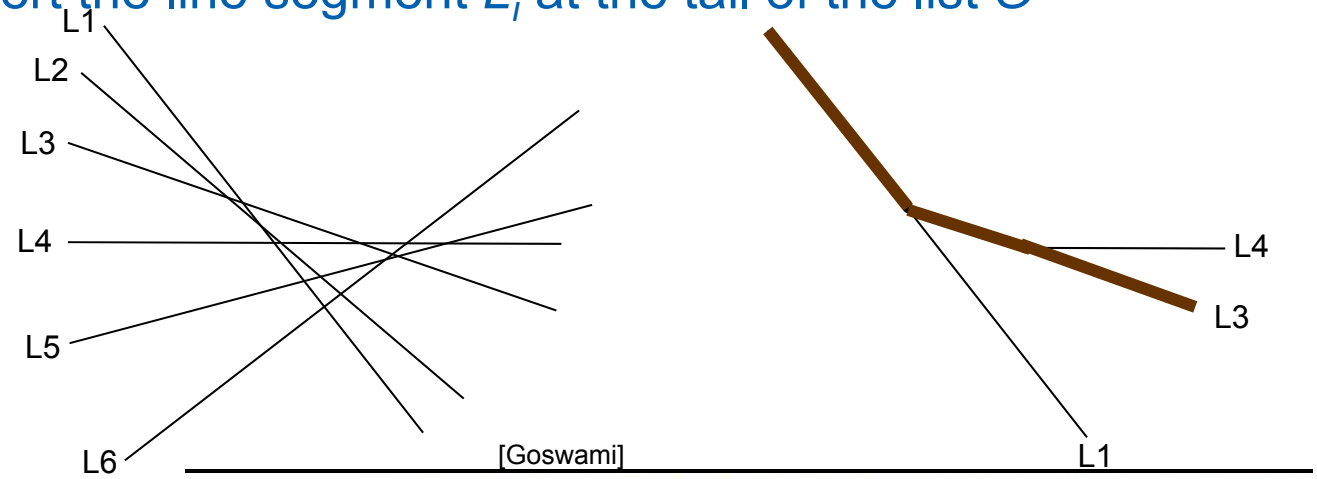

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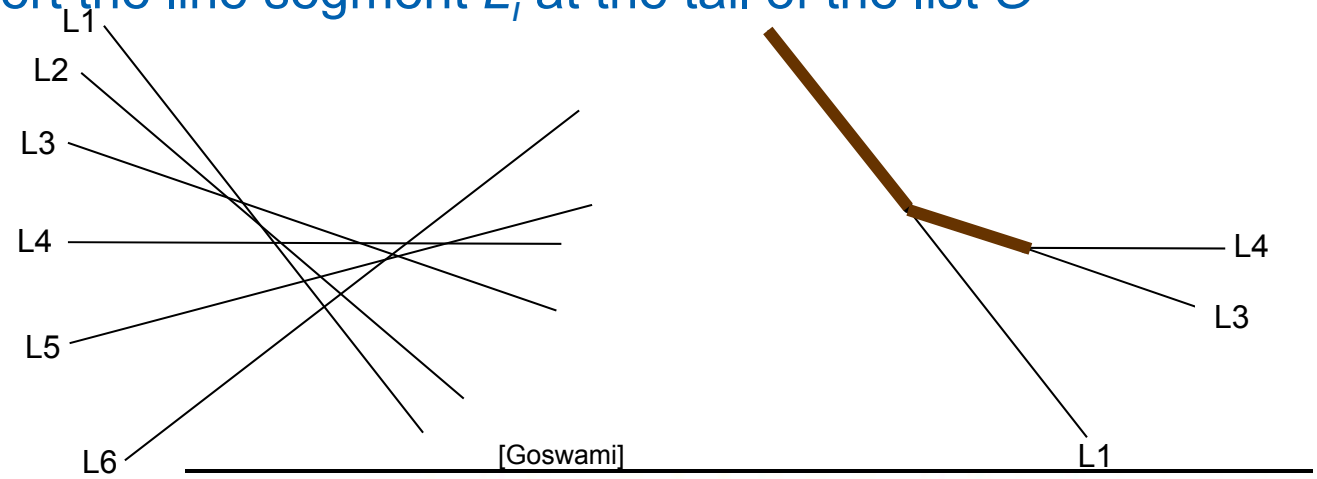

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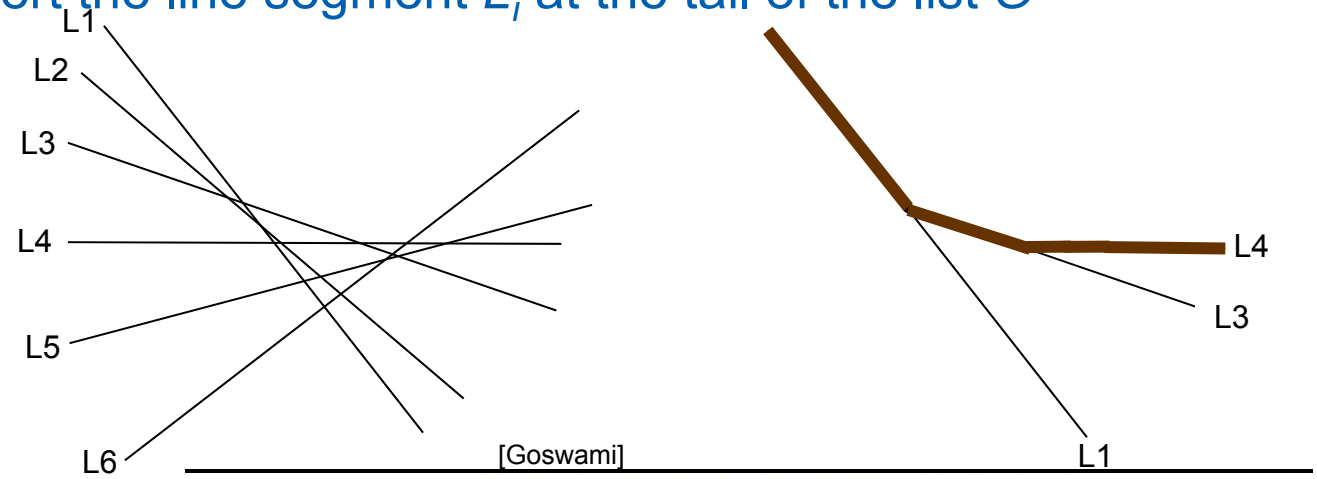

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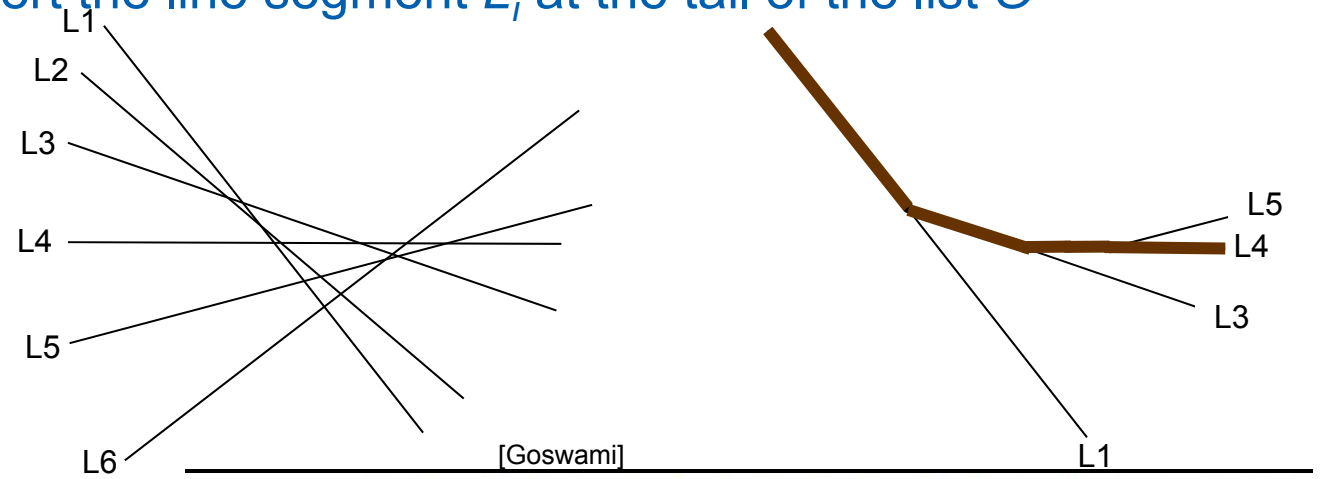

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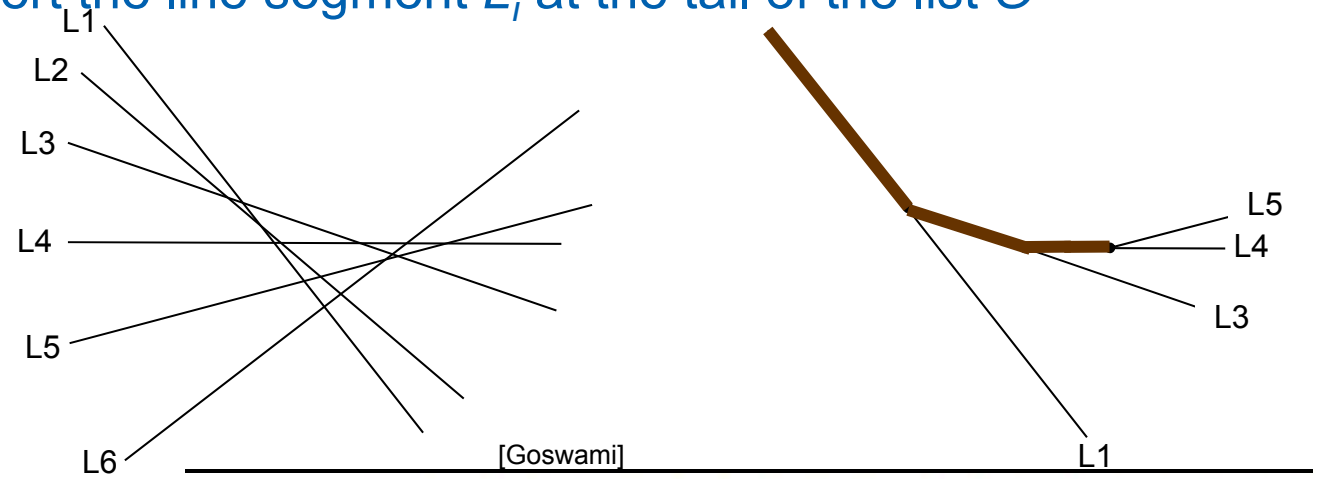

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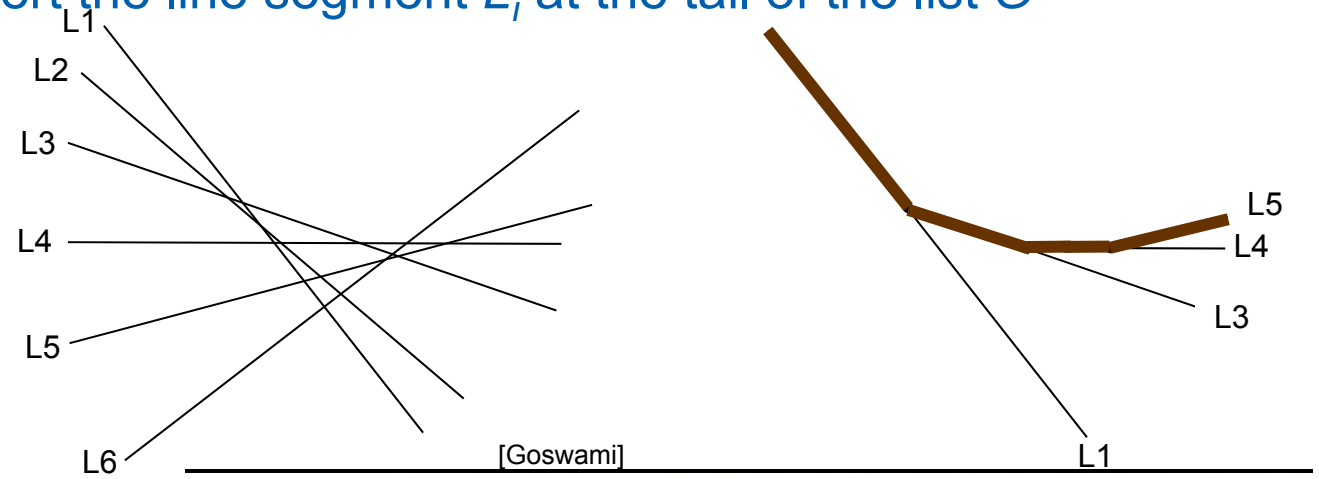

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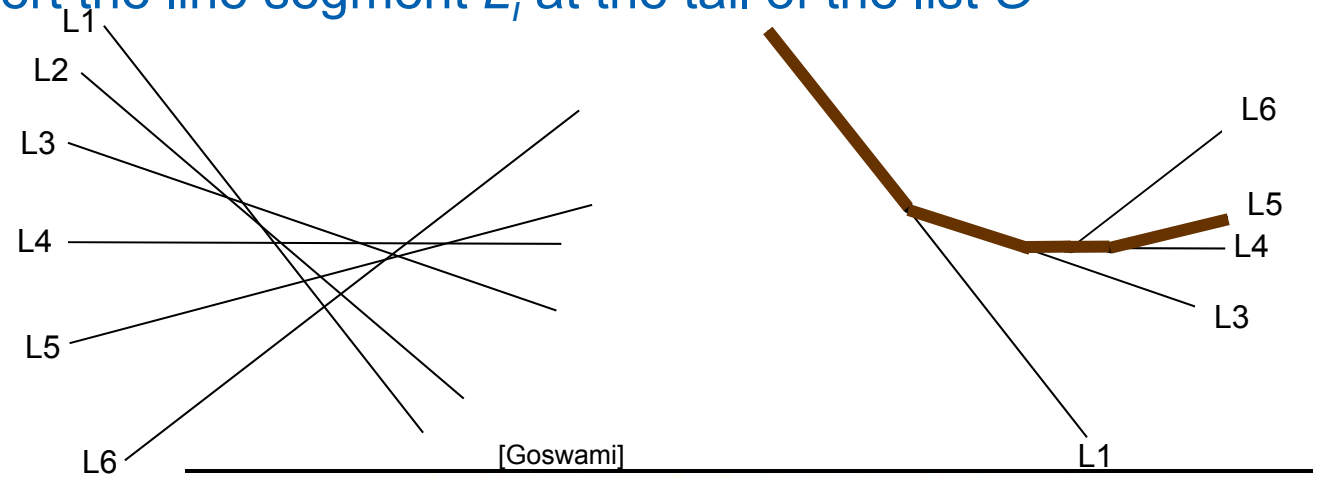

