DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

WINDOWING

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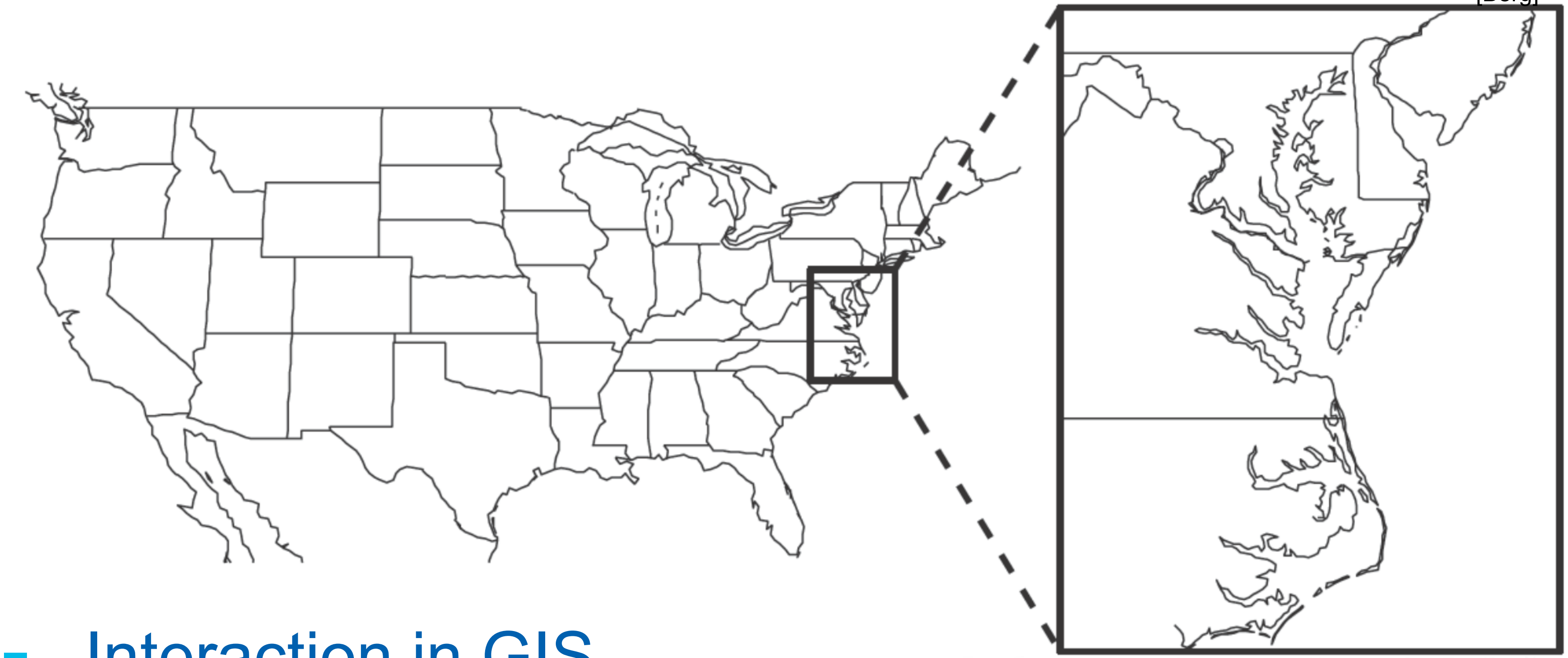
Based on [Berg], [Mount]

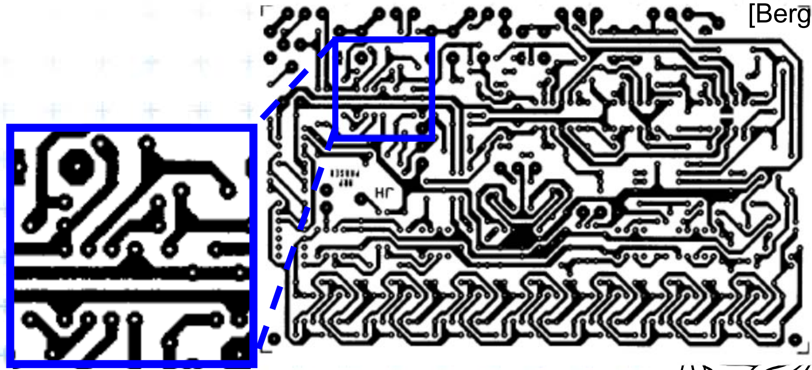
Version from 3.12.2015

Windowing queries - examples



[Berg]

- Interaction in GIS
  - Select subset by outlining
  - Zoom in and re-center
- Circuit board inspection,…



[Berg]

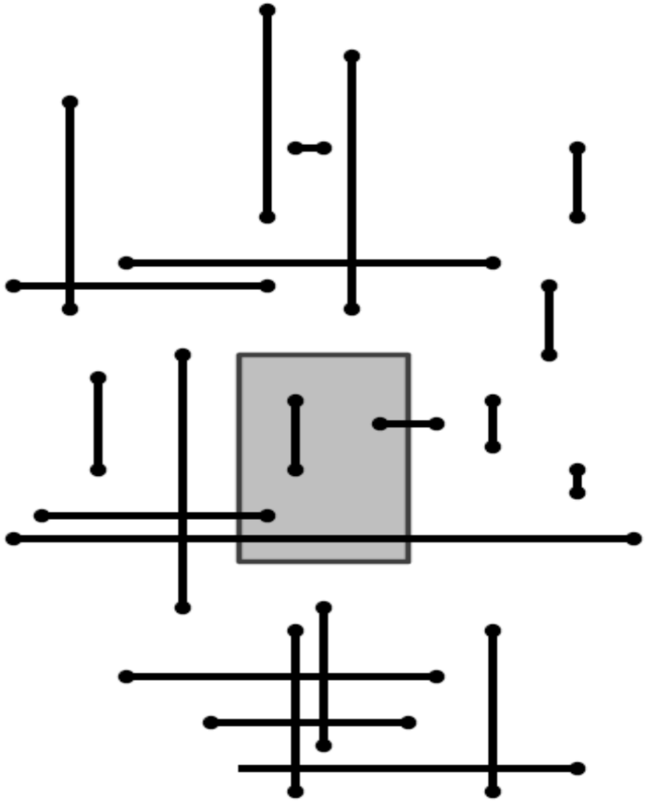
[Vakken]

Windowing versus range queries

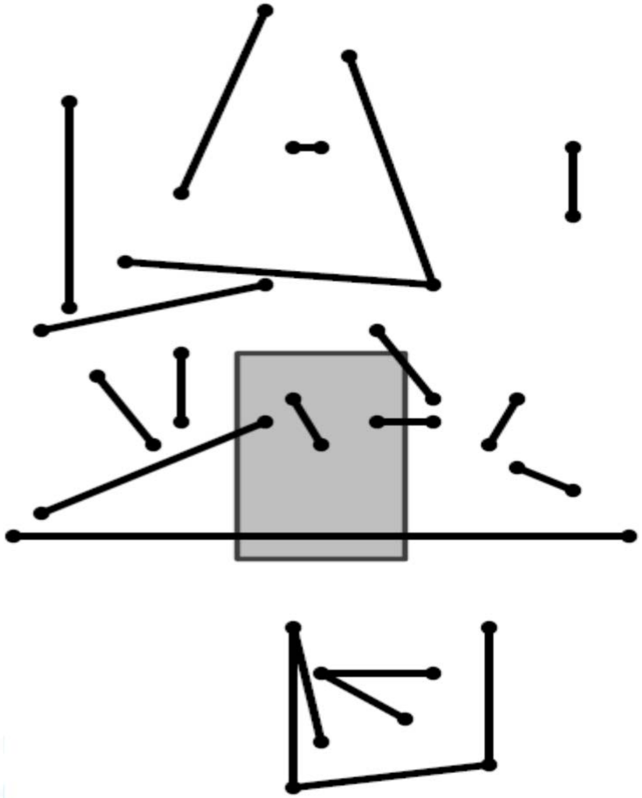
- Range queries (see range trees in Lecture 03)
  - Points
  - Often in higher dimensions
- Windowing queries
  - Line segments, curves, …
  - Usually in low dimension (2D, 3D)

- The goal for both:
  Preprocess the data into a data structure
  - so that the objects intersected by the query rectangle can be reported efficiently

Windowing queries on line segments



1. Axis parallel line segments



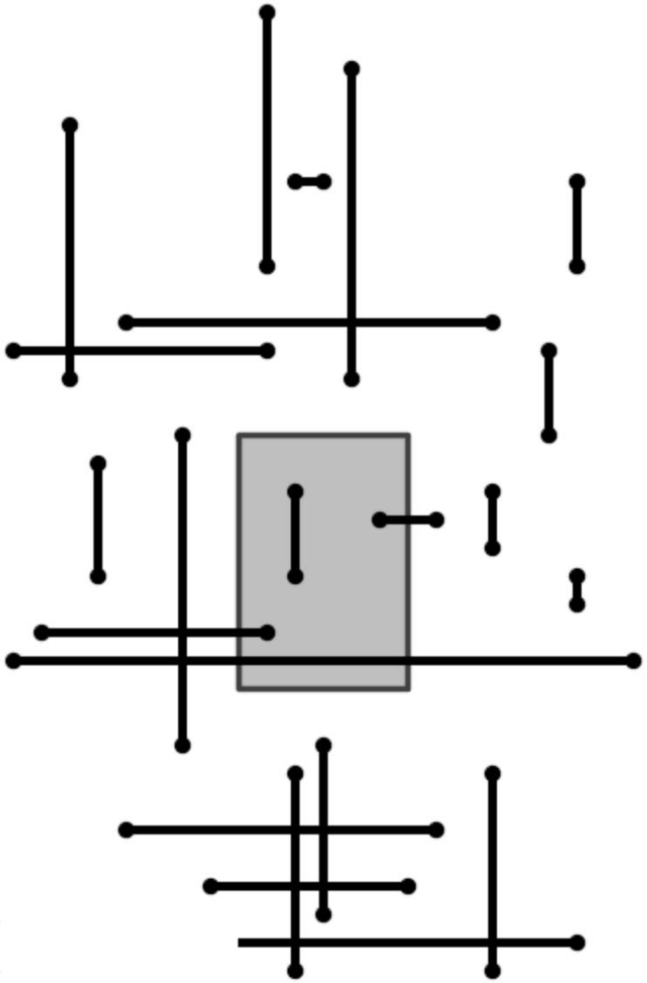
2. Arbitrary line segments (non-crossing)

[Vakken]

Talk overview

1. Windowing of axis parallel line segments in 2D (3 variants of *interval tree - IT*)
   1. Line stabbing (*IT* with *sorted lists*) lecture 9 - intersections
   2. Line segment stabbing (*IT* with *range trees*)
   3. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*

1. Windowing of axis parallel line segments

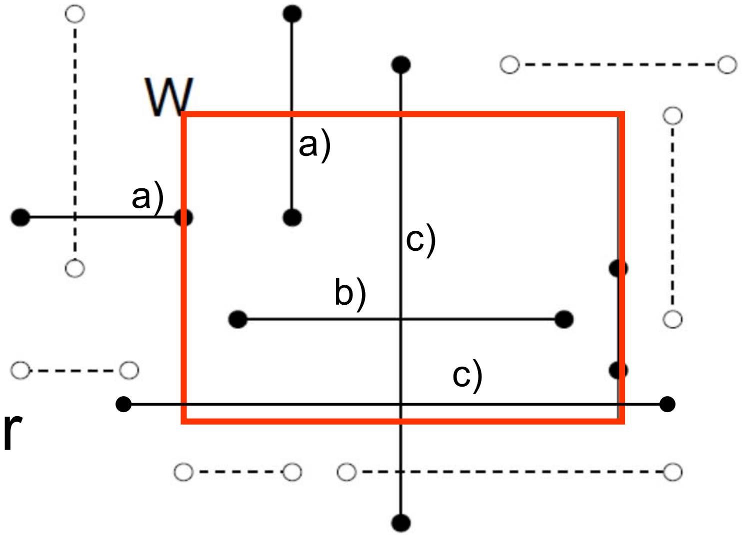


[Vakken]