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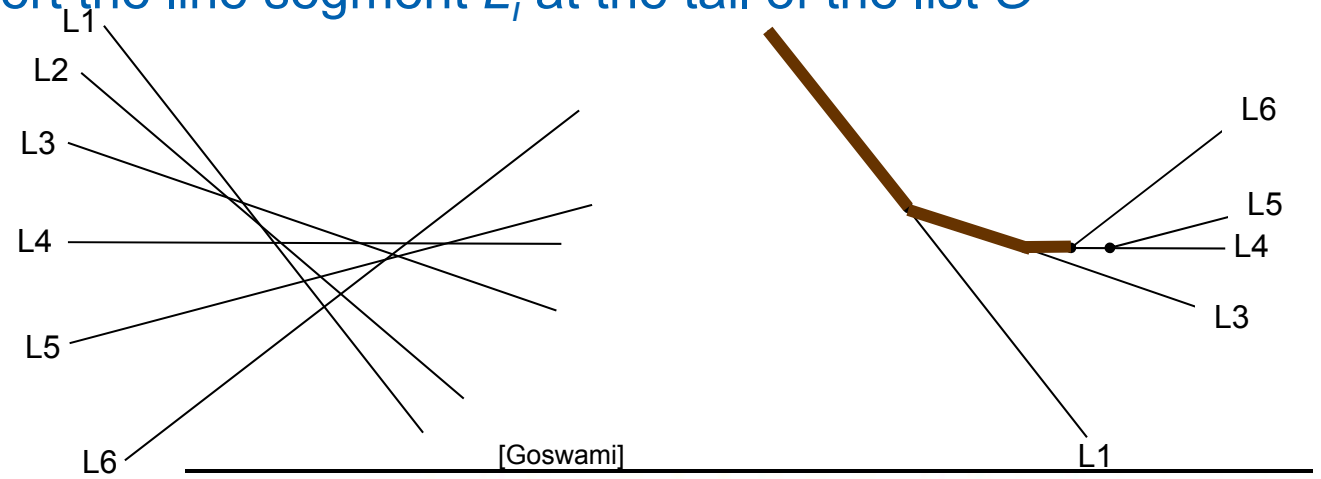

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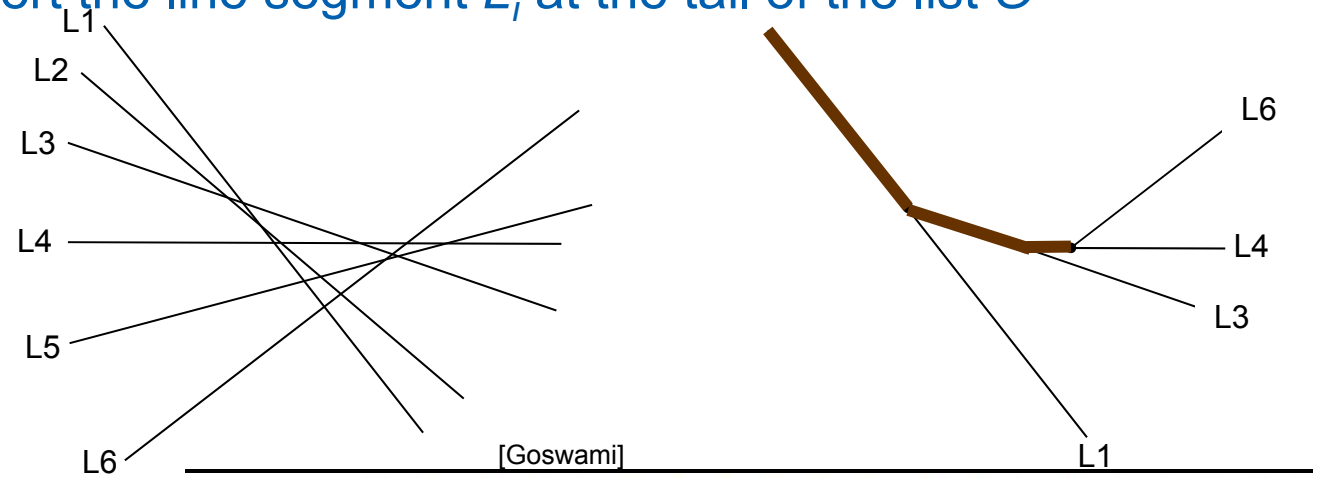

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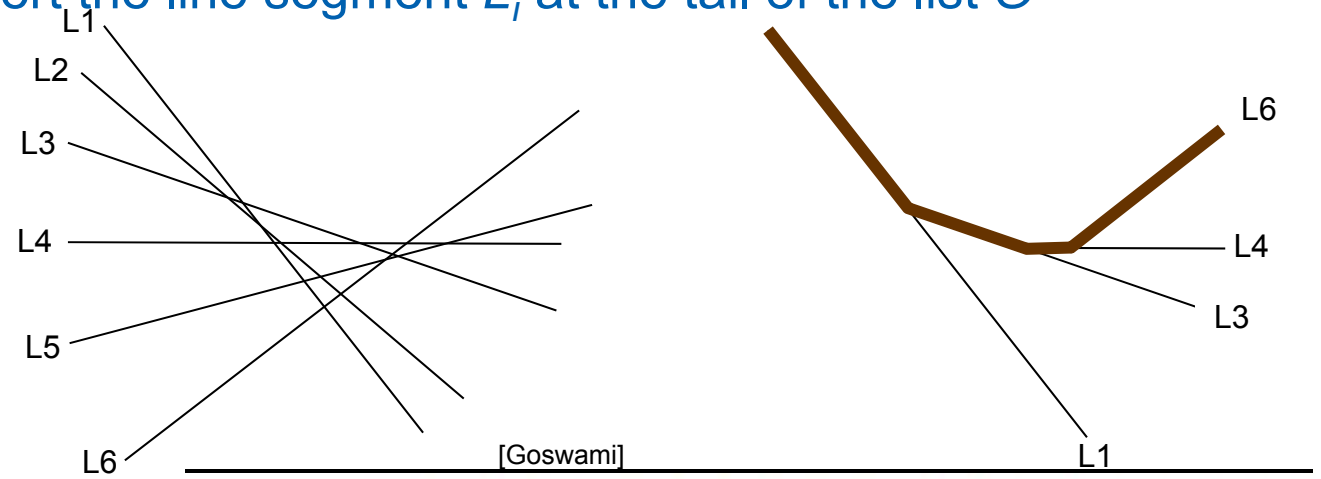

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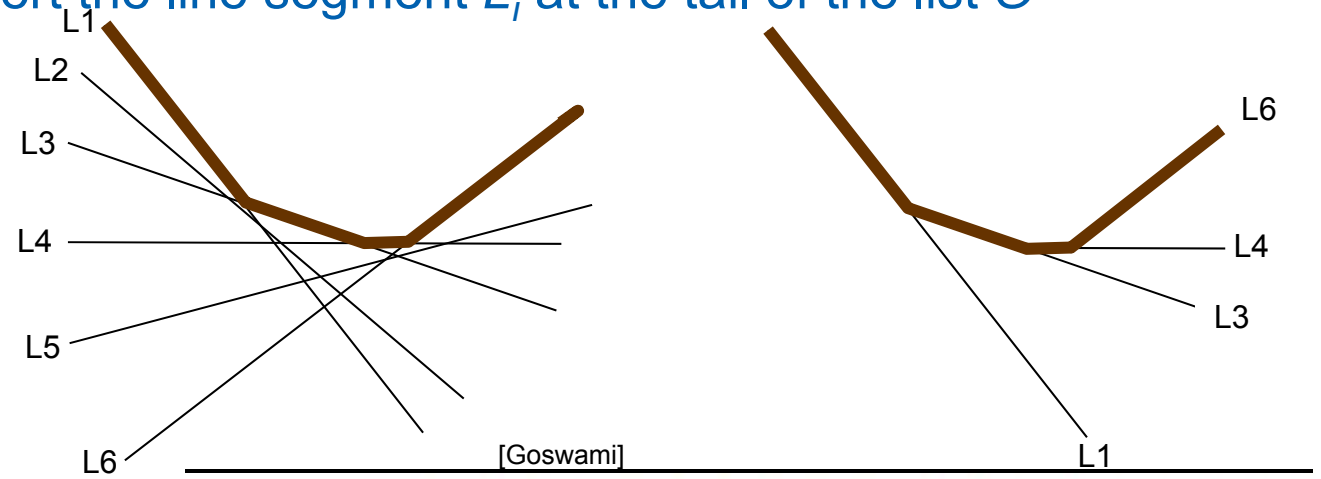


[Goswami]

Convex hull via upper and lower envelope

- Upper envelope complexity
  - After sorting n lines by their slopes in O(n log$n$) time, the upper envelope can be obtained in O(n) time
  - Proof: It may check more than one line segment when inserting a new line, but those ones checked are all removed except the last one.
    (O(n) insertions, max O(n) removals
    => O(n) all steps. Average step O(1) amortized time)
- Convex hull complexity
  - Given a set P of n points in the plane, CH(P) can be computed in O(n log n) time using O(n) space.

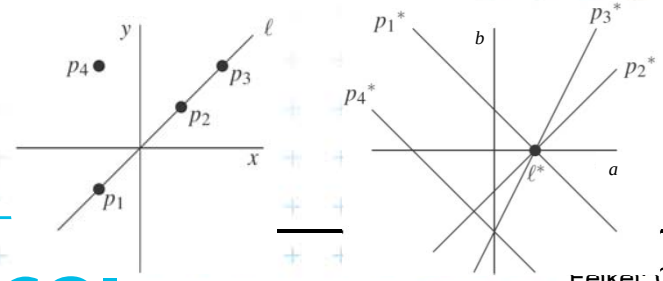
Applications of line arrangement

Examples of applications – solved in $O(n^2)$ and
- $O(n^2)$ space by constructing a line arrangement or
  $O(n)$ space through topological plain sweep.

a) General position test:

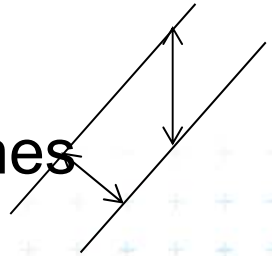
Given a set of n points in the plane, determine whether any three are collinear.

- Construct an arrangement in dual plane
- Report intersections of more than 2 lines



b) Minimum k-corridor

- Given a set of n points, and an integer $k \in [\, 1 : n \,]$, determine the narrowest pair of parallel lines that enclose at least k points of the set.
- The distance between the lines can be defined
  - either as the vertical distance between the lines
  - or as the perpendicular distance between the lines



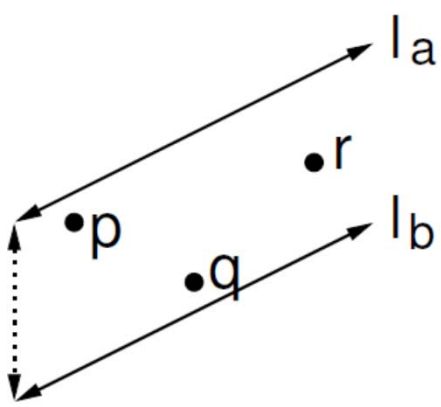
- Simplifications
  - Assume $k = 3$ and no 3 points are collinear

    => narrowest corridor - contains exactly 3 points

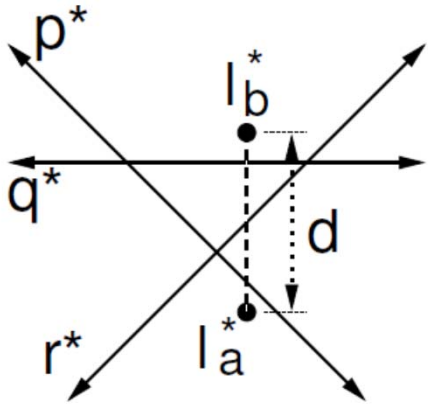
    - has width > 0

    vertical
  - No 2 points have the same x coordinate (avoid I duals)

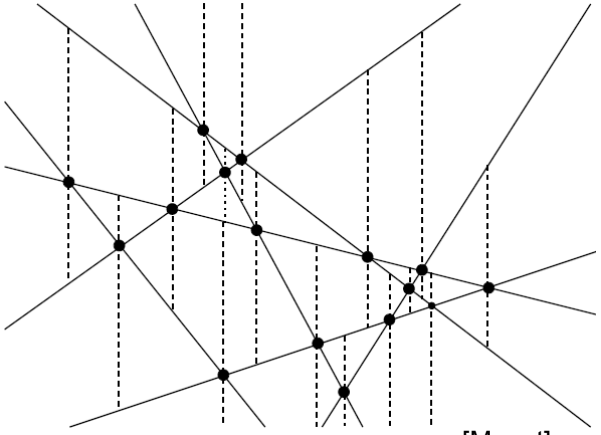
b) Minimum k-corridor



Primal Plane

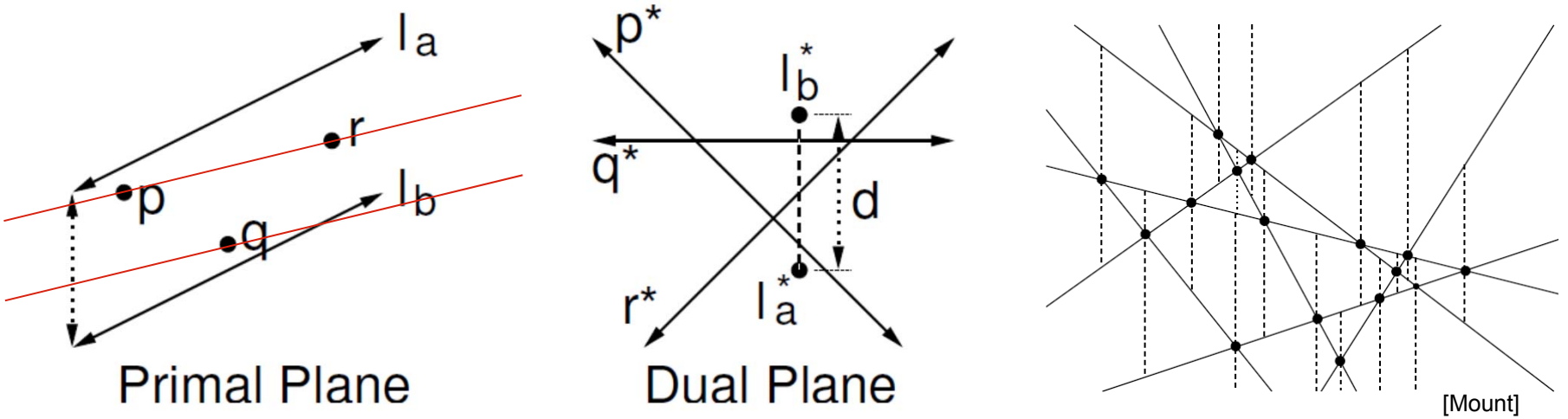


Dual Plane


[Mount]

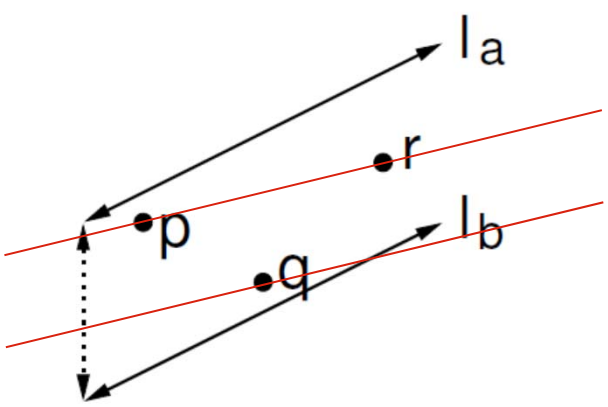
- Vertical distance of l_a,l_b = (-) distance of l_a^*,l_b^*
- Nearest lines – one passes 2 vertices, e.g., *p* & *r*
- In dual plane are represented as intersection p*× r*
- Find nearest 3-stabber similarly as trapezoidal map
- O(n^2) time and O(n) space – topological line sweep

b) Minimum k-corridor

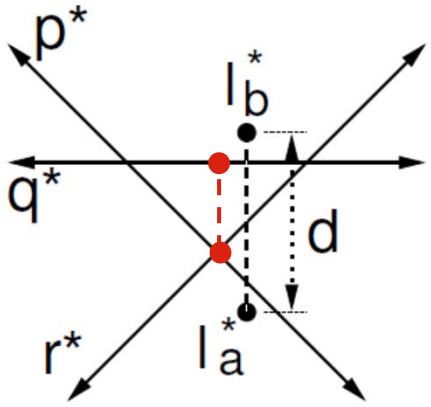


- Vertical distance of l_a, l_b = (-) distance of l_a^*, l_b^*
- Nearest lines – one passes 2 vertices, e.g., *p* & *r*
- In dual plane are represented as intersection $p^* \times r^*$
- Find nearest 3-stabber similarly as trapezoidal map
- $O(n^2)$ time and $O(n)$ space – topological line sweep

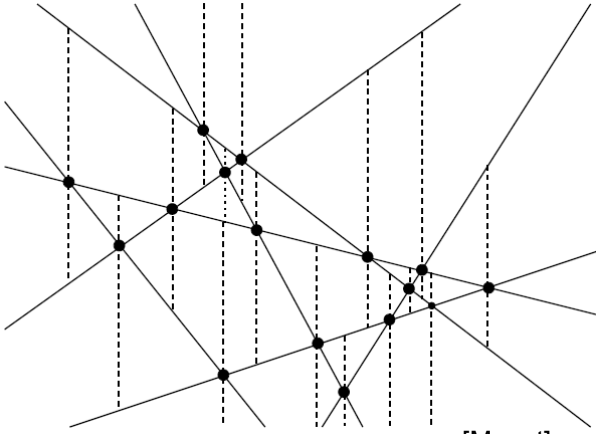
b) Minimum k-corridor



Primal Plane



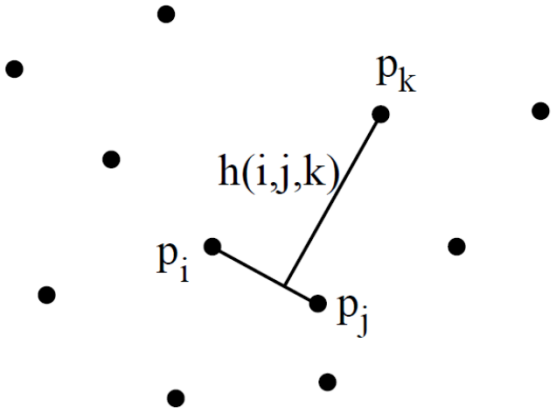
Dual Plane

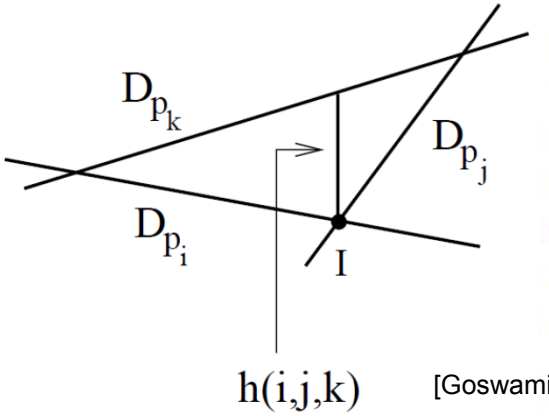

[Mount]

- Vertical distance of l_a,l_b = (-) distance of l_a*,l_b*
- Nearest lines – one passes 2 vertices, e.g., *p* & *r*
- In dual plane are represented as intersection p*× r*
- Find nearest 3-stabber similarly as trapezoidal map
- O(n^2) time and O(n) space – topological line sweep

c) Minimum area triangle [Goswami]

- Given a set of *n* points in the plane, determine the minimum area triangle whose vertices are selected from these points
- Construct “trapezoids” as in the nearest corridor
- Minimize perpendicular distances (converted from vertical) multiplied by the distance from p_i to p_j





[Goswami]

d) Sorting all angular sequences – naïve

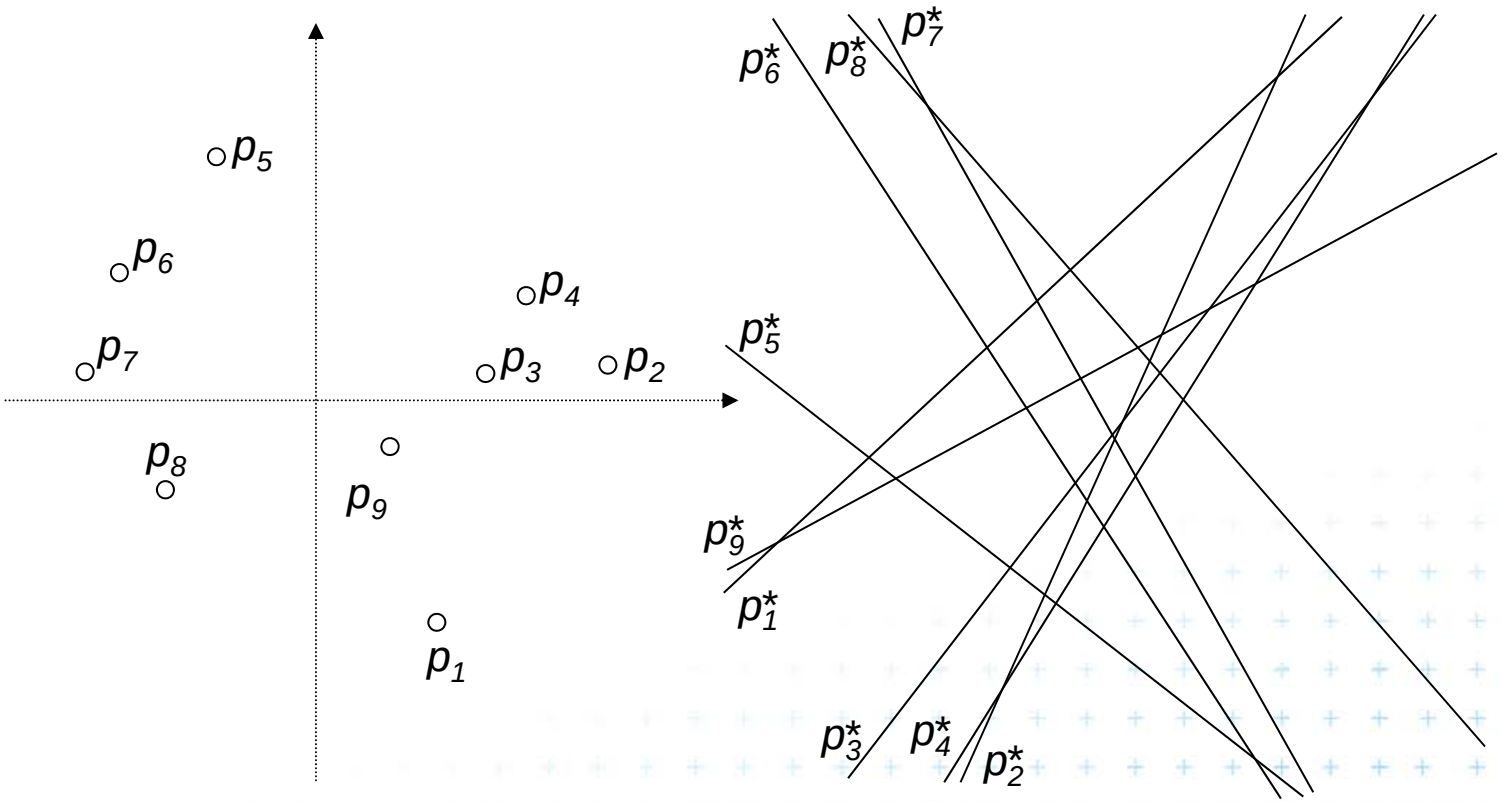
- Natural application of duality and arrangements
- Important for visibility graph computation
- Set of n points in the plane
- For each point perform an CCW angular sweep
- Naïve: for each point compute angles to remaining $n - 1$ points and sort them
- => $O(n \log n)$ time per point
- $O(n^2 \log n)$ time overall
- Arrangements can get rid of O(log n) factor

d) Sorting all angular sequences – optimal

- For point p_i
  - Dual of point p_i is line p_i*
  - Line p_i* intersects other dual lines in order of slope
    (angles from -90° to 90°) (180°)
  - We need order of angles around p_i
    (angles from -90° to 270°) (360°)

  - Split points in primal plane by vertical line through p_i
  - First, report intersections of points right of p_i
  - Second, report the intersections of points left of p_i
  - Once arrangement is constructed:
    $O(n)$ time for point, $O(n^2)$ time for all n points