1. Windowing of axis parallel line segments

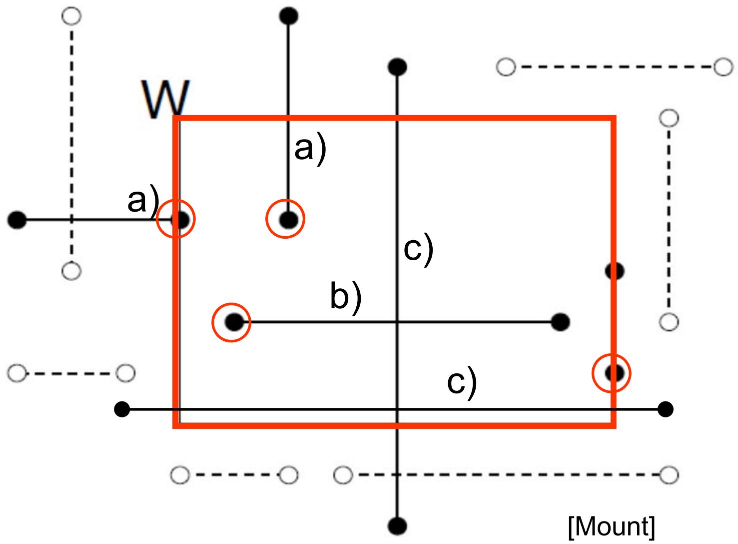
Window query

- Given
  - a set of orthogonal line segments S (preprocessed),
  - and orthogonal query rectangle $W = [\, x : x' \,] \times [\, y : y' \,]$
- Count or report all the line segments of S that intersect W
- Such segments have
  a) 1 endpoint in
  b) 2 end points in – Included
  c) no end point in – Cross over



[Mount]

Line segments with 1 or 2 points inside

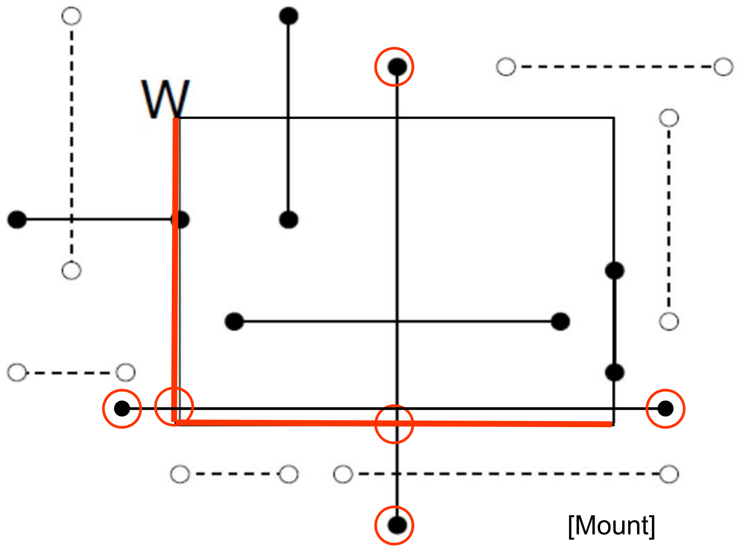


a) 1 point inside
  - Use a range tree (Lesson 3)
  - O(*n* log *n*) storage
  - O($\log^2 n + k$) query time or
  - O($\log n + k$) with fractional cascading

b) 2 points inside – as a) 1 point inside
  - Avoid reporting twice
    1. Mark segment when reported (clear after the query)
    2. When end point found, check the other end-point. Report only the leftmost or bottom endpoint

Line segments that cross over the window

c) No points inside



[Mount]

- not detected using a range tree
- Cross the boundary twice
  or
  contain one boundary edge
- It is enough to detect segments intersected by the left and bottom boundary edges (not having end point inside)
- For left boundary: Report the segments intersecting vertical query *line segment* (1/ii.)
- Let's discuss vertical query *line* first (1/i.)
- Bottom boundary is rotated 90°

Talk overview

1. Windowing of axis parallel line segments in 2D (variants of *interval tree - IT*)
   1. Line stabbing (*IT* with *sorted lists*)
   2. Line segment stabbing (*IT* with *range trees*)
   3. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*

i. Segment intersected by vertical line->1D

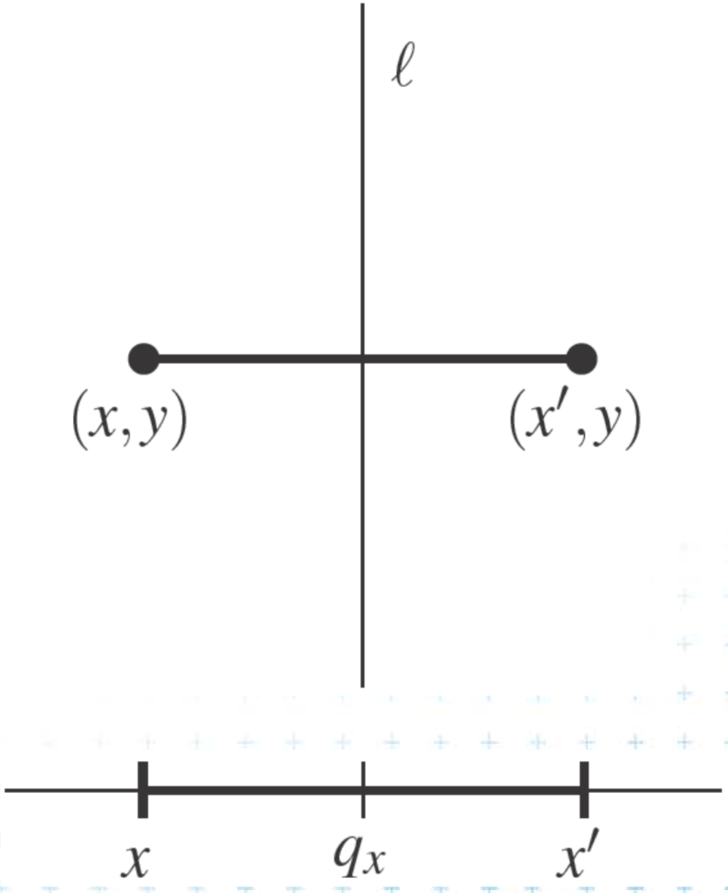
- Query line $\ell := (x=q_x)$

  Report the segments stabbed by a vertical line

  = 1 dimensional problem

  (ignore y coordinate)

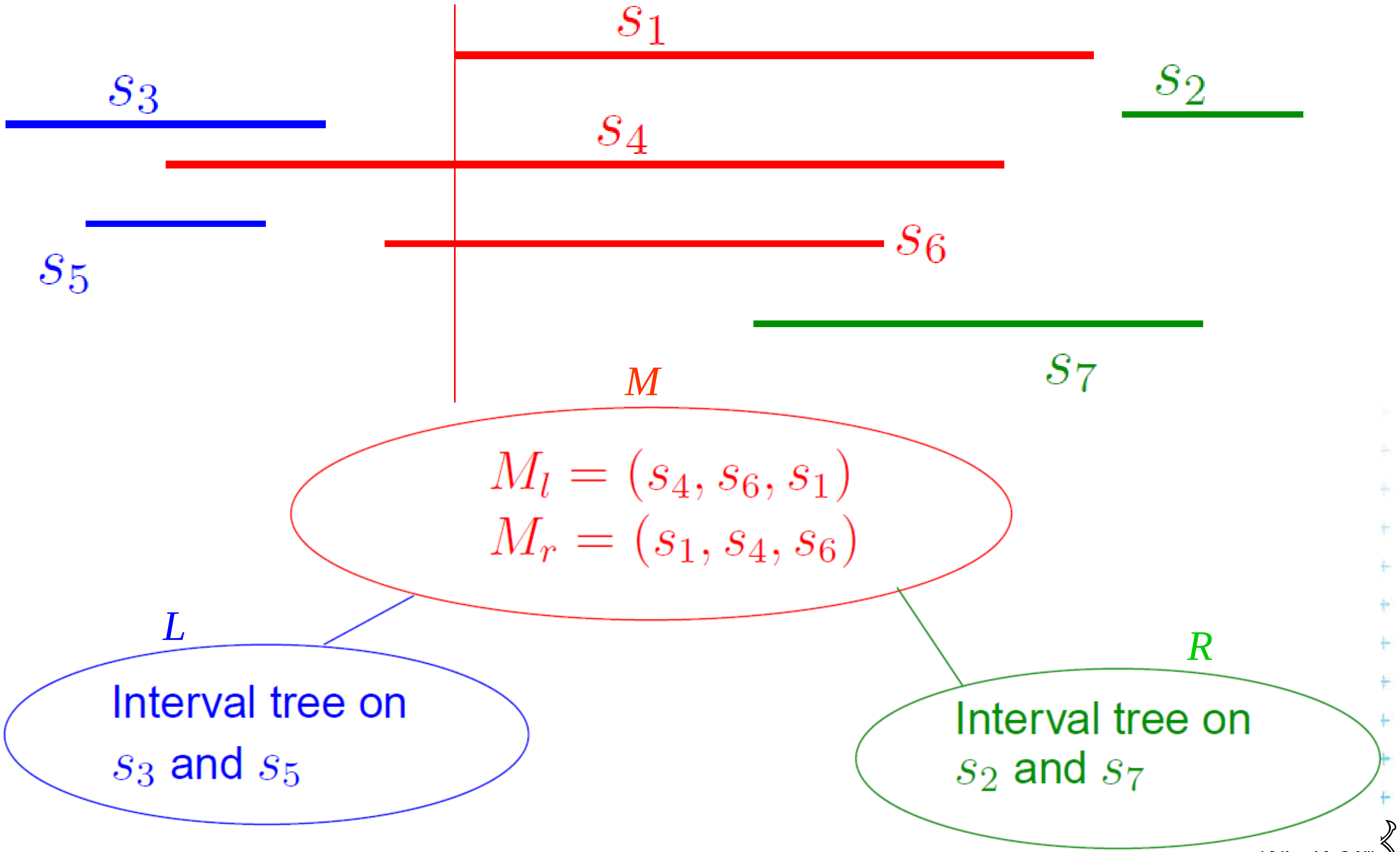
=> Report the interval containing query point q_x



[Mount]

Interval tree principle

(see lecture 9 - intersections)