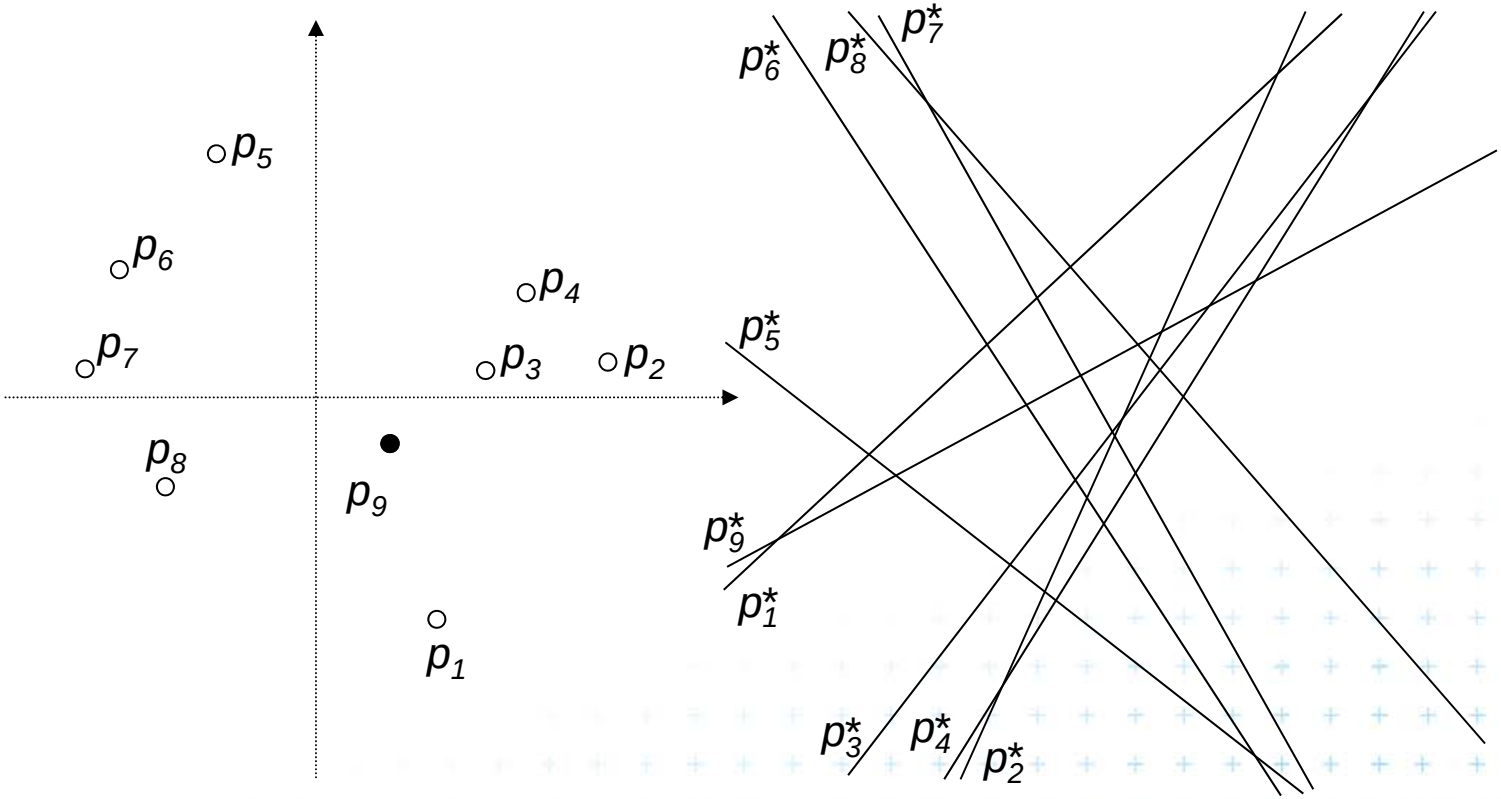
d) Angular sequence around p_9



In primal plane

In dual plane

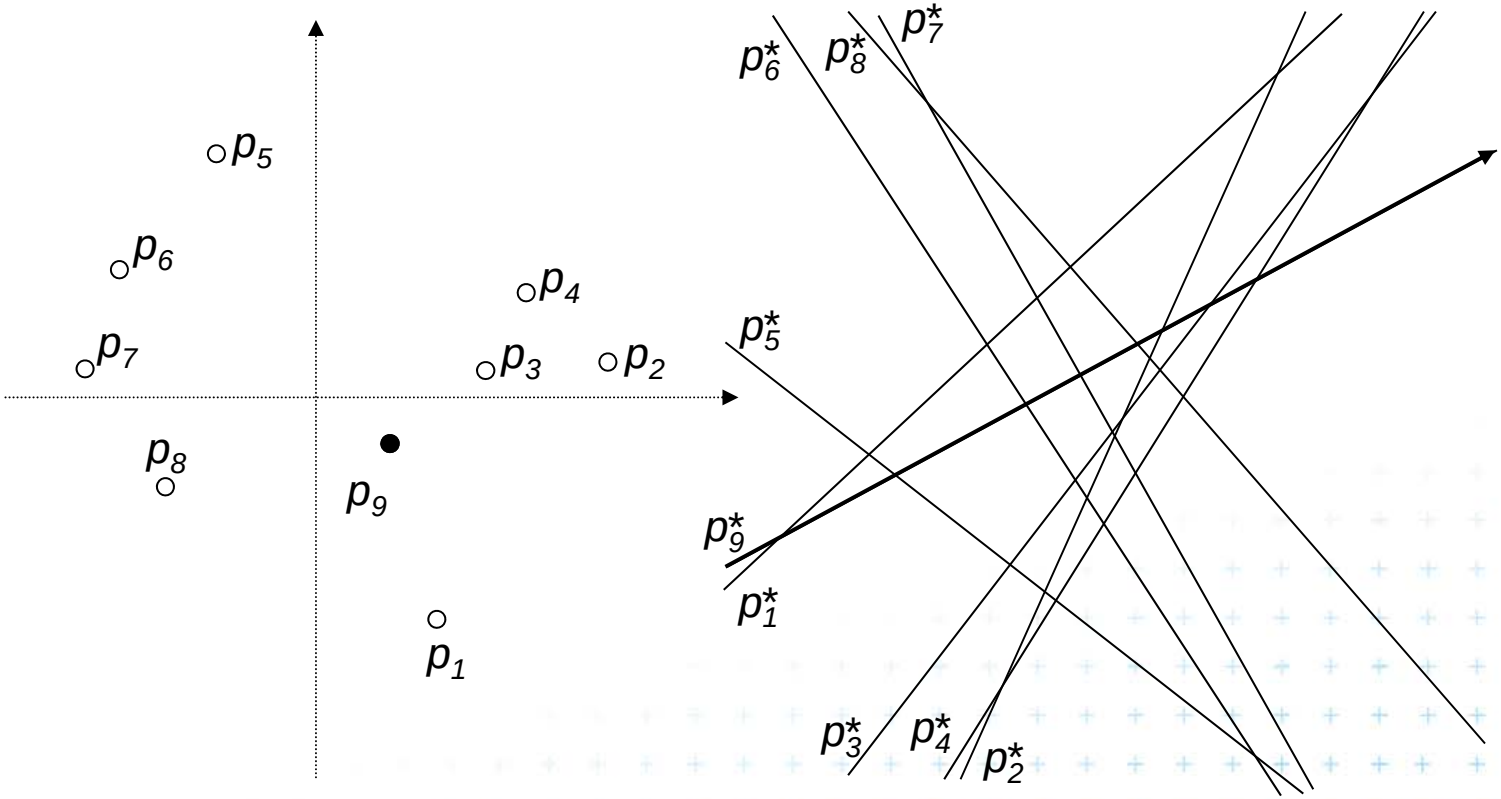
d) Angular sequence around p_9



In primal plane

In dual plane

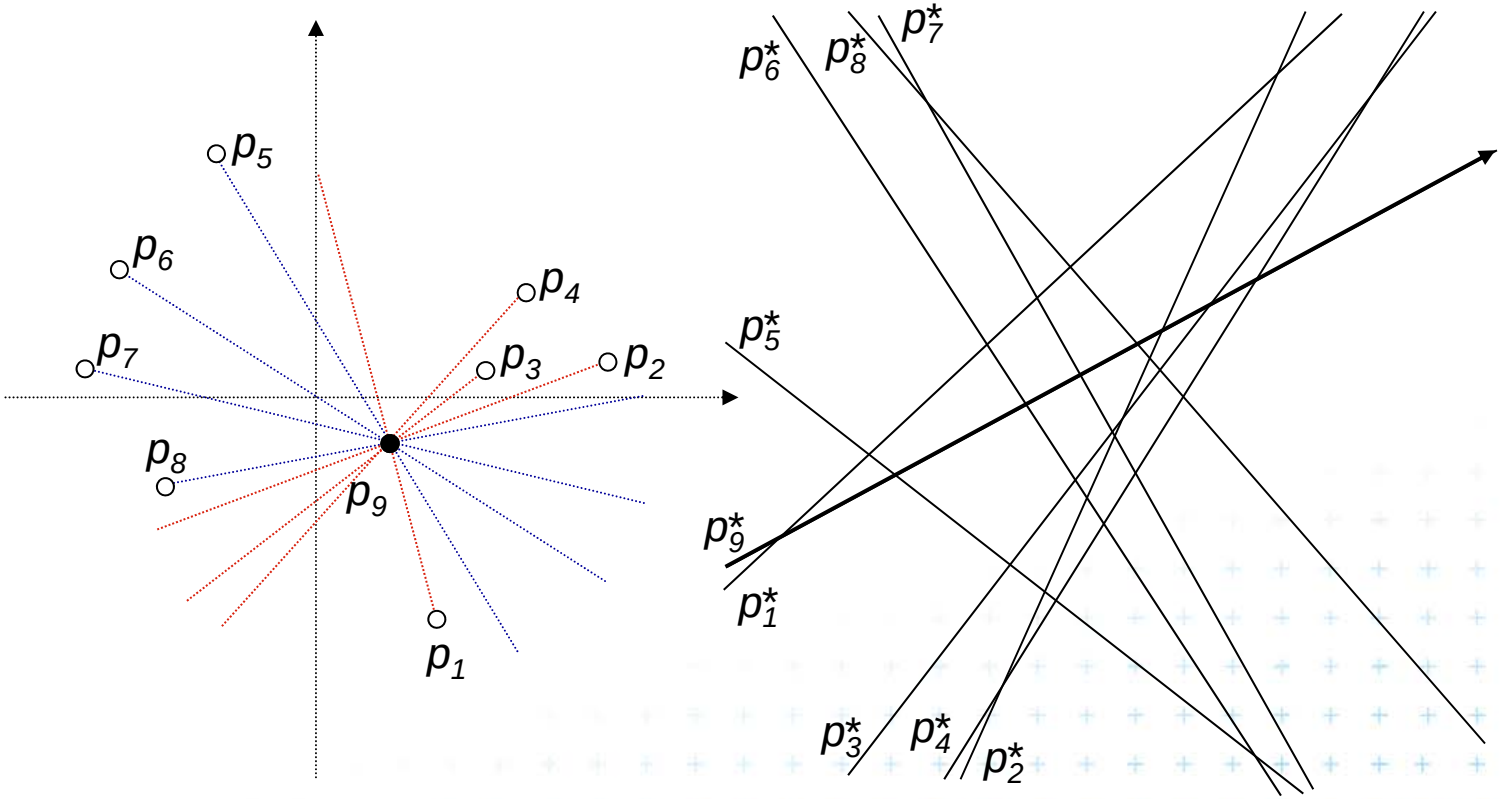
d) Angular sequence around p_9



In primal plane

In dual plane

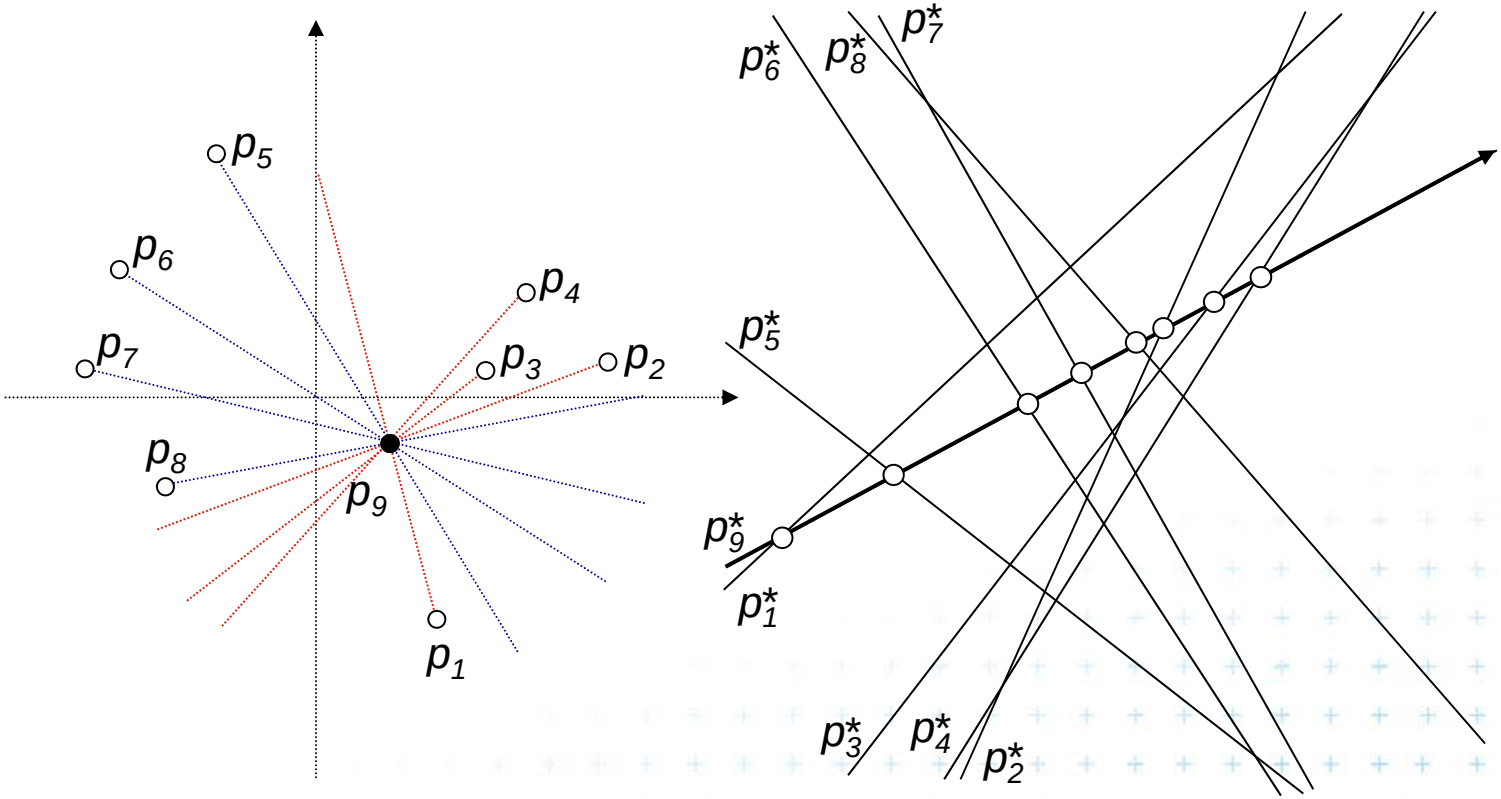
d) Angular sequence around p_9



In primal plane

In dual plane

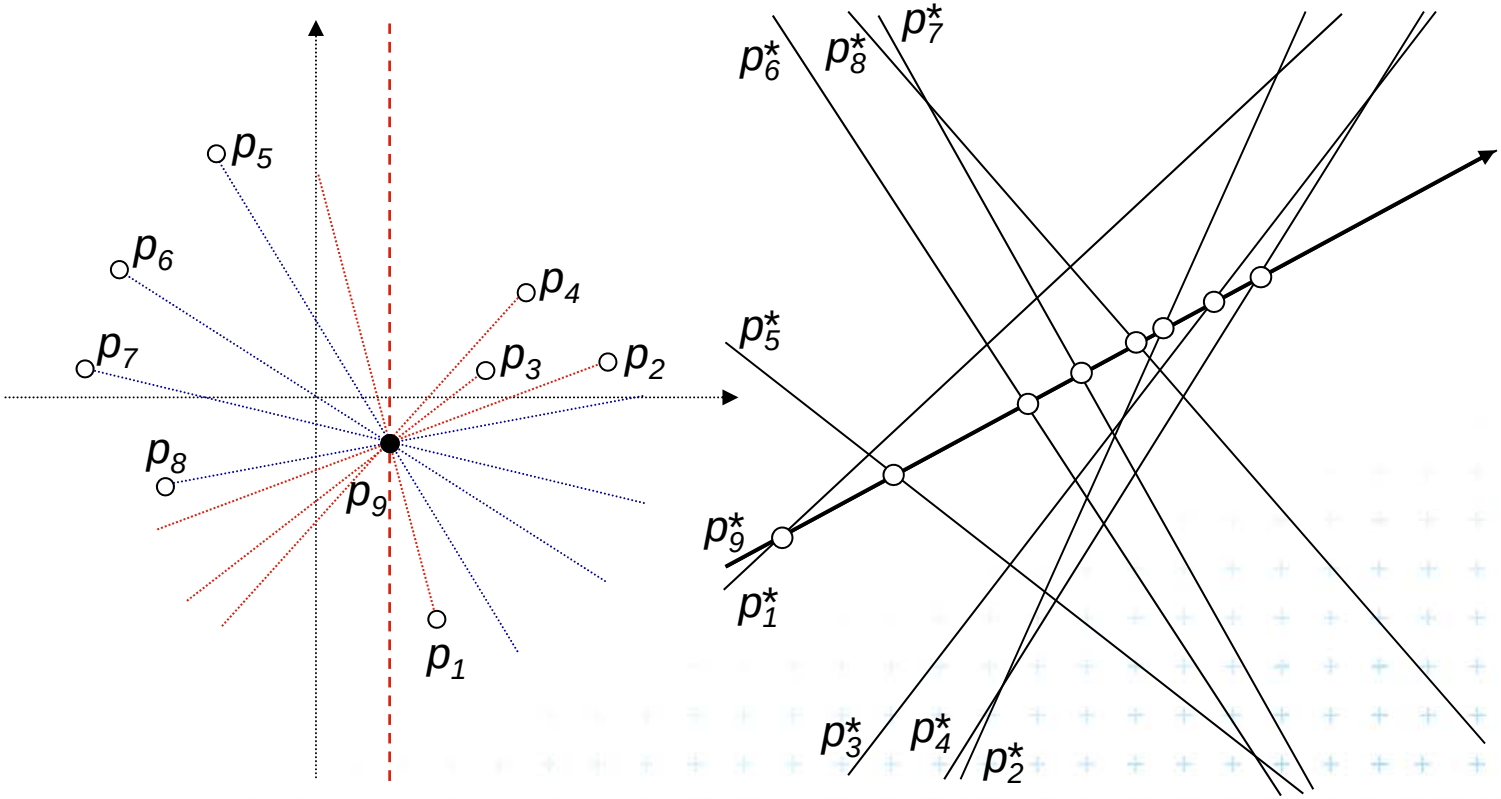
d) Angular sequence around p_9



In primal plane

In dual plane

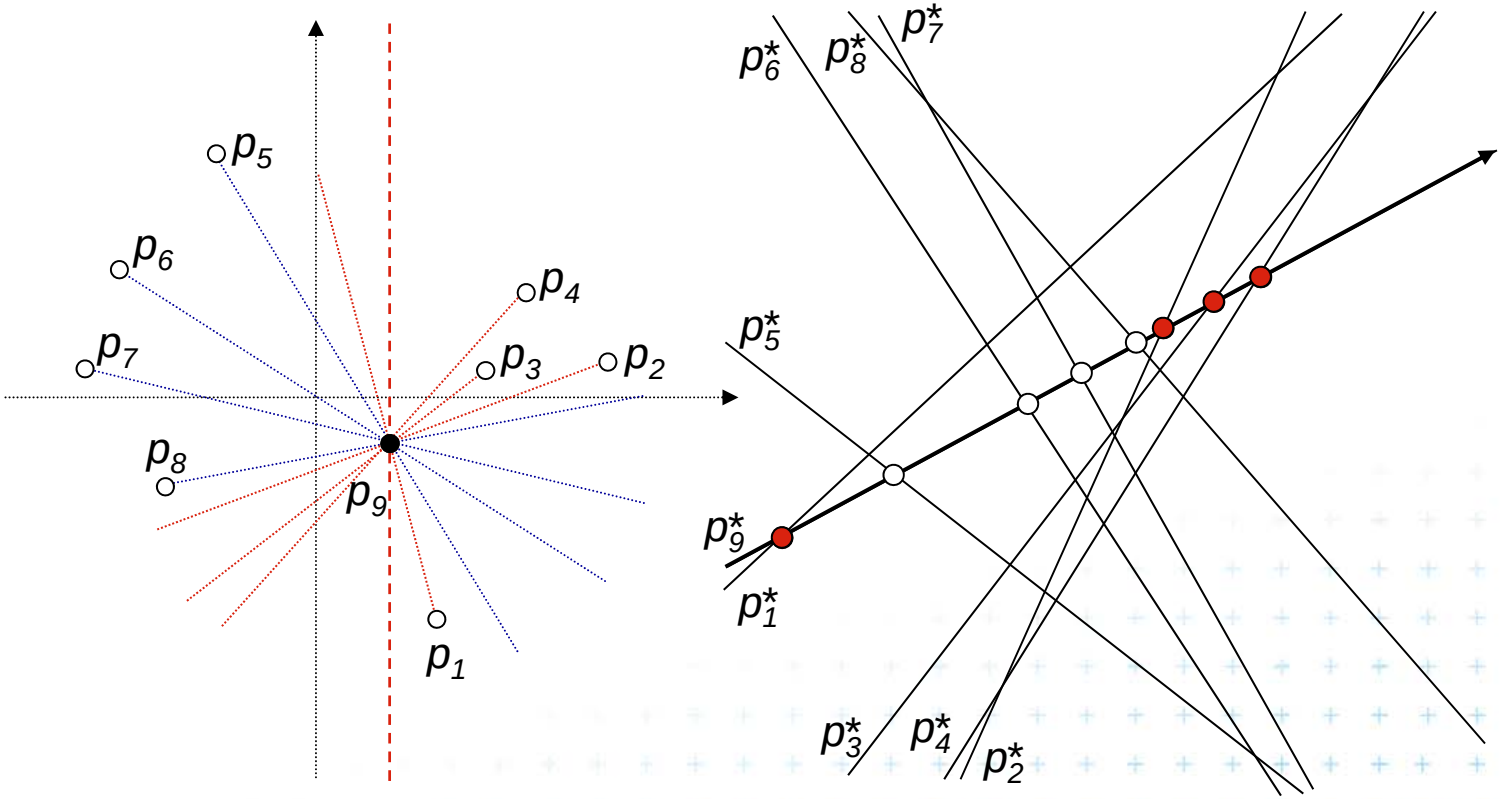
d) Angular sequence around p_9



In primal plane

In dual plane

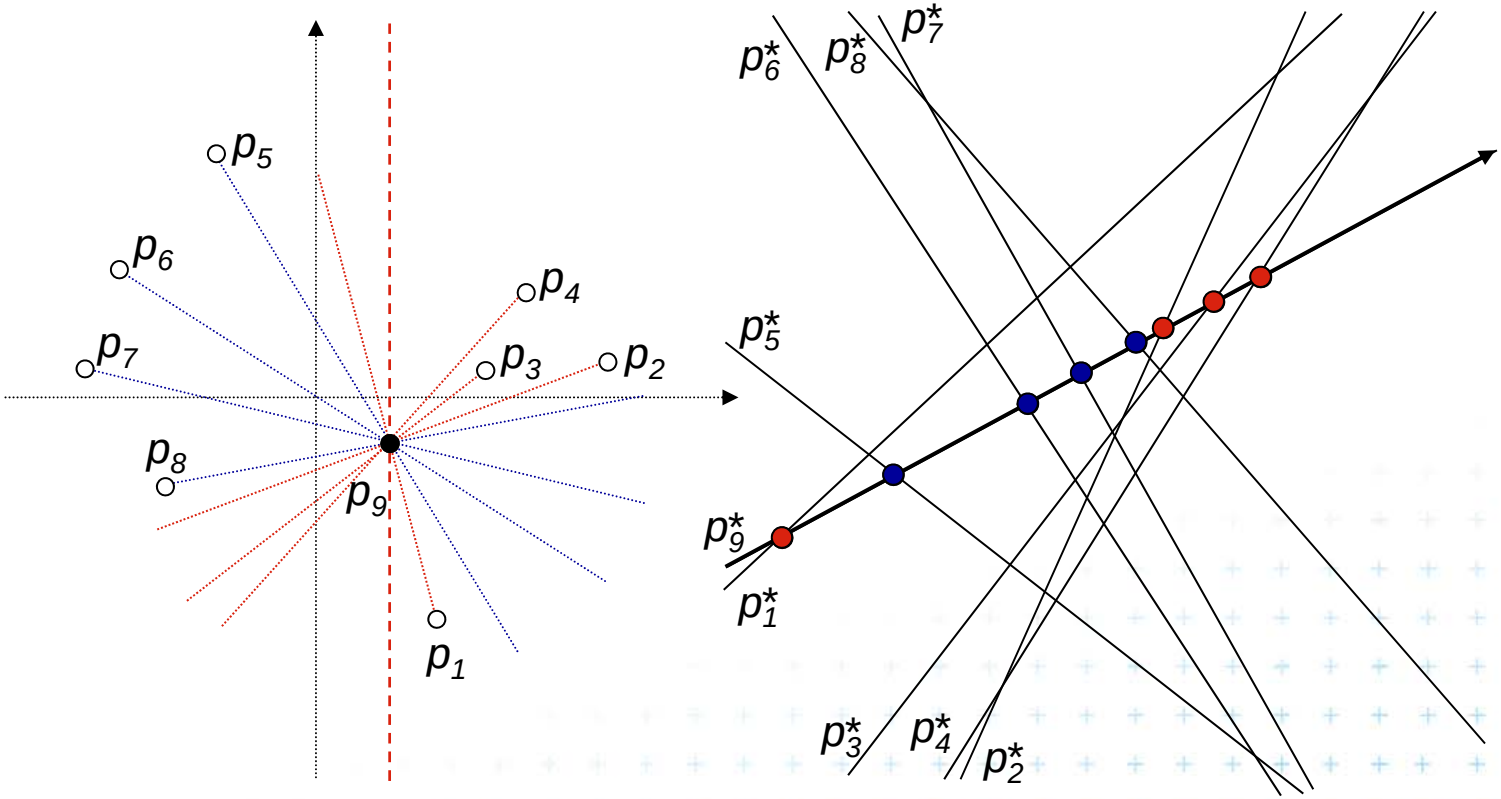