$M_l = (s_4, s_6, s_1)$

$M_r = (s_1, s_4, s_6)$

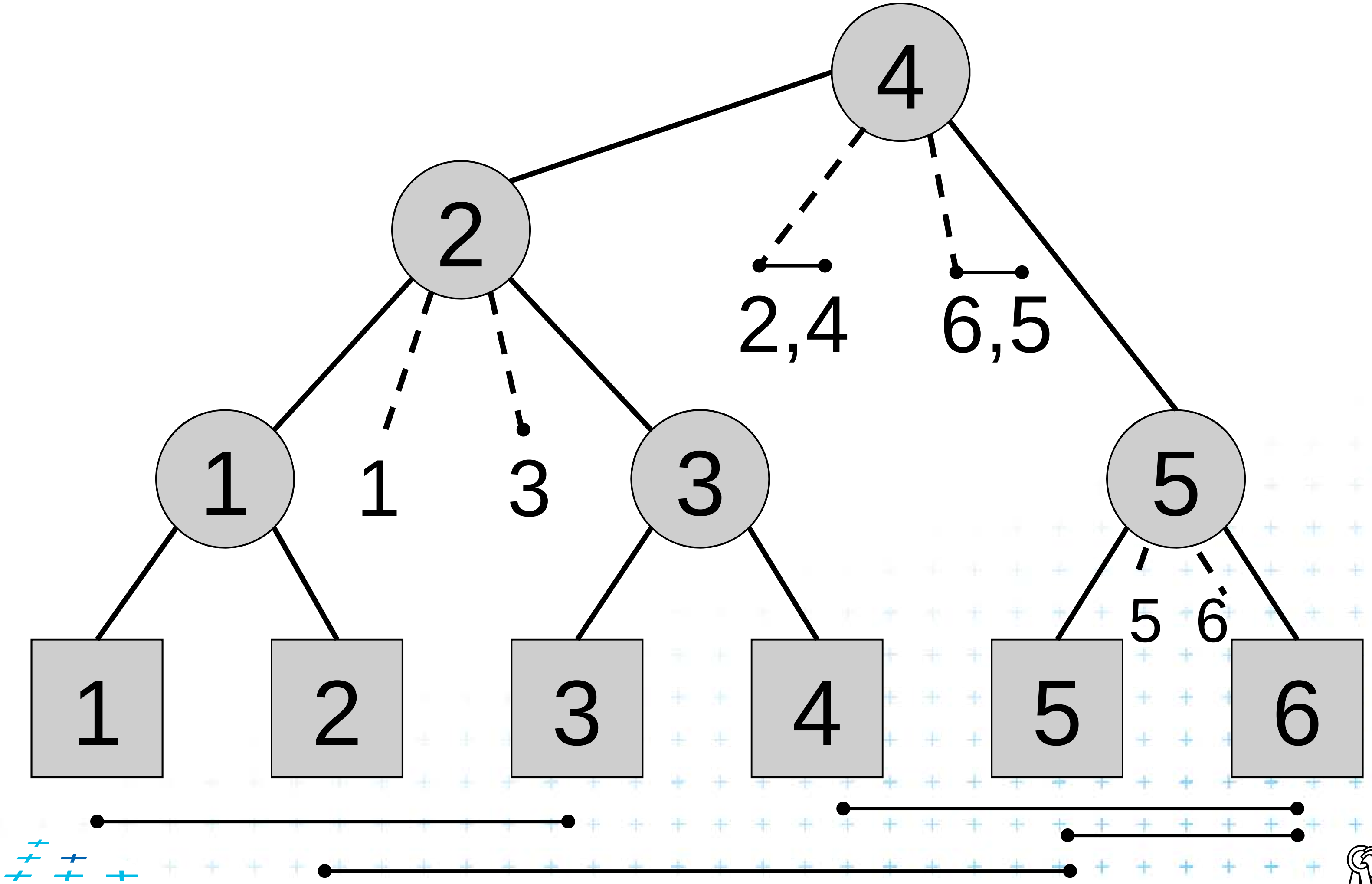
Interval tree on s_3 and s_5

Interval tree on s_2 and s_7

[Vigneron]

Static interval tree [Edelsbrunner80]

Tree over sorted segment end-points

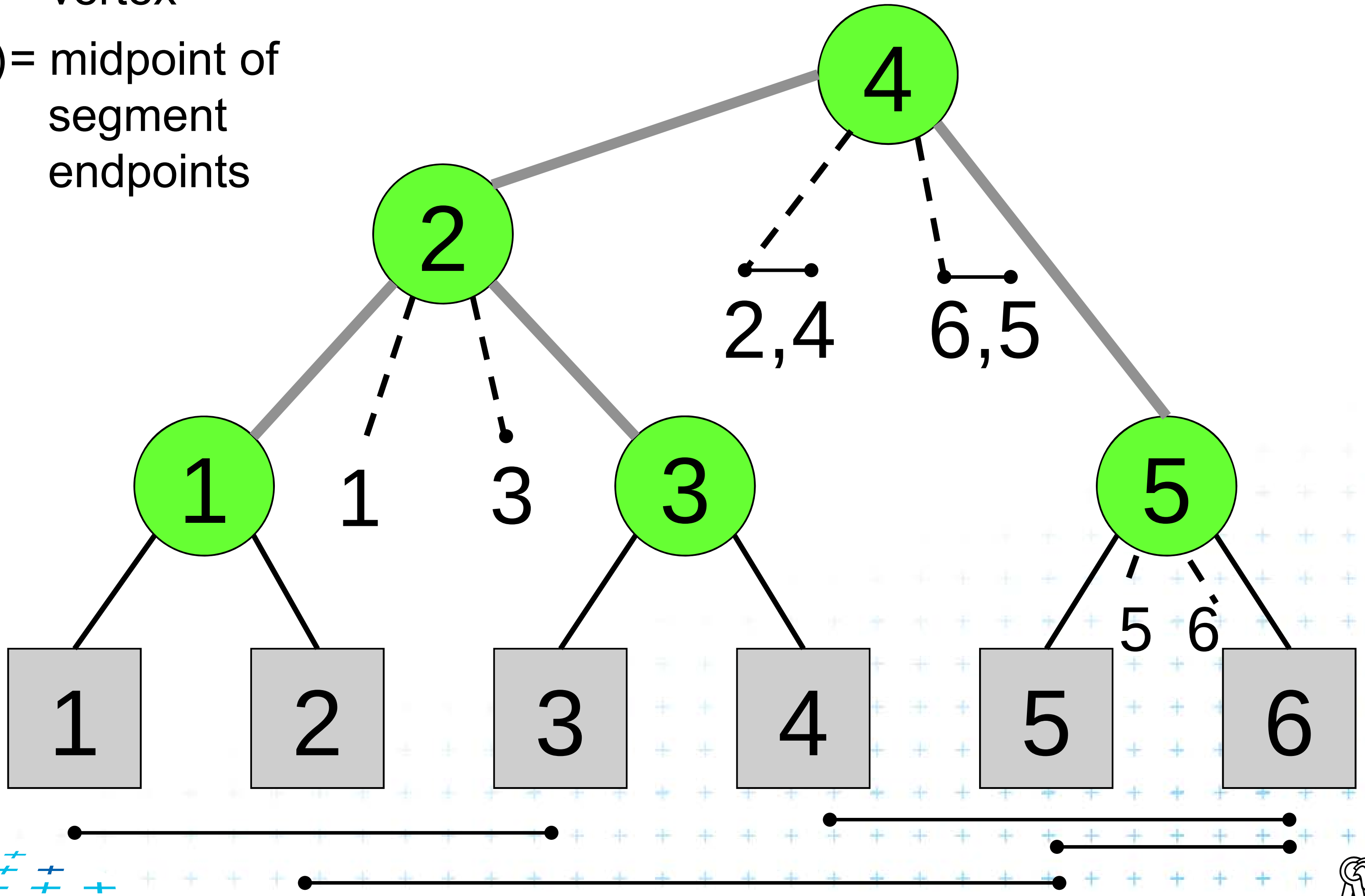


[Kukral]

Primary structure – static tree for endpoints

v = vertex

d(v)= midpoint of segment endpoints

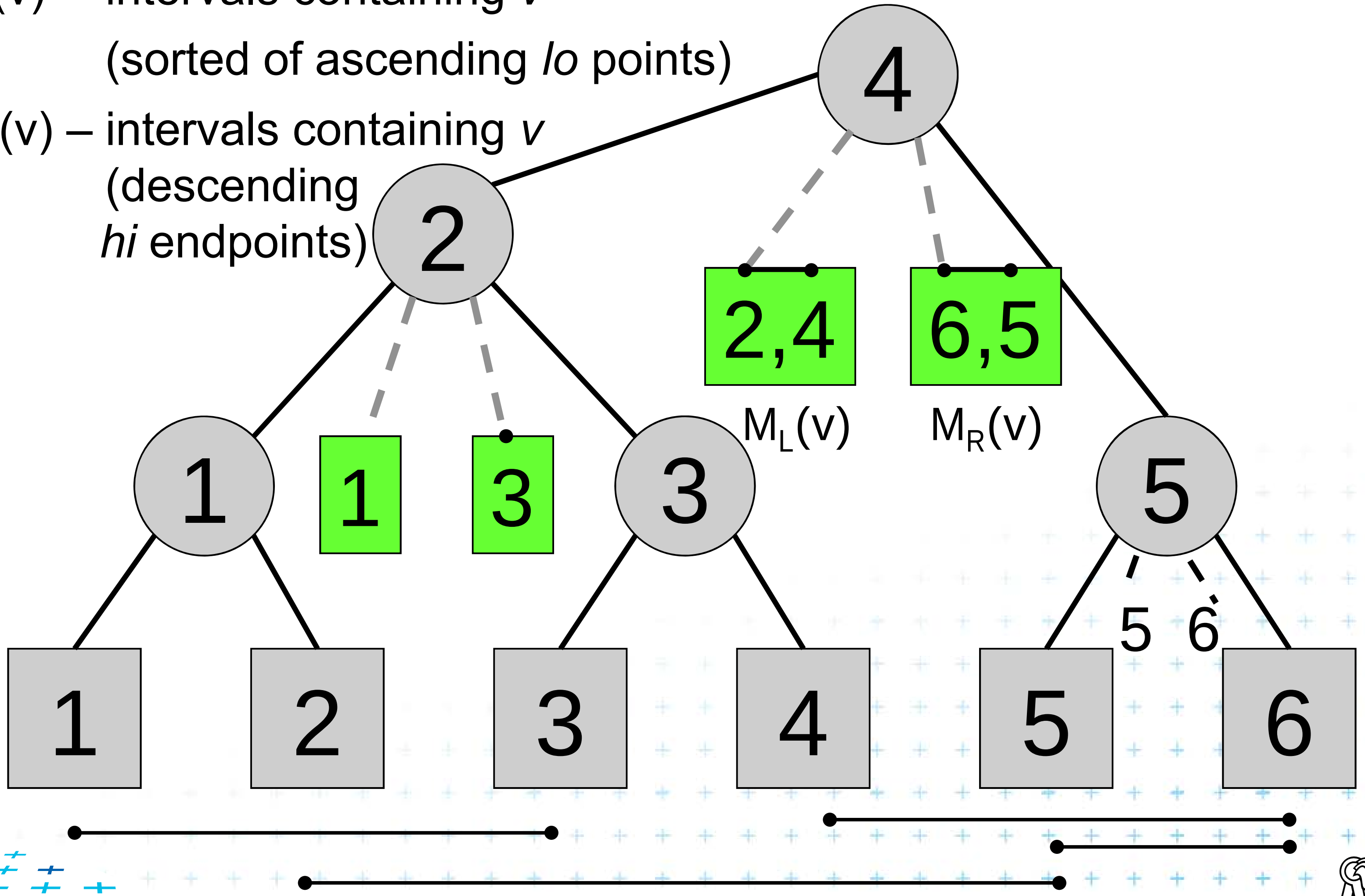


[Kukral]

Secondary lists – sorted segments in M

ML(v) – intervals containing *v*
(sorted of ascending *lo* points)

MR(v) – intervals containing *v*
(descending
hi endpoints)



[Kukral]

Interval tree construction (all intervals at once)

ConstructIntervalTree(*S*)   **// Intervals all active – no active lists**

Input: Set S of intervals on the real line – on *x-axis*

Output: The root of an interval tree for *S*

```
if (|S| == 0) return null                              // no more
else
    xMed = median endpoint of intervals in S           // median endpoint
    L = { [xlo, xhi] in S | xhi < xMed }               // left of median
    R = { [xlo, xhi] in S | xlo > xMed }               // right of median
    M = { [xlo, xhi] in S | xlo <= xMed <= xhi }       // contains median
    ML = sort M in increasing order of xlo             // sort M
    MR = sort M in decreasing order of xhi
    t = new IntTreeNode(xMed, ML, MR)                  // this node
    t.left  = ConstructIntervalTree(L)                 // left subtree
    t.right = ConstructIntervalTree(R)                 // right subtree
    return t
```

[Mount]

Line stabbing query for an interval tree

```
Stab( t, xq)
Input:    IntTreeNode t, Scalar xq
Output:   prints the intersected intervals
if (t == null) return                            // fell out of tree
if (xq < t.xMed)                                 // left of median?
    for (i = 0; i < t.ML.length; i++)            // traverse ML
            if (t.ML[i].lo <= xq) print(t.ML[i]) // ..report if in range
            else break                           // ..else done
    stab(t.left, xq)                             // recurse on left
else  // (xq ≥ t.xMed)                           // right of or equal to median
    for (i = 0; i < t.MR.length; i++) {          // traverse MR
            if (t.MR[i].hi ≥ xq) print(t.MR[i])  // ..report if in range
            else break                           // ..else done
    stab(t.right, xq)                            // recurse on right
```

Note: Small inefficiency for xq == t.xMed – recurse on right [Mount]

Complexity of line stabbing via interval tree

- Construction - O(*n* log *n*) time
  - Each step divides at maximum into two halves or less (minus elements of M) => tree height O(log *n*)
  - If presorted the endpoints in three lists L,R,M then median in O(1) and copy to new L,R,M in O(*n*)]
- Vertical line stabbing query - O(*k* + log *n*) time
  - One node processed in O(1 + k'), k'=reported intervals
  - *v* visited nodes in O(*v* + k), k=total reported intervals
  - *v* = tree height = O(log *n*)
- Storage - O(*n*)
  - Tree has O(*n*) nodes, each segment stored twice (two endpoints)

Talk overview

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   3. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*

i. Segment intersected by vertical line - 1D

- Query line $\ell := (x = q_x)$

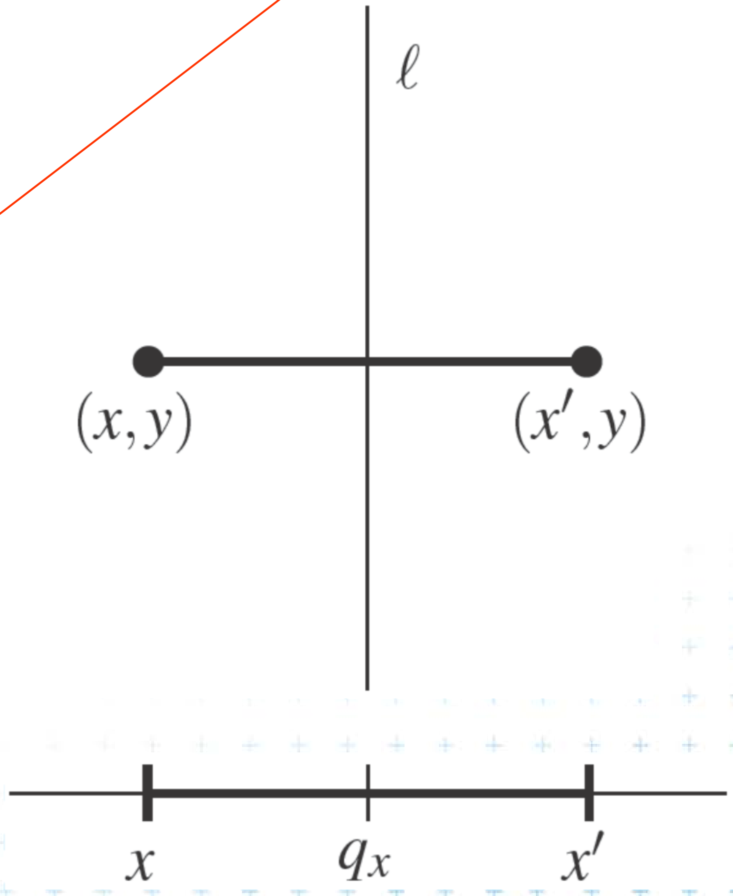
  Report the segments stabbed by a vertical line

  = 1 dimensional problem

  (ignore y coordinate)

=> Report the interval containing query point q_x

DS: Interval tree

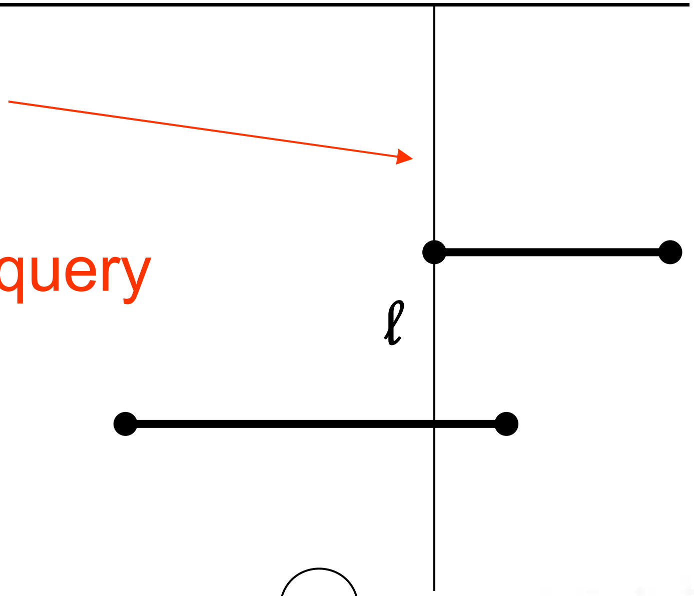


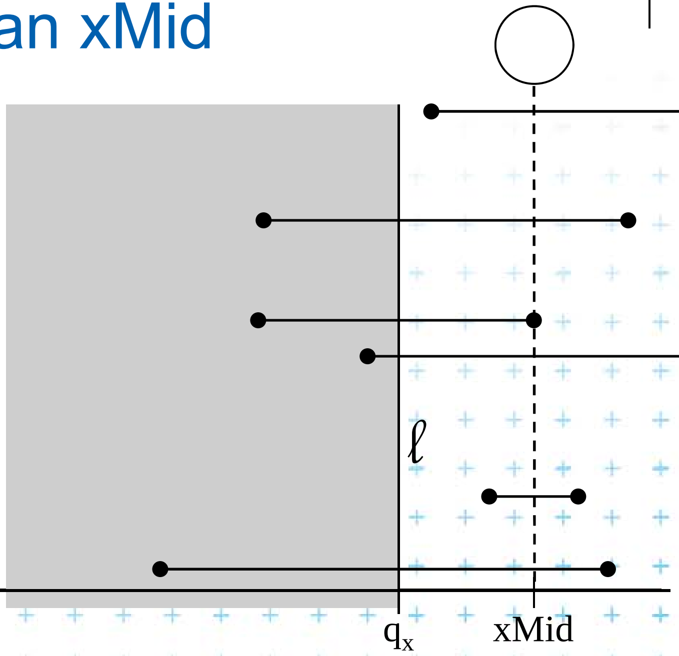
[Berg]

i. Segment intersected by vertical line - 2D

- Query line $\ell := q_x \times [-\infty : \infty]$
- Horizontal segment of M stabs the query line ℓ **iff** its left endpoint lies in halph-space

  $$(-\infty : q_x] \times [-\infty : \infty]$$
- In IT node with stored median xMid report all segments from M
  - whose left point lies in $(-\infty : q_x]$ if ℓ lies left from xMid
  - whose right point lies in $(q_x : +\infty]$ if ℓ lies right from xMid

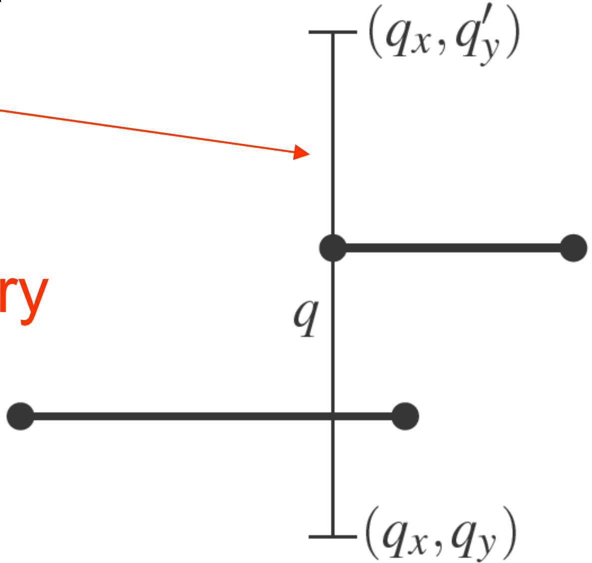




Inspired by [Berg]

ii. Segment intersected by vertical line segment

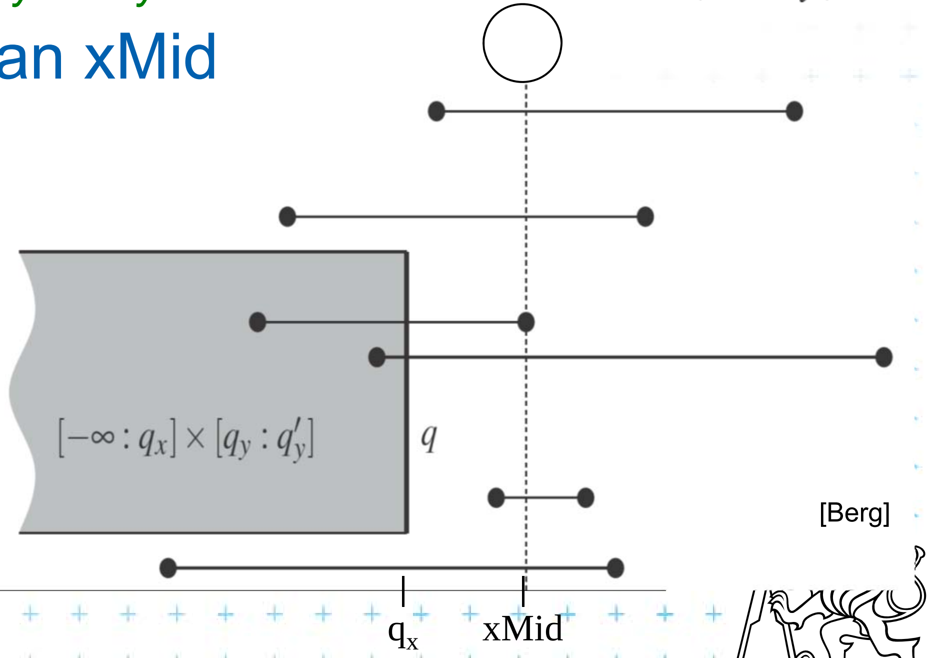
- Query segment $q := q_x \times [q_y : q'_y]$



- Horizontal segment of *M* stabs the query segment *q* **iff** its left endpoint lies in semi-infinite rectangular region

$$(-\infty : q_x] \times [q_y ; q'_y]$$

- In IT node with stored median xMid report all segments
  - whose left point lies in $(-\infty : q_x] \times [q_y ; q'_y]$ if *q* lies left from xMid
  - whose right point lies in $(q_x : +\infty] \times [q_y ; q'_y]$ if *q* lies right from xMid