d) Angular sequence around p_9



In primal plane

In dual plane

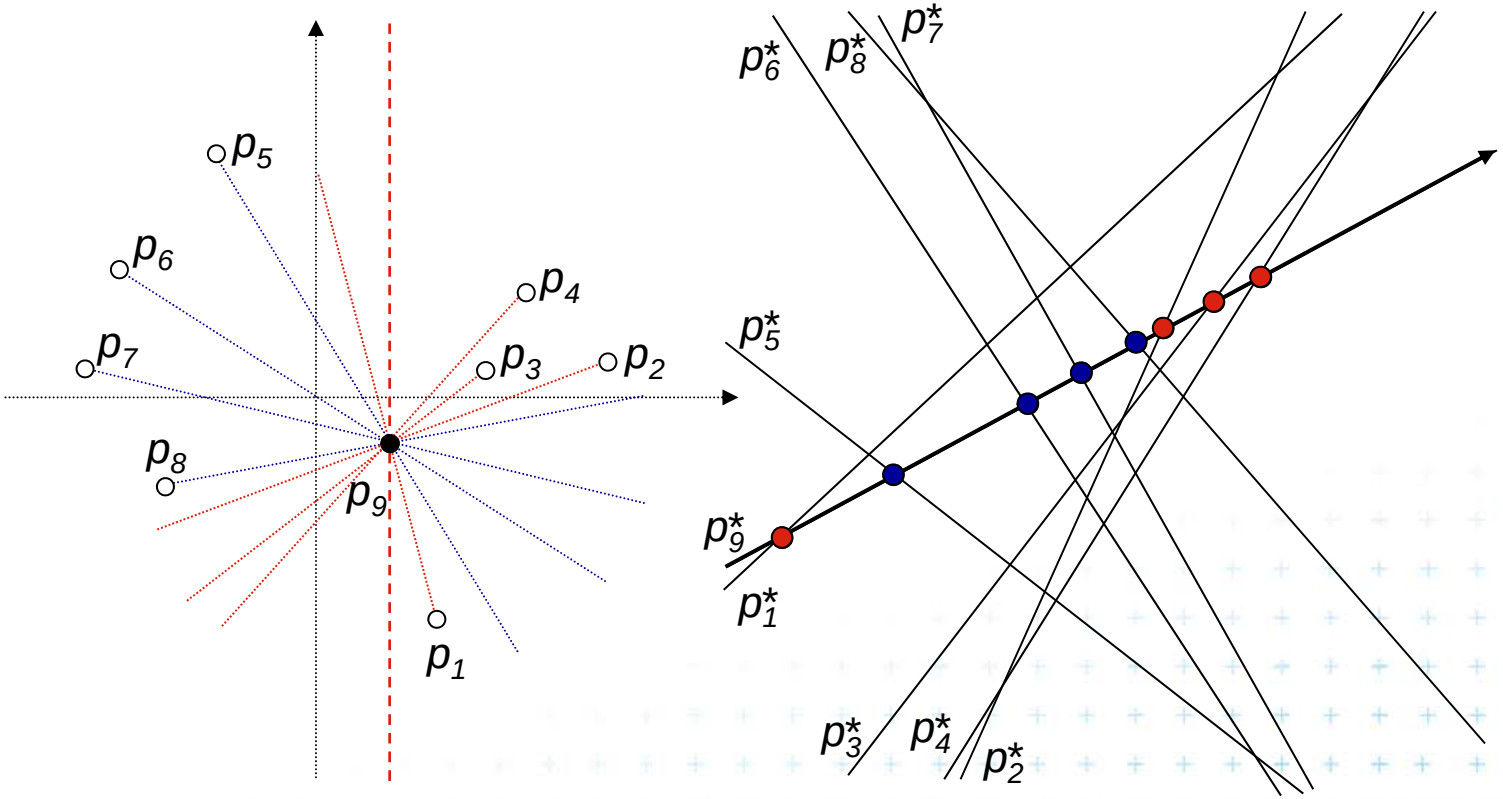
d) Angular sequence around p_9



In primal plane

In dual plane

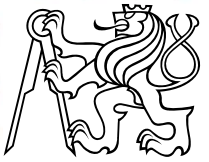
d) Angular sequence around p_9



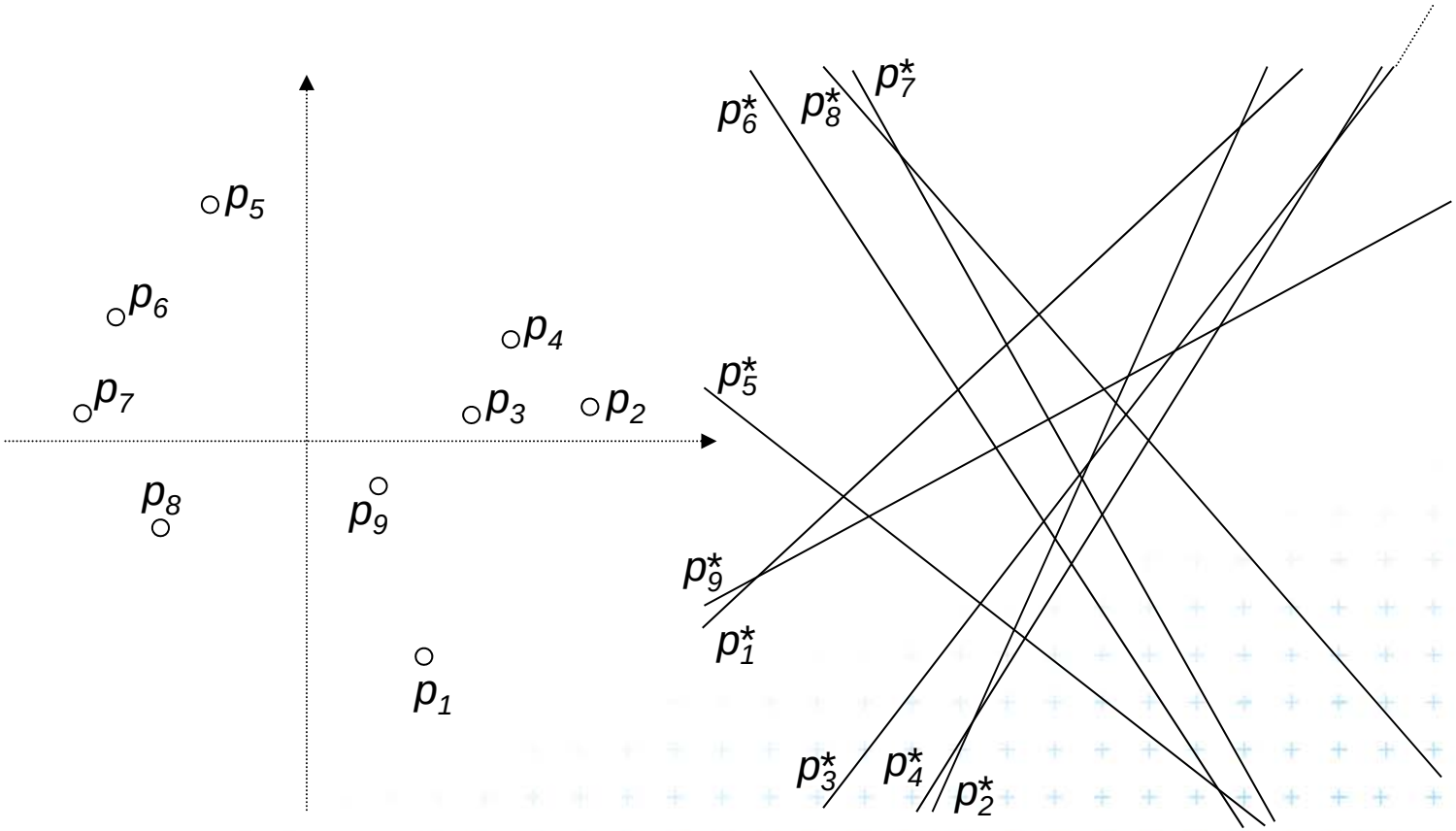
In primal plane

In dual plane

Point order around p_9 : p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8



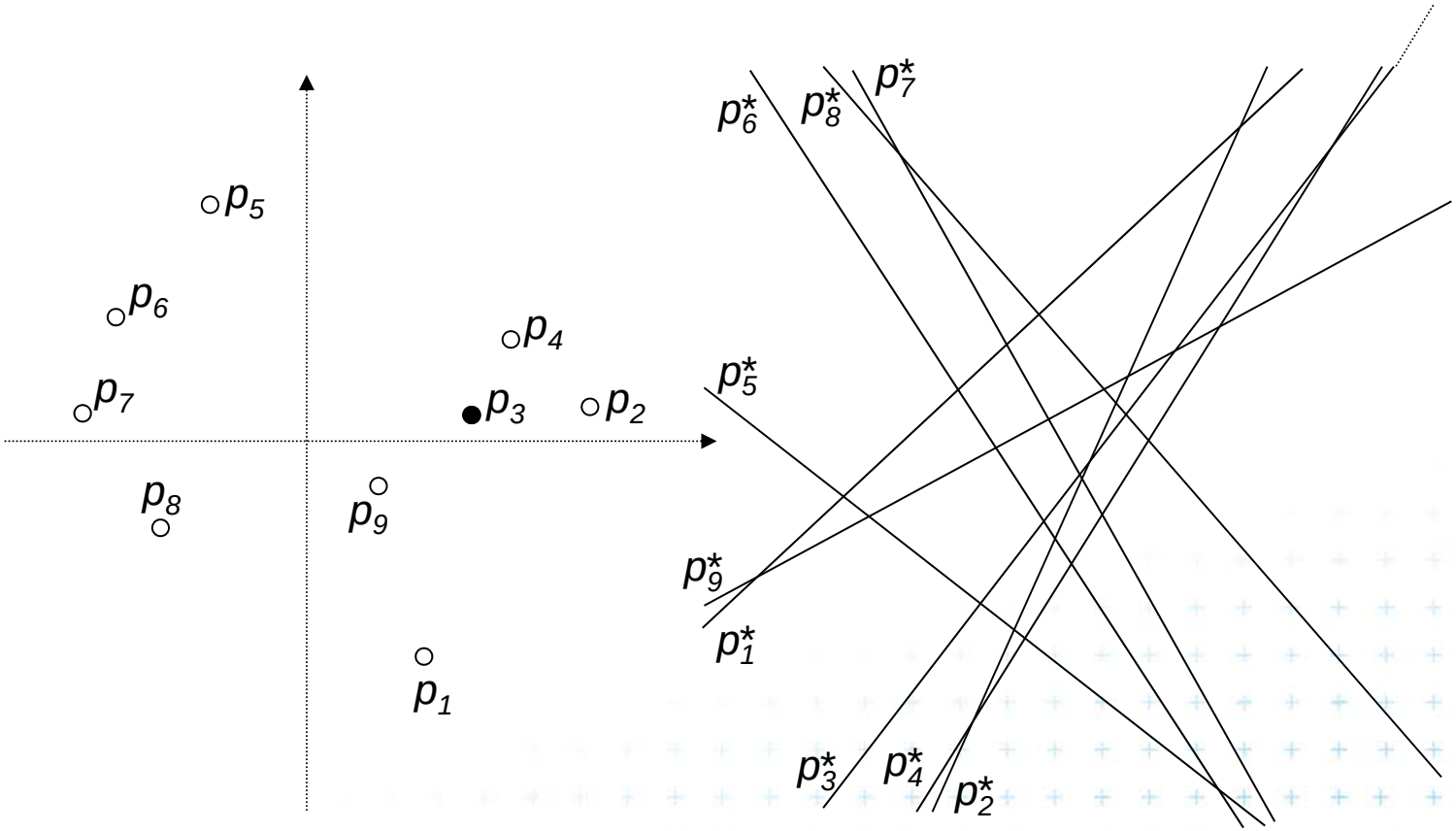
d) Angular sequences around p_3



In primal plane

In dual plane

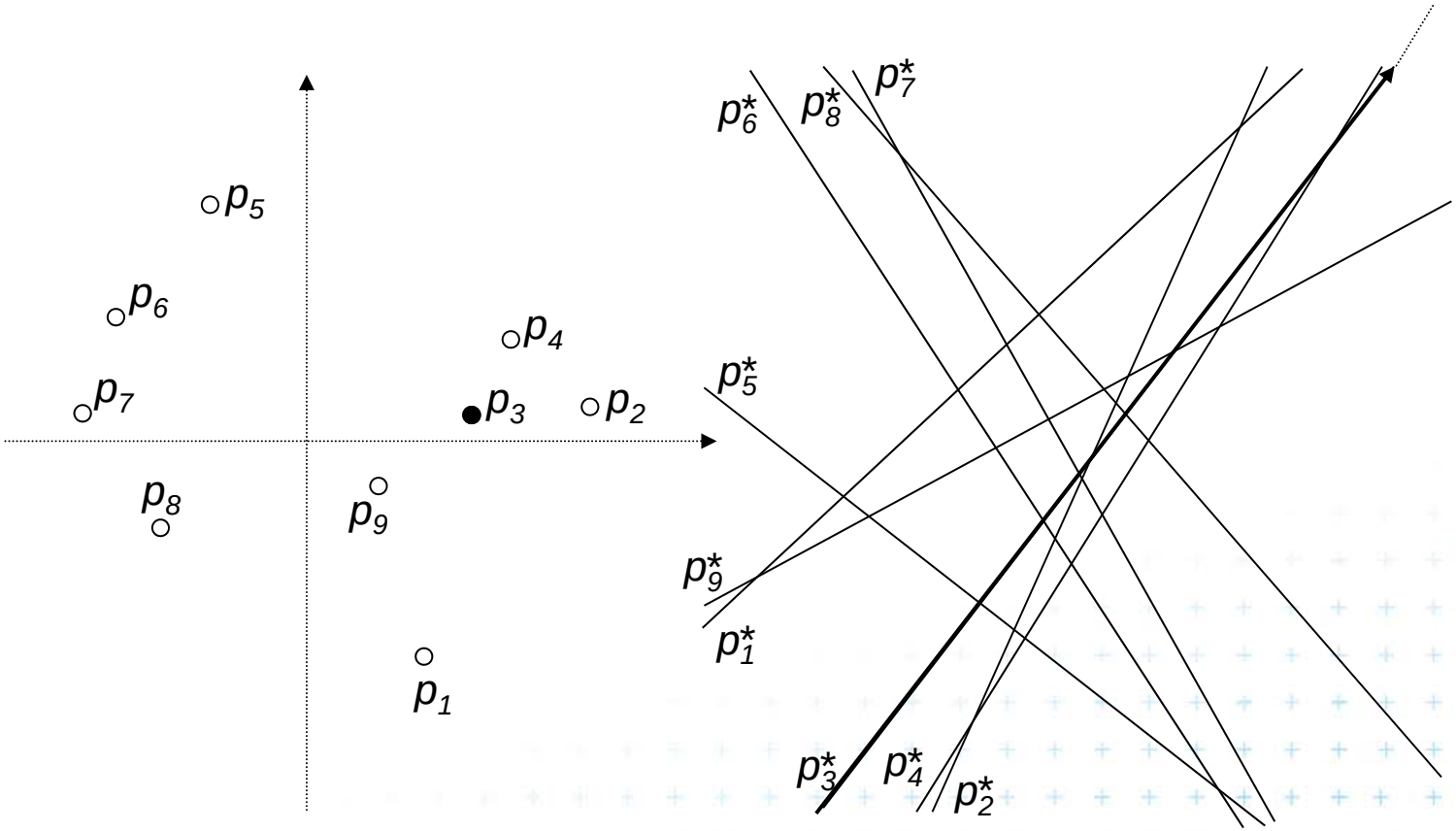
d) Angular sequences around p_3



In primal plane

In dual plane

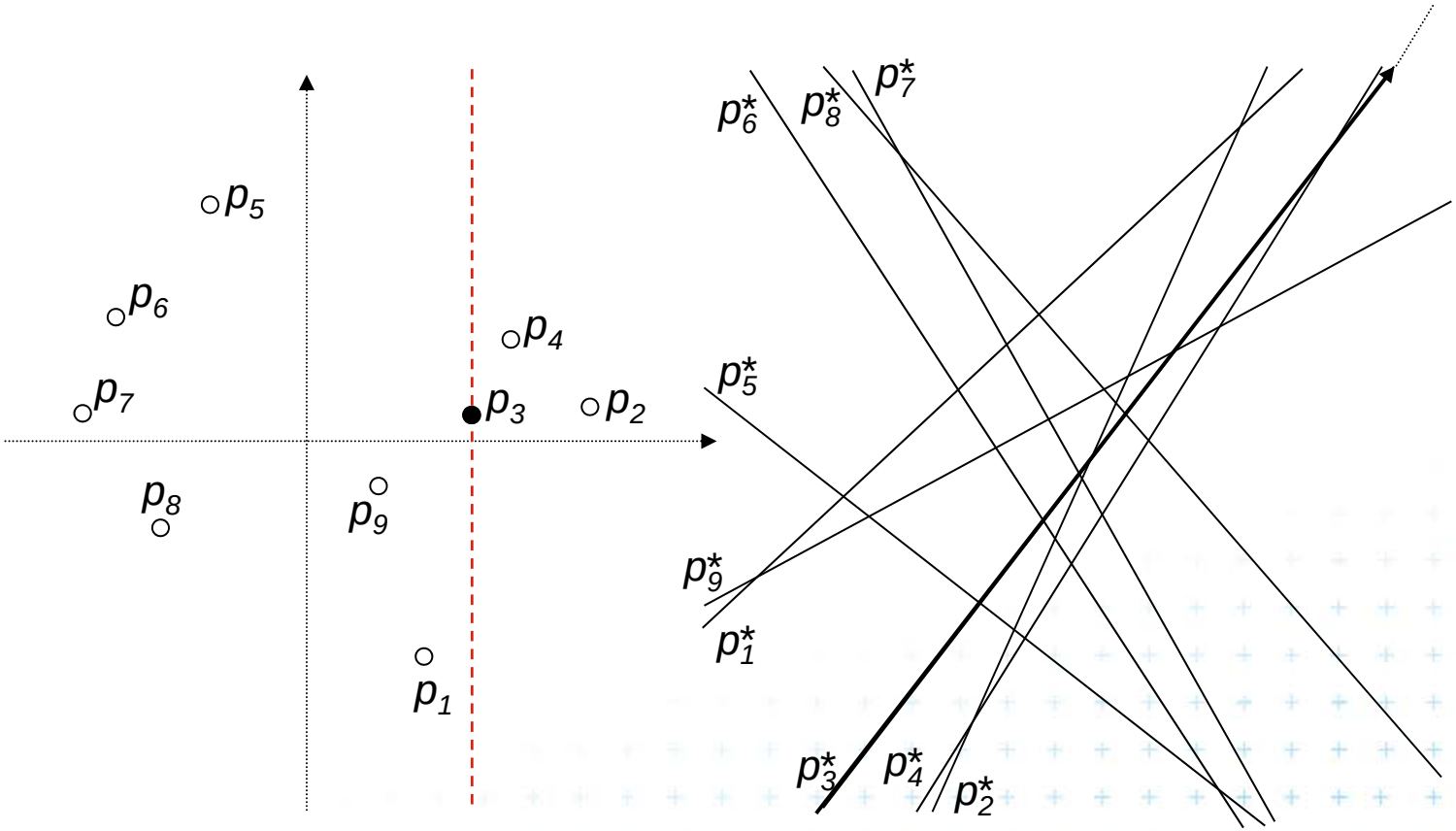
d) Angular sequences around p_3



In primal plane

In dual plane

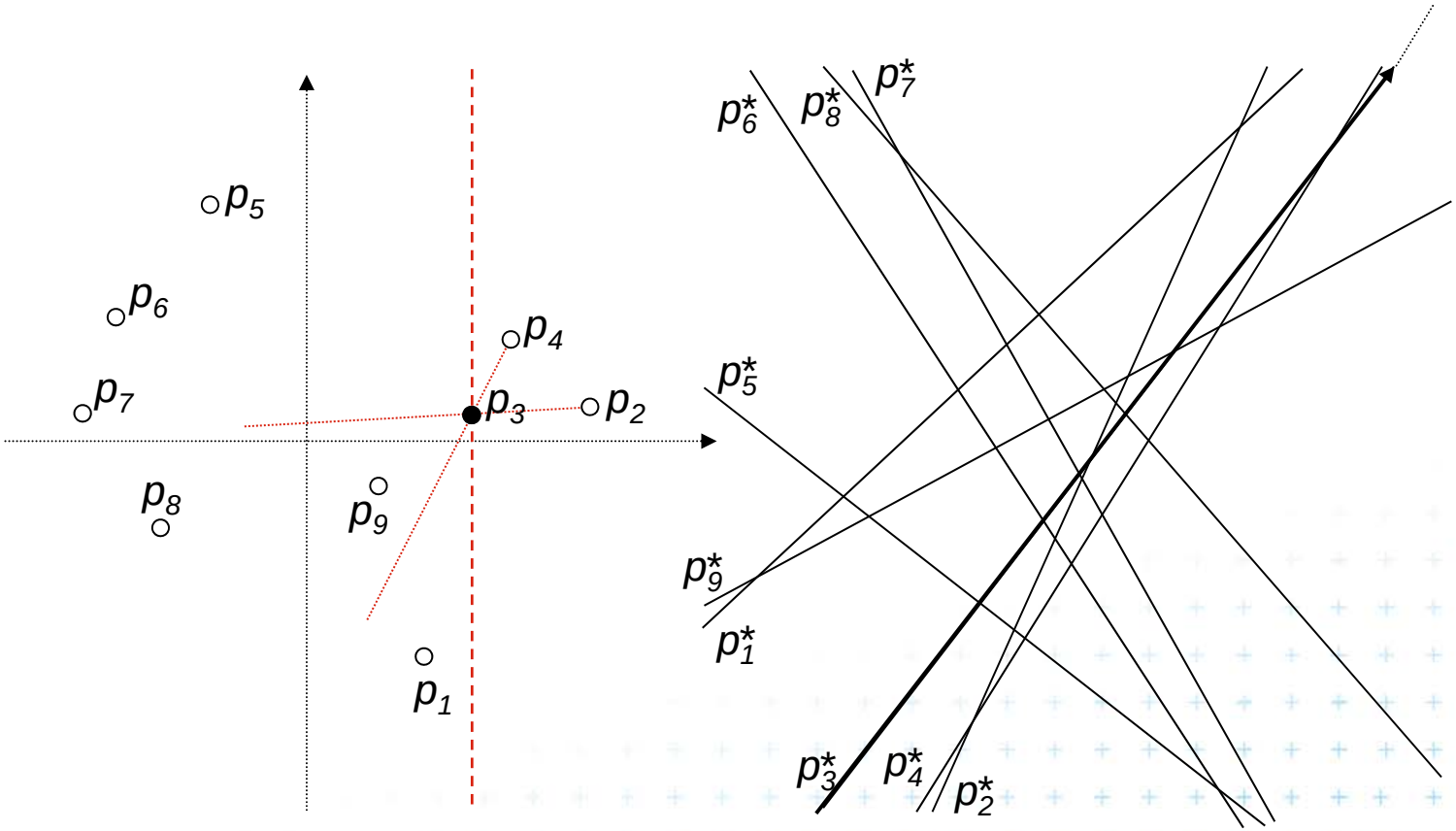
d) Angular sequences around p_3



In primal plane

In dual plane

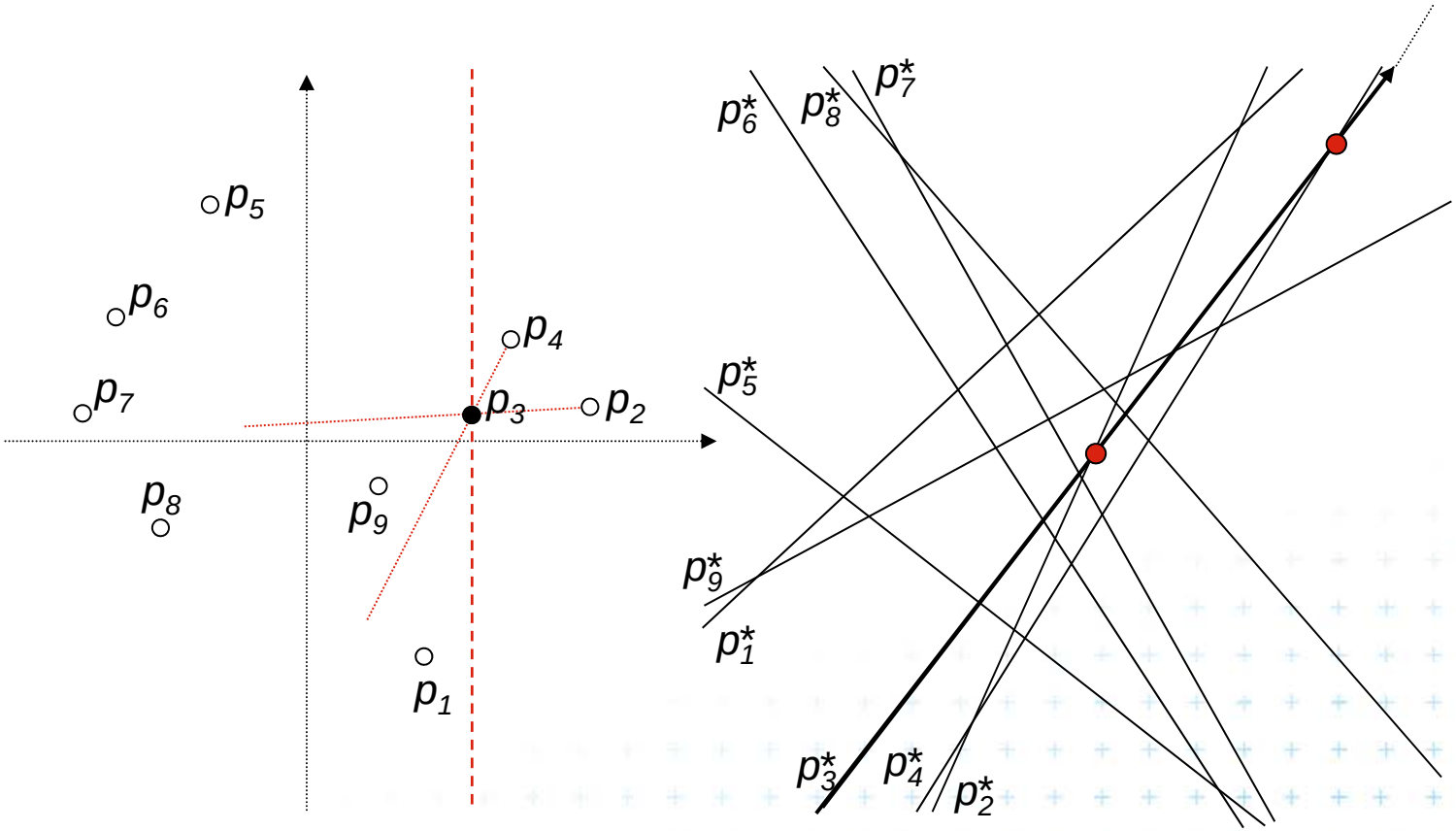
d) Angular sequences around p_3



In primal plane

In dual plane

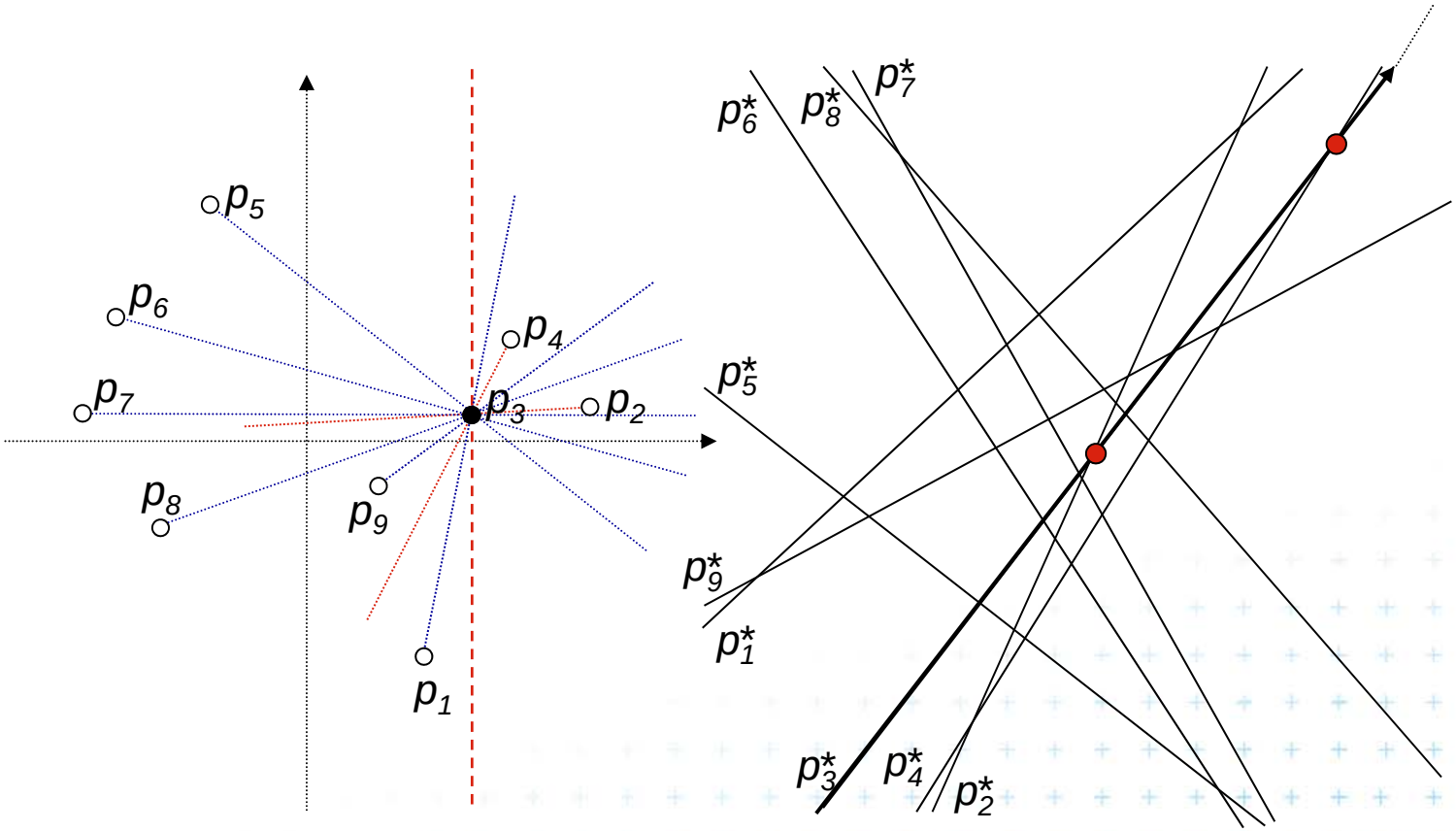
d) Angular sequences around p_3



In primal plane

In dual plane

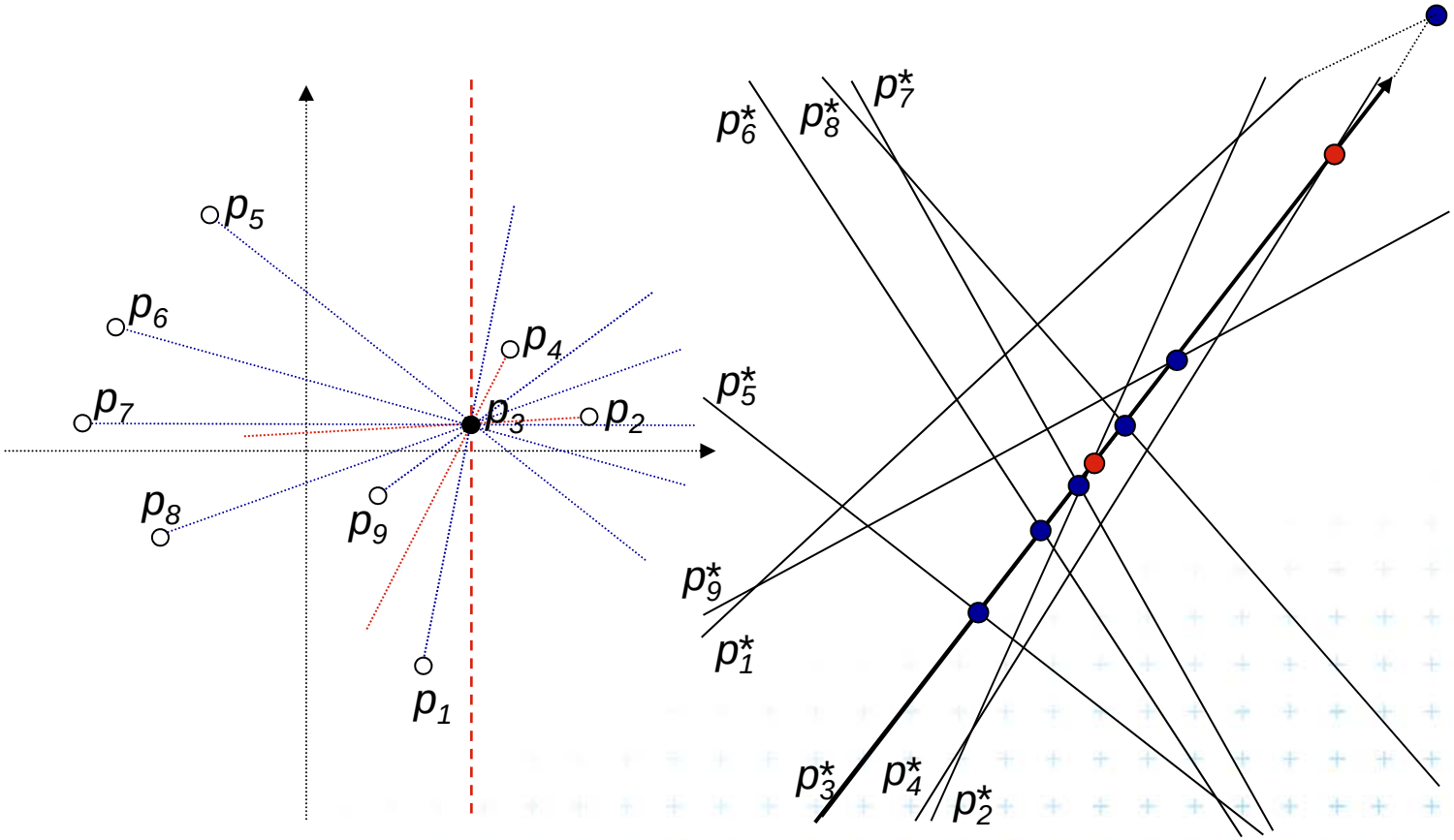
d) Angular sequences around p_3



In primal plane

In dual plane

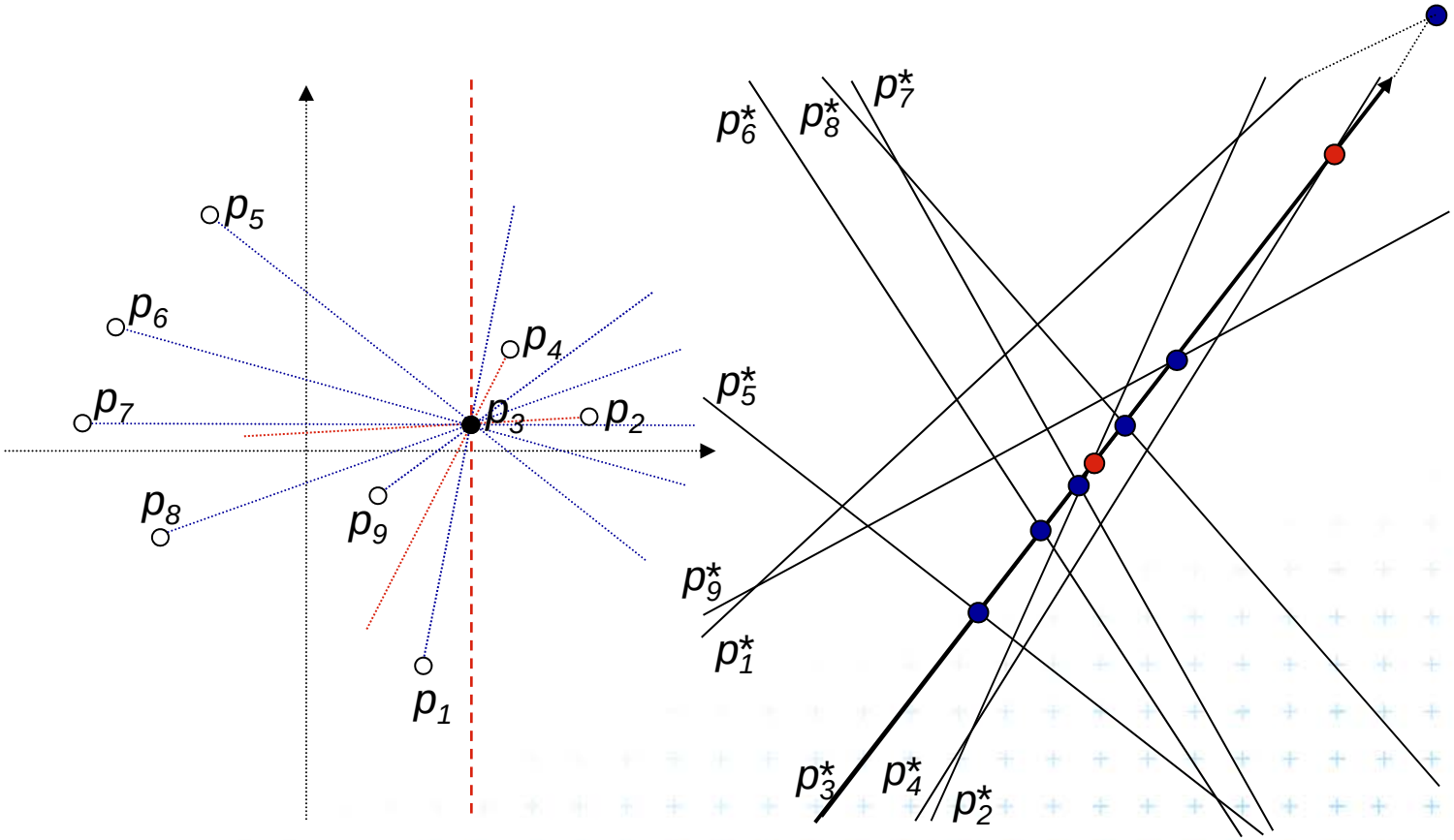
d) Angular sequences around p_3



In primal plane

In dual plane

d) Angular sequences around p_3

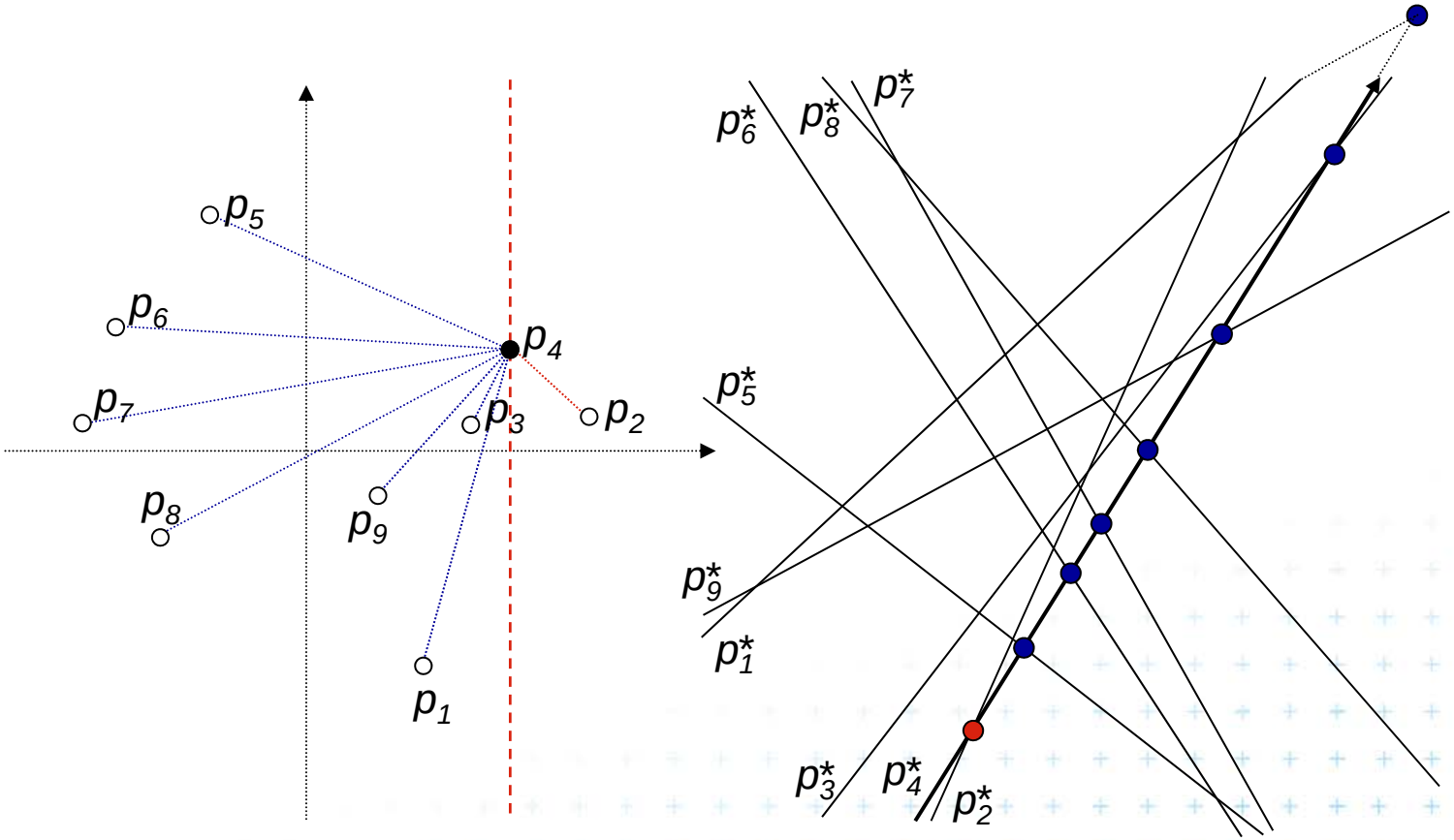