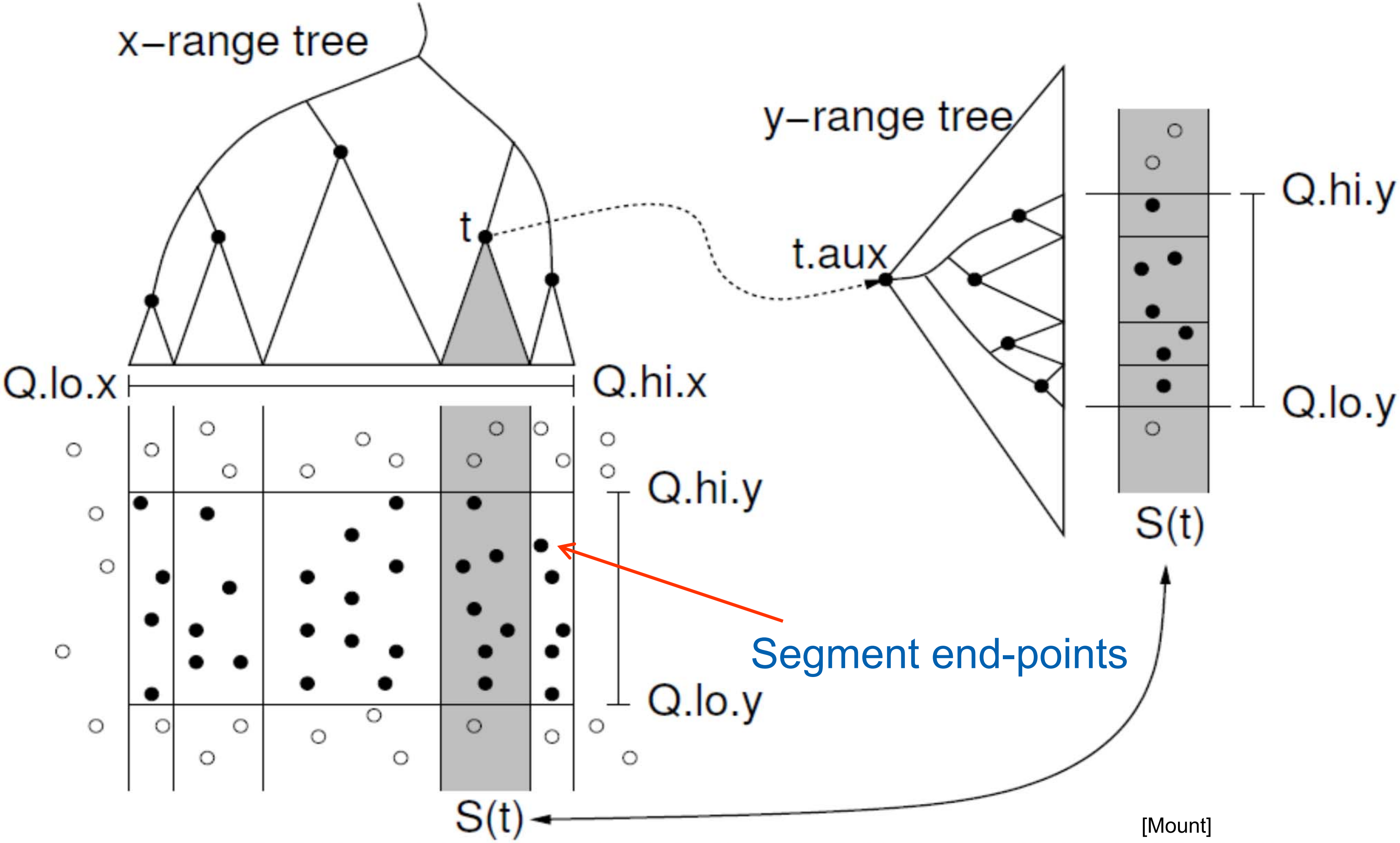


[Berg]

Data structure for endpoints

- Storage of ML and MR
  - Sorted lists not enough for line segments
  - Use two 2D range trees
- Instead O(*n*) sequential search in ML and MR perform O(log *n*) search in range tree with fractional cascading

2D range tree (without fractional casc. - see more in Lecture 3)



Segment end-points

[Mount]

Complexity of line segment stabbing

- Construction - O(n log n) time
  - Each step divides at maximum into two halves L,R or less (minus elements of M) => tree height O(log n)
  - If the range trees are efficiently build in O(n) after points sorted
- Vertical line segment stab. q. - O(k + log^2 n) time

  2D range tree search with FC
  - One node processed in O(log n + k'), k'=reported inter.
  - v-visited nodes in O(v log n + k), k=total reported inter.
  - v = interval tree height = O(log n)
  - O(k + log^2 n) time - range tree with fractional cascading
  - O(k + log^3 n) time - range tree without fractional casc.
- Storage - O(n log n)
  - Dominated by the range trees

Talk overview

1. Windowing of axis parallel line segments in 2D (variants of *interval tree - IT*)
   - i. Line stabbing (*IT* with *sorted lists*)
   - ii. Line segment stabbing (*IT* with *range trees*)
   - iii. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*

iii. Priority search trees

[McCreight85]

- Priority search trees – in case c) on slide 8
  - Exploit the fact that query rectangle in range queries is unbounded
  - Can be used as secondary data structures for both left and right endpoints (ML and MR) of segments (intervals) in nodes of interval tree
  - Improve the storage to O(*n*) for horizontal segment intersection with window edge (Range tree has O(*n* log *n*))
- For cases a) and b) - O(*n* log *n*) remains
  - we need range trees for windowing segment endpoints

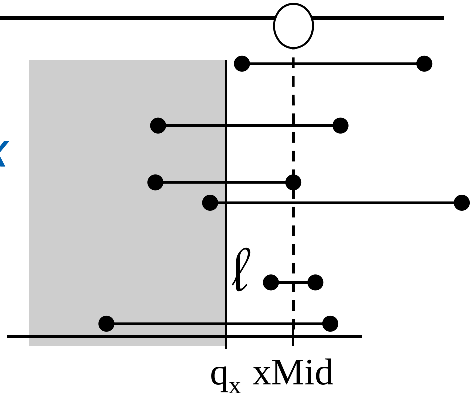
Rectangular range queries variants

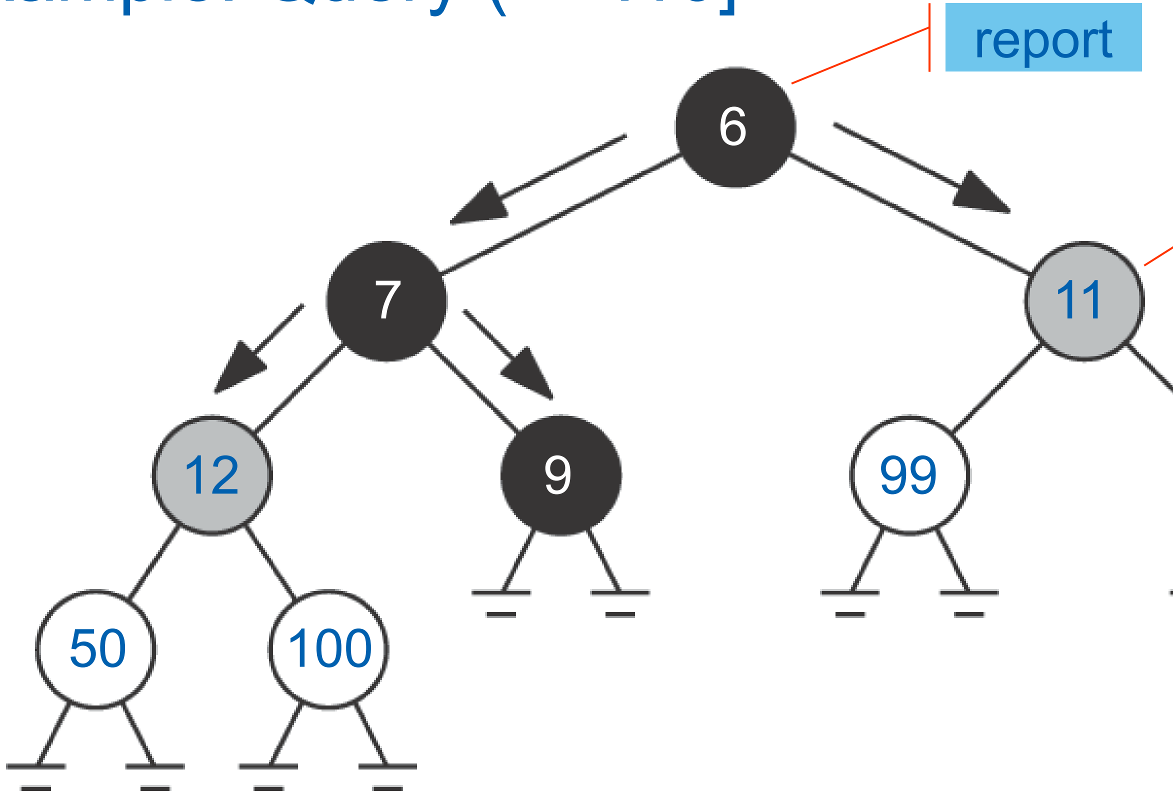
- Let $P = \{ p_1, p_2, \ldots, p_n \}$ is set of points in plane
- Goal: rectangular range queries of the form $(-\infty : q_x] \times [q_y ; q'_y]$
- In 1D: search for nodes v with $v_x \in (-\infty : q_x]$
  - range tree — $O(\log n + k)$ time
  - ordered list — $O(1 + k)$ time (start in the leftmost, stop on v with $v_x > q_x$)
  - use heap — $O(1 + k)$ time (traverse all children, stop when $v_x > q_x$)
- In 2D – use heap for points with $x \in (-\infty : q_x]$

  + integrate information about y-coordinate

Heap for 1D unbounded range queries

- Traverse all children, stop when $v_x > q_x$
- Example: Query $(-\infty:10]$



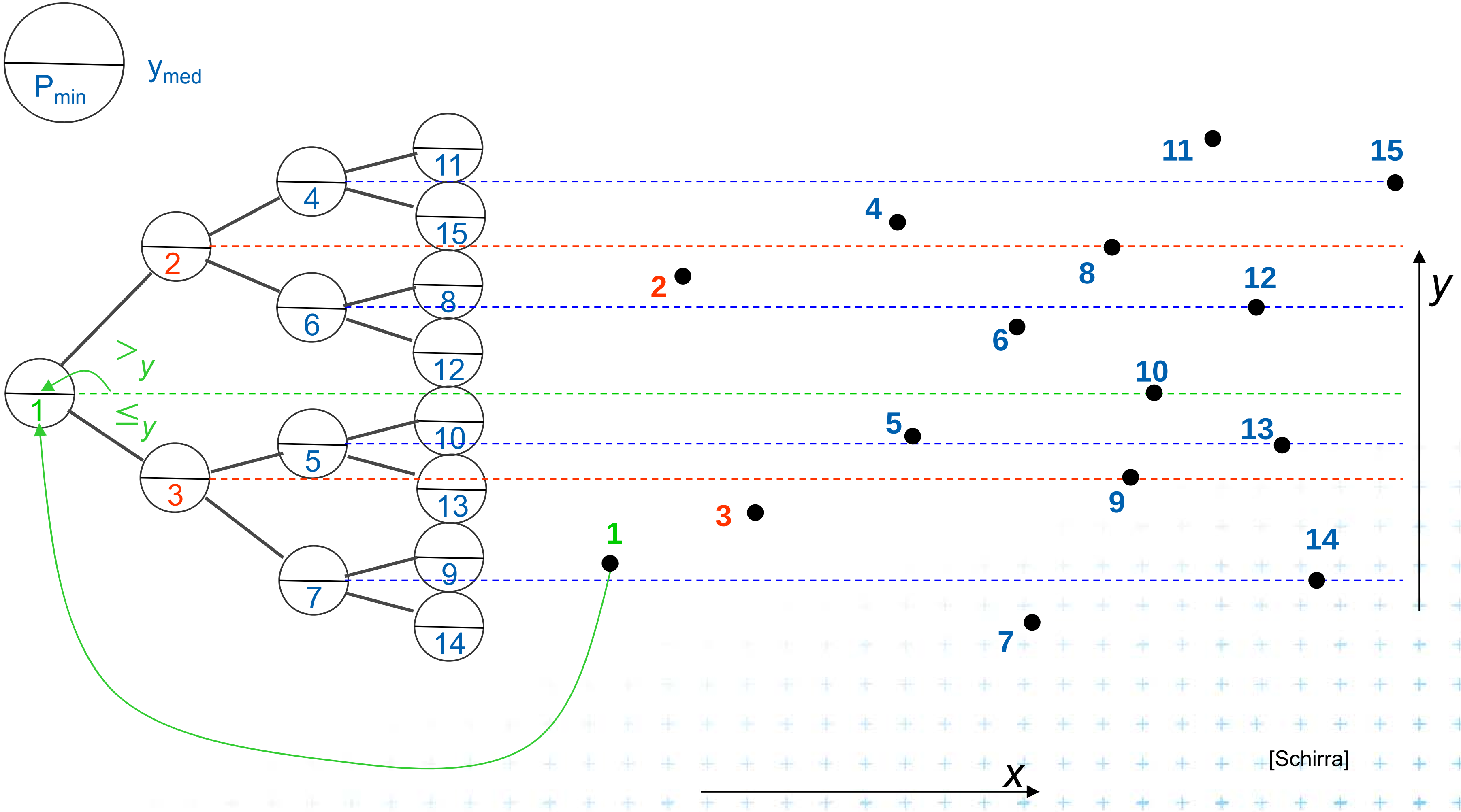


[Berg]