In primal plane

In dual plane

Point order around p_3 : p_2, p_4, p_5, p_6, p_7, p_8, p_3, p_1

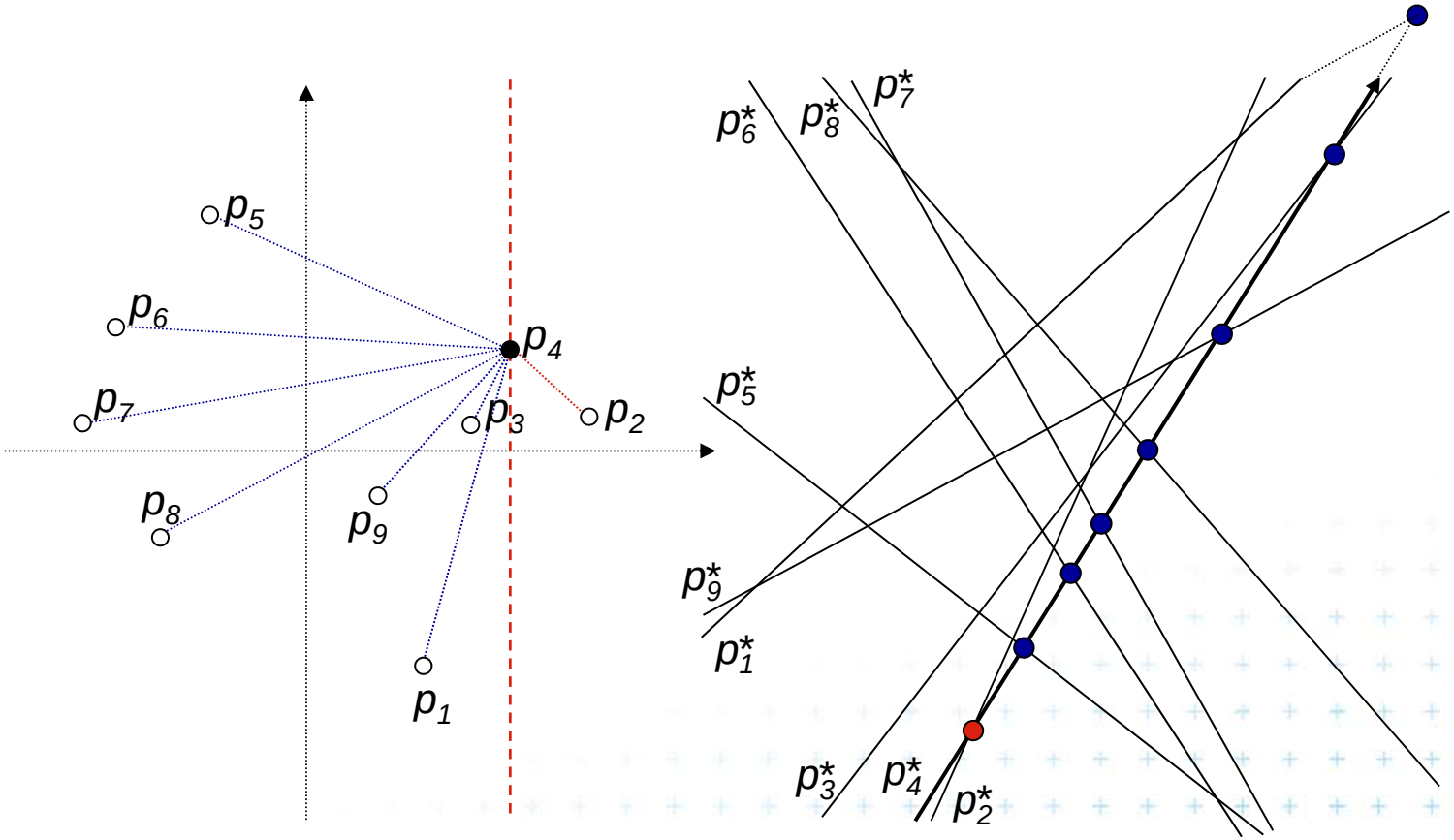
d) Angular sequences around p_4



In primal plane

In dual plane

d) Angular sequences around p_4



In primal plane

In dual plane

Point order around p_4 : p_2, p_5, p_6, p_7, p_8, p_9, p_3, p_1

e) More applications of line arrangement

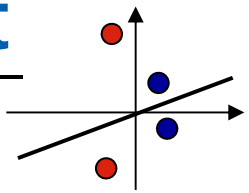
Visibility graph

Given a set of *n* non-intersecting line segments, compute the *visibility graph*, whose vertices are the endpoints of the segments, and whose edges are pairs of visible endpoints
(use angular sequences).

Maximum stabbing line