Priority search tree (PST)

- Heap in 2D can incorporate info about both *x,y*
  - BST on *y*-coordinate (horizontal slabs) ~ range tree
  - Heap on *x*-coordinate (minimum *x* from slab along *x*)
- If P is empty, PST is empty leaf
- else
  - p_{min} = point with smallest x-coordinate in P --- a heap root
  - y_{med} = y-coord. median of points $P \setminus \{p_{min}\}$ --- BST root
  - $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med}\}$
  - $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med}\}$
- Point p_{min} and scalar y_{med} are stored in the PST root
- The left subtree is PST of P_{below}
- The right subtree is PST of P_{above}

Priority search tree construction example



[Schirra]

Priority search tree definition

```
**PrioritySearchTree( P )**
Input:    set P of points in plane
Output:   priority search tree  T
if $P=\emptyset$ then PST is an empty leaf
else
    $p_{min}$ = point with smallest x-coordinate in $P$     // heap on x root
    $y_{med}$ = $y$-coord. median of points $P \setminus \{p_{min}\}$      // BST on y root
    Split points $P \setminus \{p_{min}\}$ into two subsets – according to $y_{med}$
        $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med} \}$
        $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med} \}$
    $T$ = newTreeNode()                                  Notation in alg:
    $T.p = p_{min}$      // point [ $x, y$ ]                     … p(v)
    $T.y = y_{mid}$      // skalar                        … y(v)
    $T.left$  = PrioritySearchTree( $P_{below}$ )             … lc(v)
    $T.rigft$ = PrioritySearchTree( $P_{above}$ )             … rc(v)
```

O(n log n) , but O(n) if presorted on y-coordinate and bottom up

Query Priority Search Tree

QueryPrioritySearchTree($T, (-\infty : q_x] \times [q_y ; q'_y]$)

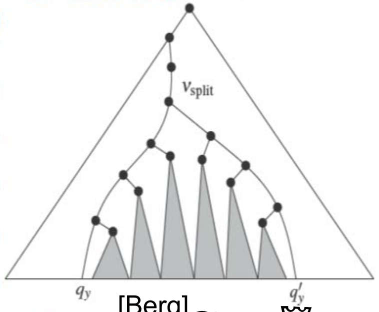
Input: A priority search tree and a range, unbounded to the left

Output: All points lying in the range

```
Search with $q_y$ and $q'_y$ in $T$      // BST on $y$-coordinate – select $y$ range
Let $v_{split}$ be the node where the two search paths split (split node)

for each node $v$ on the search path of $q_y$ or $q'_y$   // points along the paths
    if $p(v) \in (-\infty : q_x] \times [q_y ; q'_y ]$ then report $p(v)$  // starting in tree root

for each node $v$ on the path of $q_y$ in the left subtree of $v_{split}$   // inner trees
    if the search path goes left at $v$
        ReportInSubtree( $rc(v), q_x$ )   // report right subtree
for each node $v$ on the path of $q'_y$ in right subtree of $v_{split}$
    if the search path goes right at $v$
        ReportInSubtree( $lc(v), q_x$ )   // rep. left subtree
```



[Berg]

Reporting of subtrees between the paths

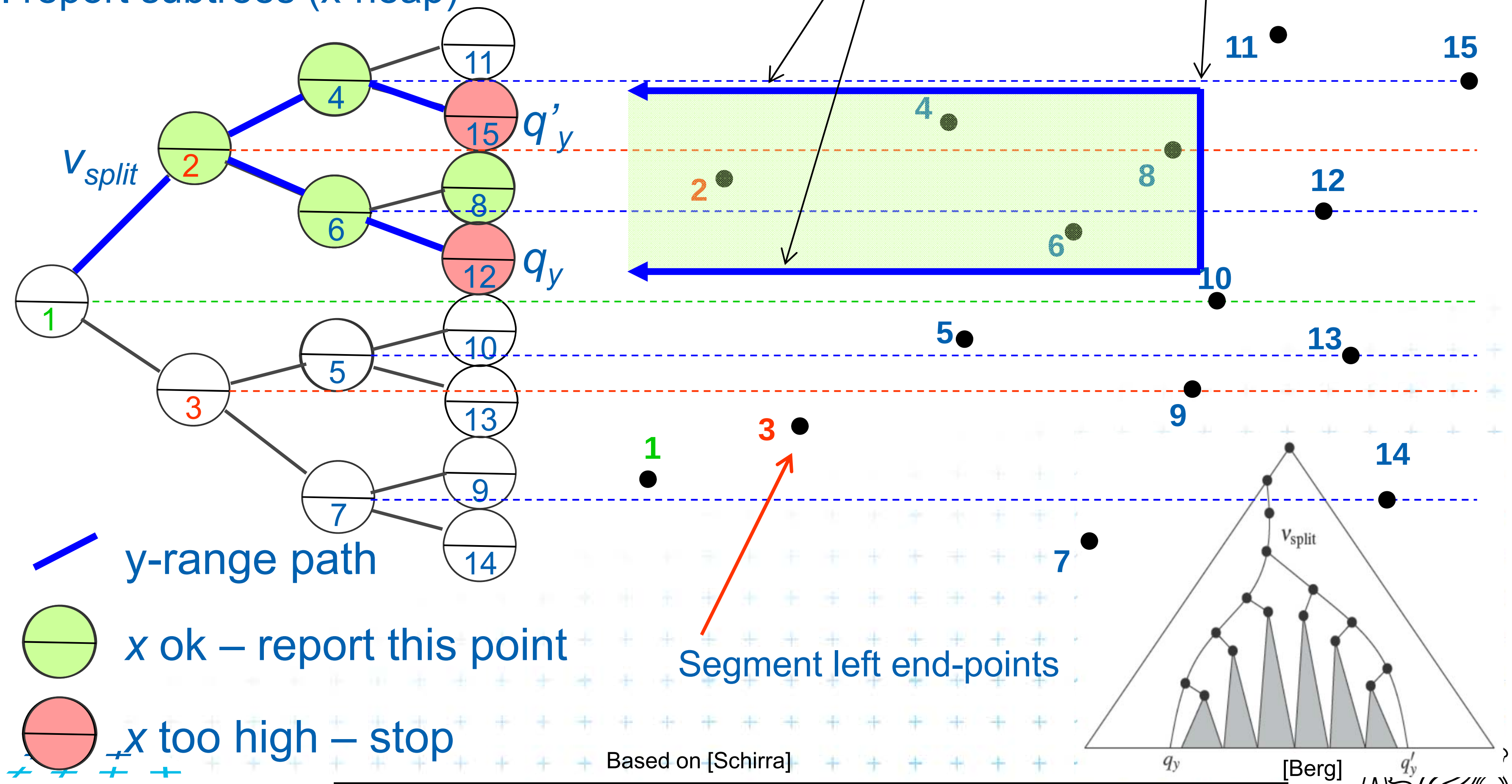
ReportInSubtree(v, q_x)

Input: The root v of a subtree of a priority search tree and a value q_x.
Output: All points in the subtree with x-coordinate at most q_x.

```
1.  if v is not a leaf and $x( p(v) ) \leq q_x$        // $x \in (-\infty : q_x]$   -- heap condition
2.      Report $p(v)$.
3.      ReportInSubtree( $lc(v)$, $q_x$ )
4.      ReportInSubtree( $rc(v)$, $q_x$ )
```

Priority search tree query

1. select *y* range (y-BVS~ 1D range tree)
2. report points on paths (x-heap)
3. report subtrees (x-heap)



Given interval $y_{min}..y_{max}$

Given x_{max}

Segment left end-points

y-range path

🟩 *x* ok – report this point

🟥 *x* too high – stop

Based on [Schirra]

[Berg]

Priority search tree complexity

For set of *n* points in the plane

- Build $O(n \log n)$
- Storage $O(n)$
- Query $O(k + \log n)$
  - points in query range $(-\infty : q_x] \times [q_y ; q'_y])$
  - *k* is number of reported points

- Use PST as associated data structure for interval trees for storage of M

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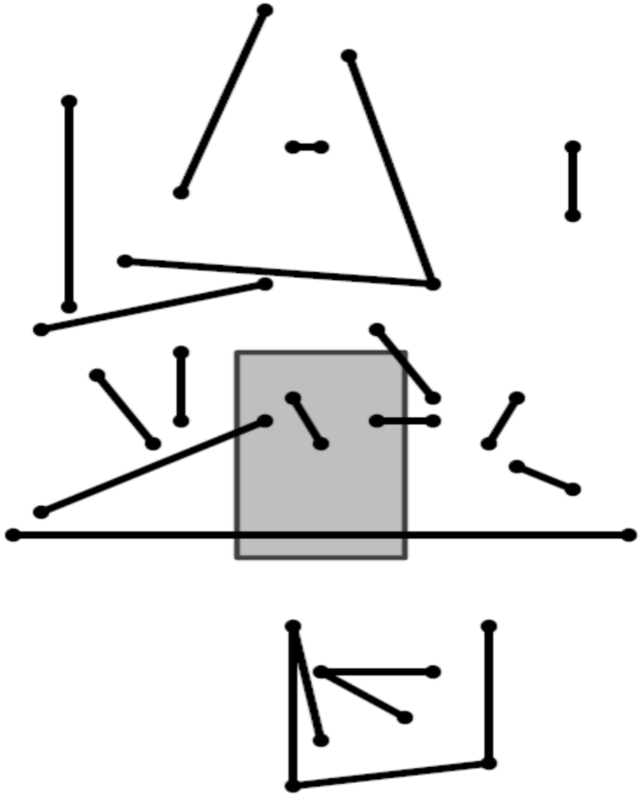
2. Windowing of line segments in general position



[Vakken]

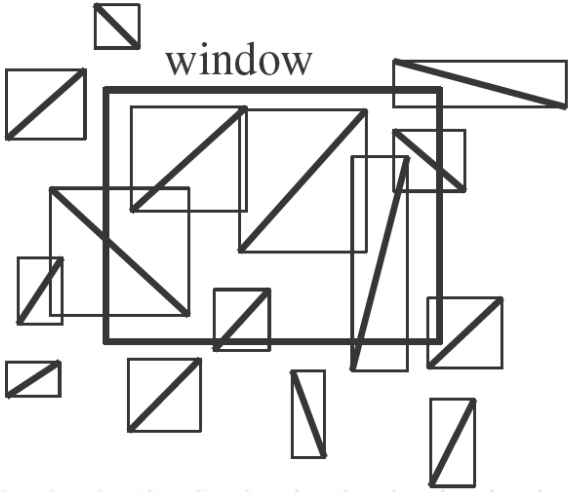
Windowing of arbitrary oriented line segments

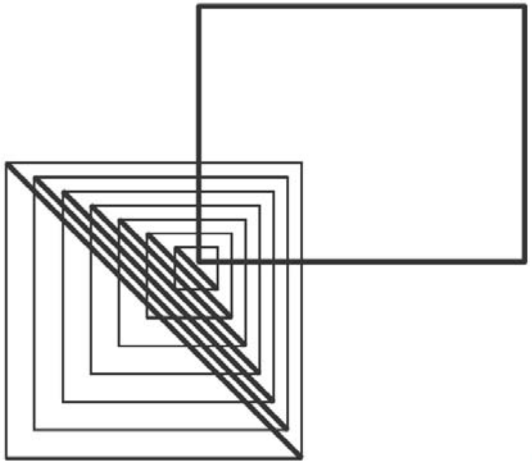
- Two cases of intersection

  a,b) Endpoint inside the query window => range tree

  c) Segment intersects side of query window => ???

- Intersection with BBOX (segment bounding box)?
  - Intersection with 4n sides
  - But segments may not intersect the window –> query y





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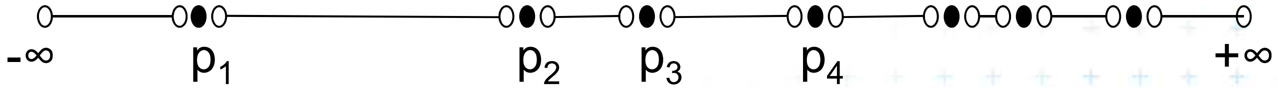
Segment tree

[Bentley, 1977]

- Exploits locus approach
  - Partition parameter space into regions of same answer
  - Localization of such region = knowing the answer
- For given set *S* of *n* intervals (segments) on real line
  - Finds *m* elementary intervals (induced by interval end-points)
  - Partitions 1D parameter space into these elementary intervals



$$(-\infty : p_1), [p_1 : p_1], (p_1 : p_2), [p_2 : p_2], \ldots, \\ (p_{m-1} : p_m), [p_m : p_m], (p_m : +\infty)$$

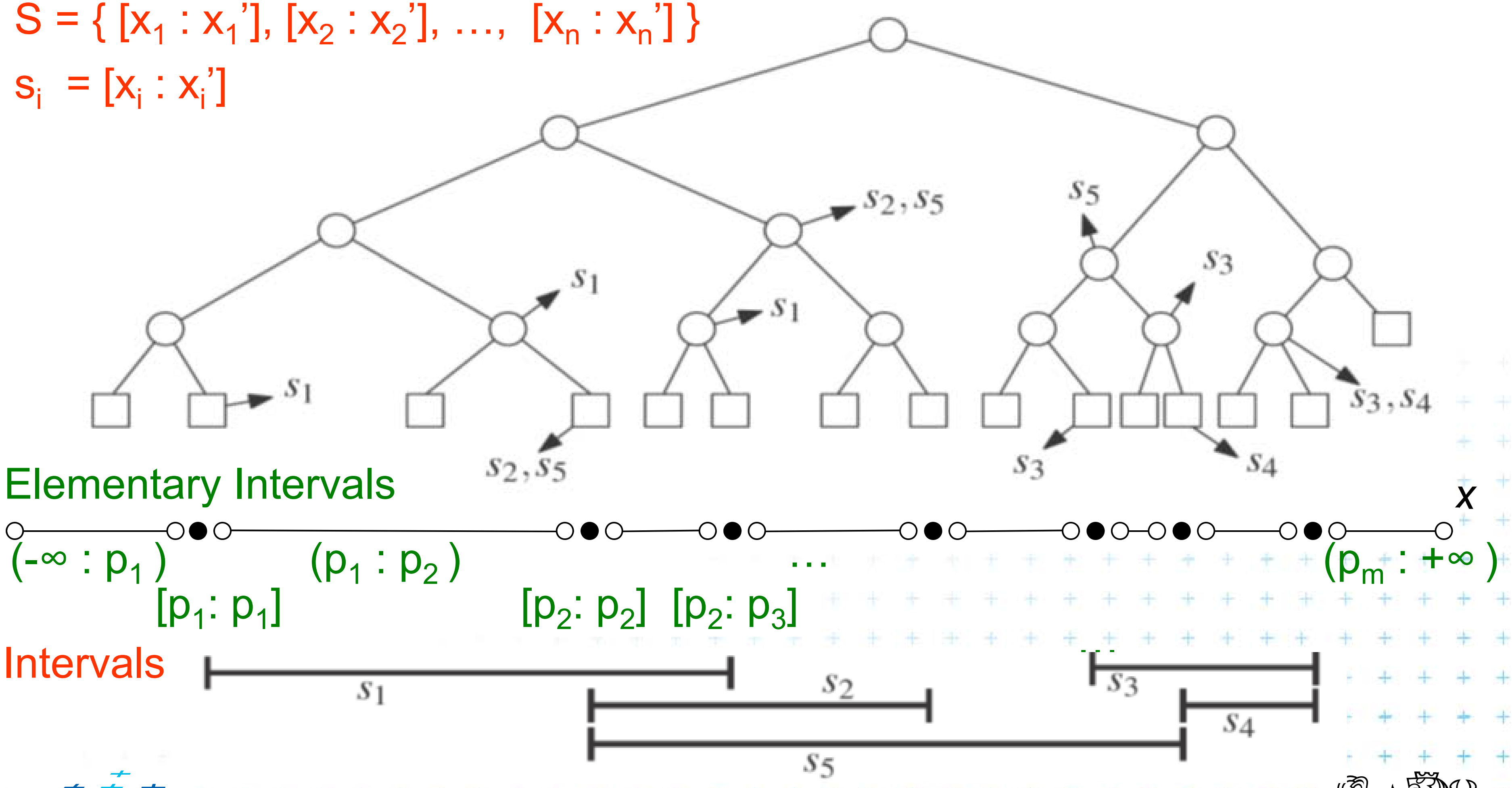
  - Stores intervals s_i with the elementary intervals
  - Reports the intervals s_i containing query point q_x.

Segment tree example

Intervals

$S = \{ [x_1 : x_1'], [x_2 : x_2'], \dots, [x_n : x_n'] \}$

$s_i = [x_i : x_i']$



Elementary Intervals

$(-\infty : p_1)$ $(p_1 : p_2)$ … $(p_m : +\infty)$

$[p_1 : p_1]$ $[p_2 : p_2]$ $[p_2 : p_3]$

Intervals

[Berg]

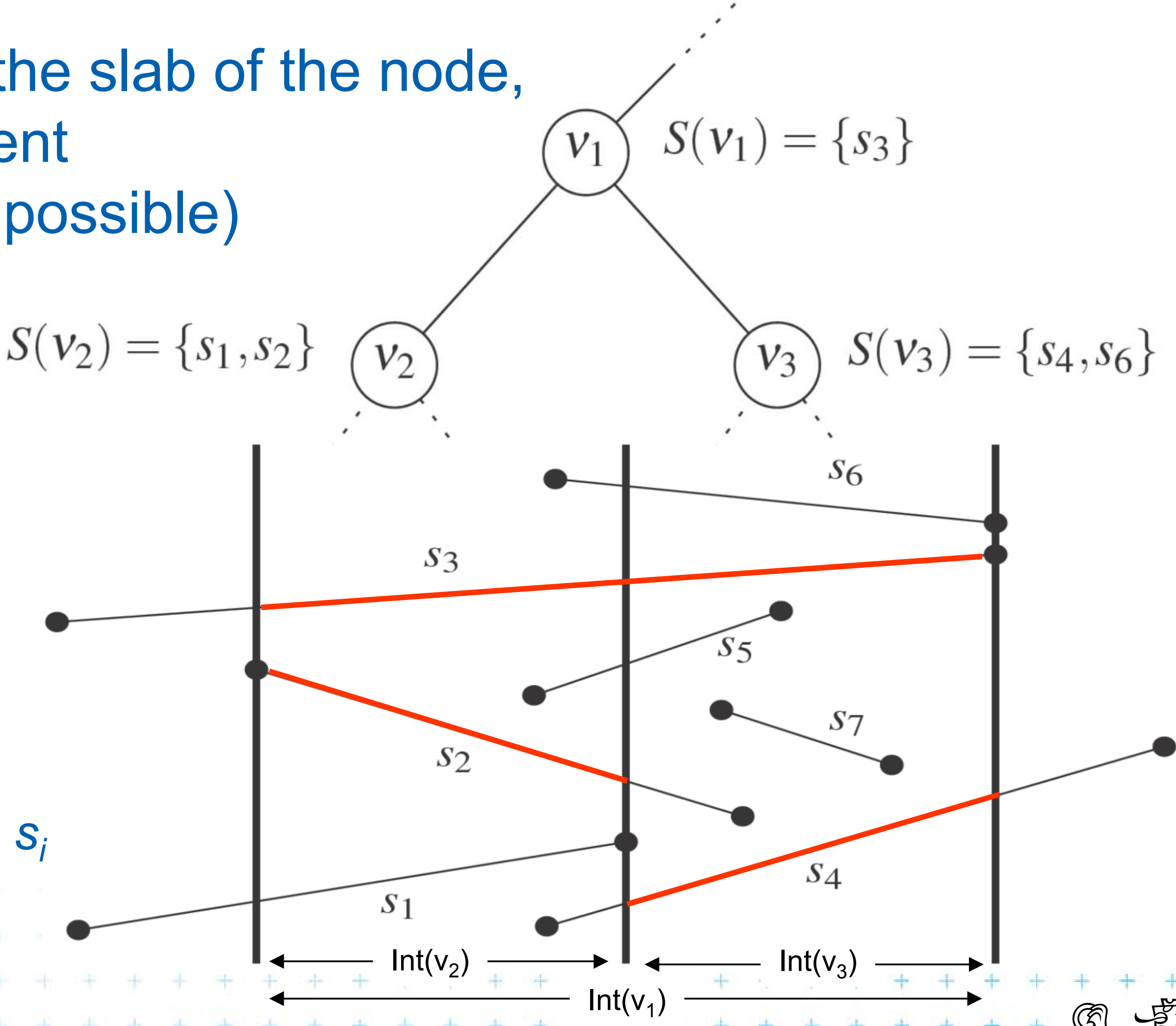
Segment tree definition

Segment tree

- Skeleton is a balanced binary tree T
- Leaves ~ elementary intervals Int(v)
- Internal nodes v
  ~ union of elementary intervals of its children
  - Store: 1. interval Int(v) = union of elementary intervals of its children
    2. canonical set $S(v)$ of intervals $[x : x'] \in S$ segments s_i
  - Holds $\mathrm{Int}(v) \subseteq [x : x']$ and $\mathrm{Int}(\mathrm{parent}(v)) \not\subseteq [x : x']$
    (node interval is not larger than a segment)
  - Intervals $[x : x']$ are stored as high as possible, such that $\mathrm{Int}(v)$ is completely contained in the segment

Segments span the slab

Segments span the slab of the node,
but not of its parent
(stored as up as possible)



Int(v_j) $\subseteq s_i$

and

Int(parent(v_j)] $\not\subseteq s_i$

[Berg]

Query segment tree

QuerySegmentTree(v, q_x)
Input: The root of a (subtree of a) segment tree and a query point q_x
Output: All intervals in the tree containing q_x.

```
Report all the intervals $s_i$ in $S(v)$.        // current node
if $v$ is not a leaf
    if $q_x \in$ Int( $lc(v)$ )                  // go left
        QuerySegmentTree( $lc(v)$, $q_x$ )
    else                                         // or go right
        QuerySegmentTree( $rc(v)$, $q_x$ )
```

Query time O(log n + k), where k is the number of reported intervals

Height O(log n), O(1 + k_v) for node

Segment tree construction

```
ConstructSegmentTree( $S$ )
Input:   Set of intervals $S$ - segments
Output:  segment tree

Sort endpoints of segments in $S$ -> get elemetary intervals …O($n$ log $n$)
Construct a binary search tree $T$ on elementary intervals    …O($n$)
 (bottom up) and determine the interval Int($v$) it represents
Compute the canonical subsets for the nodes (lists of their segments):
    v = root( $T$ )
    for all segments $s_i = [x : x'] \in S$
        InsertSegmentTree( $v$, $[x : x']$ )
```

Segment tree construction – interval insertion

```
InsertSegmentTree( $v$, $[x : x']$ )
Input:   The root of (a subtree of) a segment tree and an interval.
Output:  The interval will be stored in the subtree.
if Int($v$) $\subseteq [x : x']$                    // Int(v) contains $s_i = [x : x']$
    store $[x : x']$ at $v$
else if Int( $lc(v)$ ) $\cap [x : x'] \neq \emptyset$
        InsertSegmentTree( $lc(v)$, $[x : x']$ )
    if Int( $rc(v)$ ) $\cap [x : x'] \neq \emptyset$
        InsertSegmentTree( $rc(v)$, $[x : x']$ )
```