Given a set of *n* line segments in the plane, compute the line that stabs (intersects) the maximum number of these line segments.

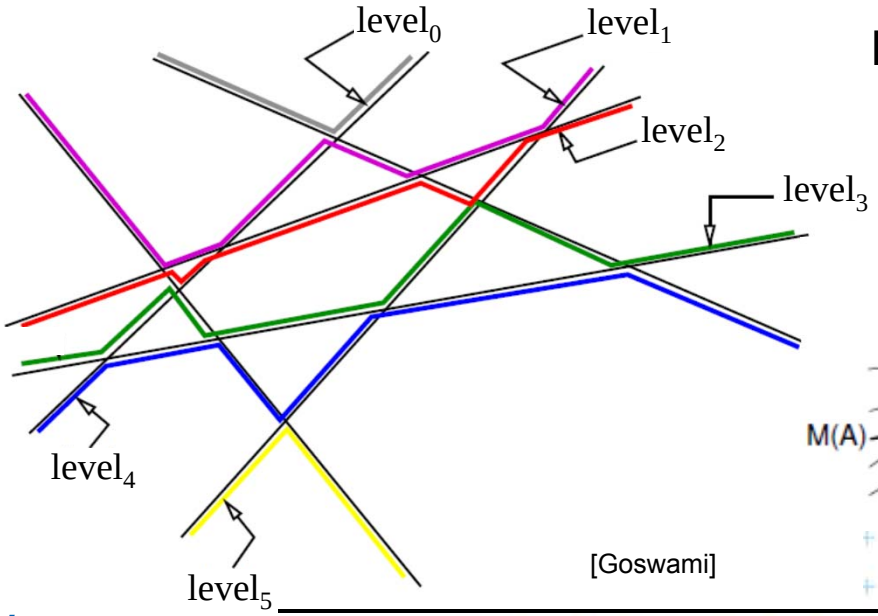
More applications of line arrangement



Ham-Sandwich cut

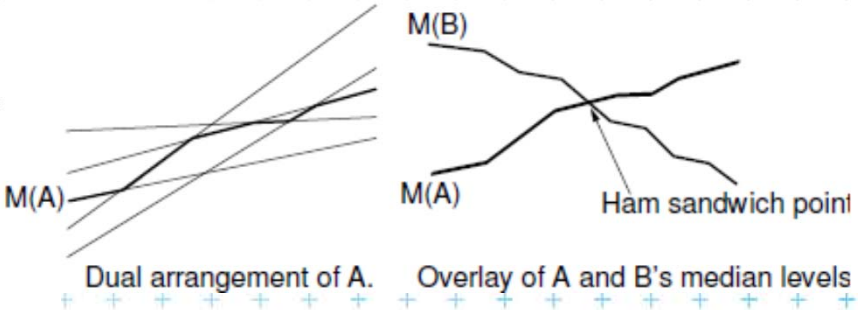
Given two sets of points, *n* red and *m* blue points compute a single line that simultaneously bisects both sets

Principle – intersect middle levels of arrangements



[Goswami]

Point at *k-th* level L_k has
at most k lines above and
at most n – k – 1 lines below



Dual arrangement of A. Overlay of A and B's median levels

[Mount]

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