One interval is stored at most twice in one level =>

Single interval insert $O(\log n)$, insert n intervals

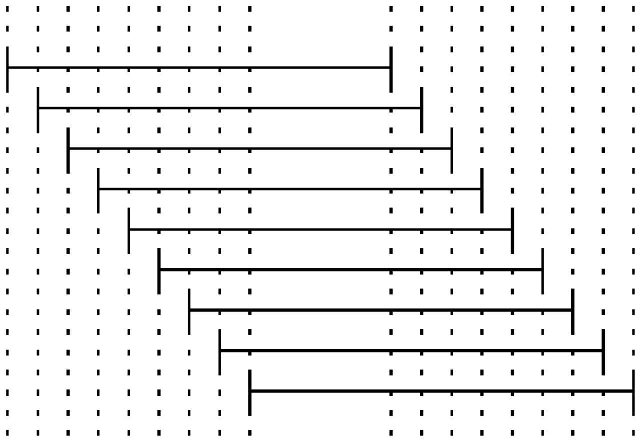
Construction total $O(n \log n)$

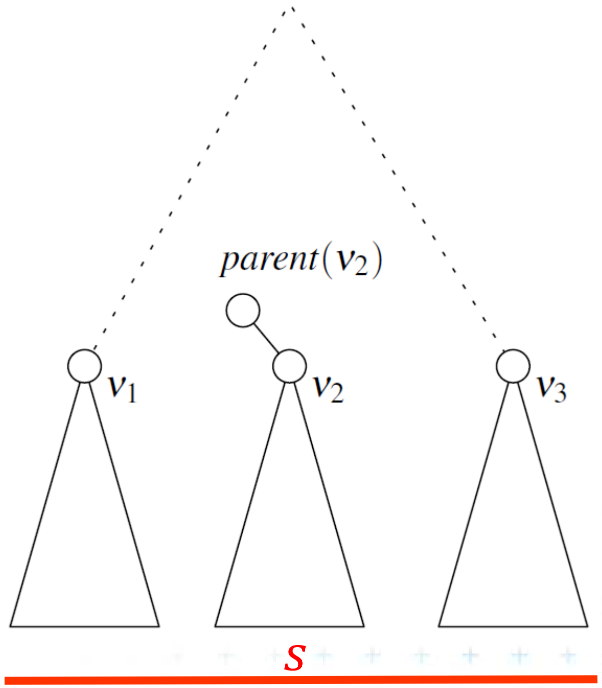
Storage $O(n \log n)$

Tree height $O(\log n)$, name stored max 2x in one level

Storage total $O(n \log n)$ – see next slide

Space complexity - notes





Worst case – $O(n^2)$ segments in leaf

But

Store segments as high, as possible

Segment max 2 times in one level

$\max 4n+1$ elementary intervals (leaves)

$\Rightarrow O(n)$ space for the tree

$\Rightarrow O(n \log n)$ space for interval names

s covered by v_1 and v_3

$\Rightarrow v_2$ covered - $Int(v_2) \in s$

$\Rightarrow$ segment must be stored in parent(v_2)

Segment tree complexity

A segment tree for set *S* of *n* intervals in the plane,

Build	O(*n* log *n*)
Storage	O(*n* log *n*)
Query	O(*k* + log *n*)

- Report all intervals that contain a query point
- *k* is number of reported intervals

Segment tree versus Interval tree

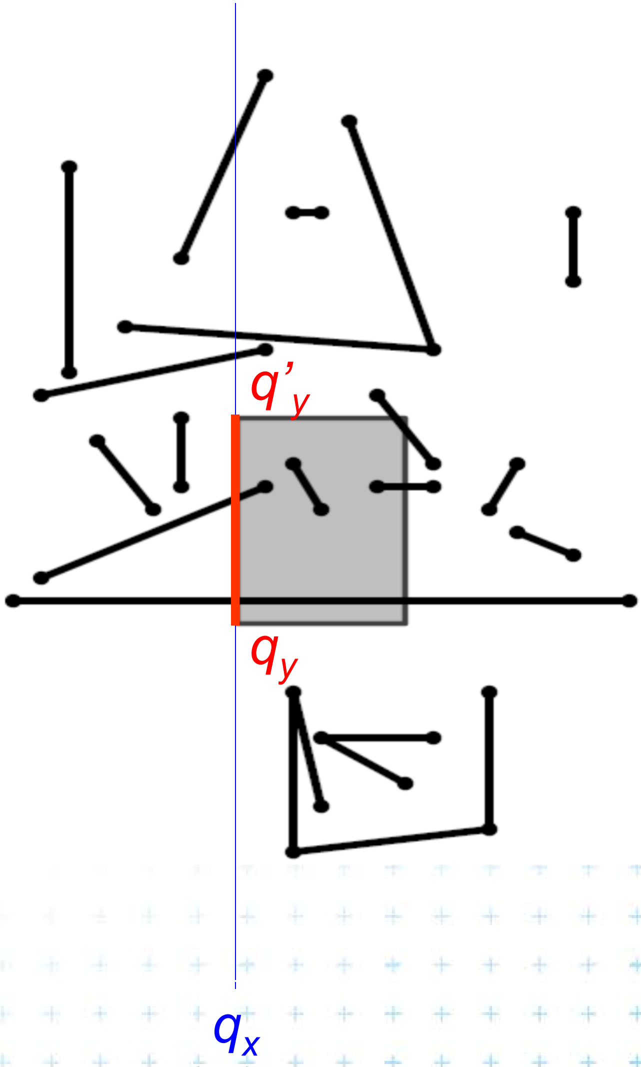
- Segment tree
  - – O(*n* log *n*) storage x O(*n*) of Interval tree
  - – But returns exactly the intersected segments s_i, interval tree must search the lists ML and/or MR
- Good for
  1. extensions (allows different structuring of intervals)
  2. stabbing counting queries
     - – store number of intersected intervals in nodes
     - – O(n) storage and O(log *n*) query time = optimal
  3. higher dimensions – multilevel segment trees

     (Interval and priority search trees do not exist in ^dims)

Talk overview

1. Windowing of axis parallel line segments in 2D (variants of *interval tree - IT*)
   1. Line stabbing (*IT* with *sorted lists*)
   2. Line segment stabbing (*IT* with *range trees*)
   3. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*
   - the algorithm

2. Windowing of line segments in general position

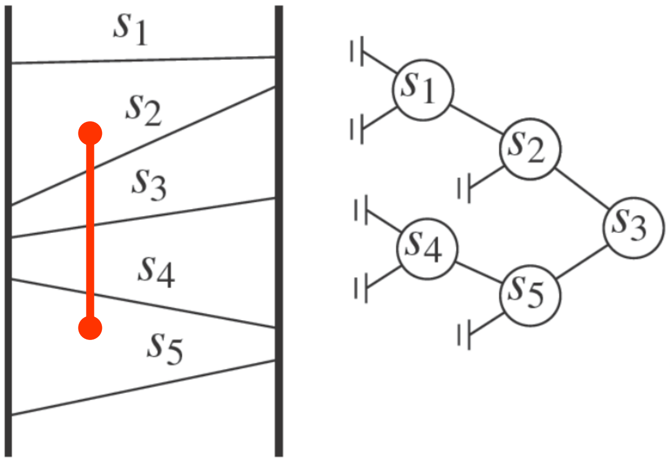


[Vakken]

Windowing of arbitrary oriented line segments

- Let *S* be a set of arbitrarily oriented line segments in the plane.
- Report the segments intersecting a vertical query segment $q := q_x \times [q_y : q'_y]$
- Segment tree *T* on *x* intervals of segments in *S*
  - node *v* of *T* corresponds to vertical slab Int(*v*) × (-∞ : ∞)
  - segments span the slab of the node, but not of its parent
  - segments do not intersect

  => segments in the slab (node) can be vertically ordered – BST



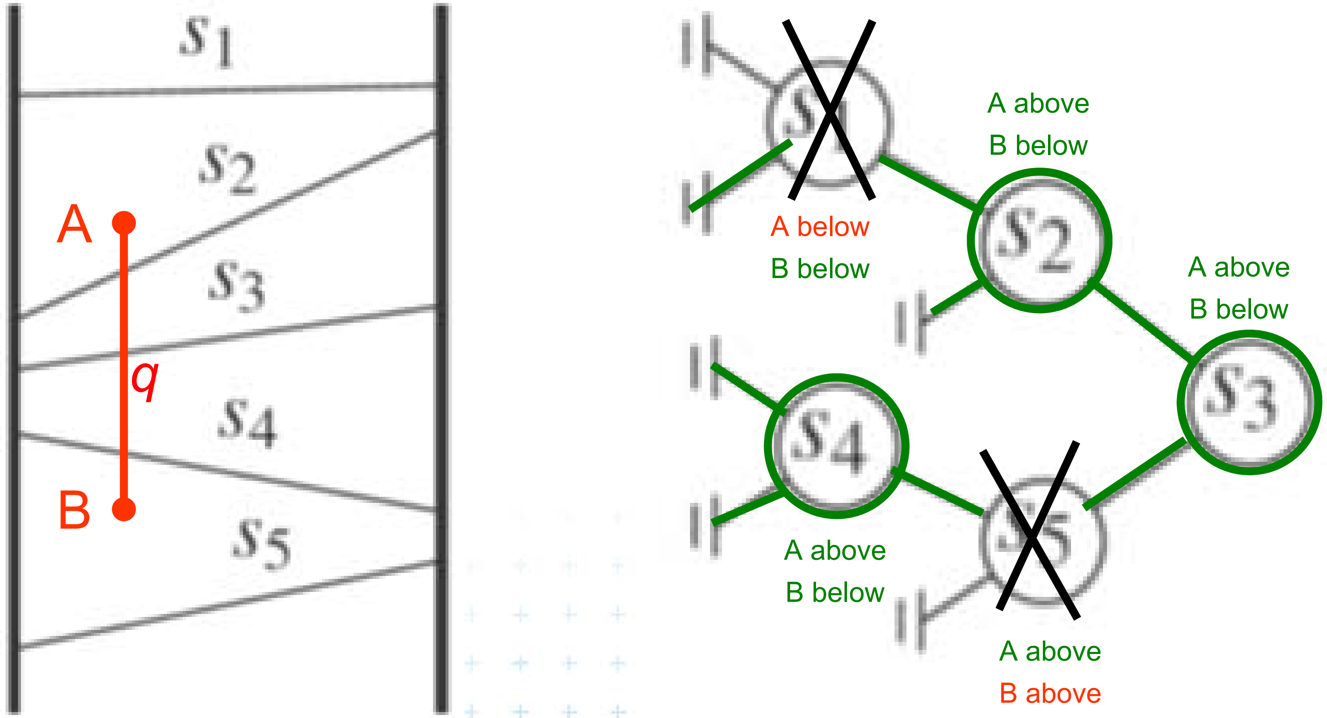
[Berg]

Segments between vertical segment endpoints

- Segments (in the slab) do not mutually intersect
  => segments can be vertically ordered and stored in BST
  - Each node *v* of the x segment tree has an associated y BST
  - BST *T*(*v*) of node *v* stores the canonical subset *S*(*v*) according to the vertical order
  - Intersected segments can be found by searching *T*(*v*) in O(k_v + log *n*), k_v is the number of intersected segments

Segments between vertical segment endpoints

- Segment *s* is intersected by vert.query segment *q* iff
  - The lower endpoint (B) of *q* is below *s* and
  - The upper endpoint (A) of *q* is above *s*



[Berg]

Windowing of arbitrary oriented line segments complexity

Structure associated to node (BST) uses storage linear in the size of $S(v)$

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log^2 n)$
  - Report all segments that contain a query point
  - k is number of reported segments

Windowing of line segments in 2D – conclusions

Construction: all variants O(n logn)

1. Axis parallel	Search	Memory
i. Line (*sorted lists*)	$O(k + \log n)$	$O(n)$
ii. Segment (*range trees*)	$O(k + \log^2 n)$	$O(n \log n)$
iii. Segment (*priority s. tr.*)	$O(k + \log n)$	$O(n)$
2. In general position		
– *segment tree*	$O(k + \log^2 n)$	$O(n \log n)$

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