DCGI
DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

WINDOWING

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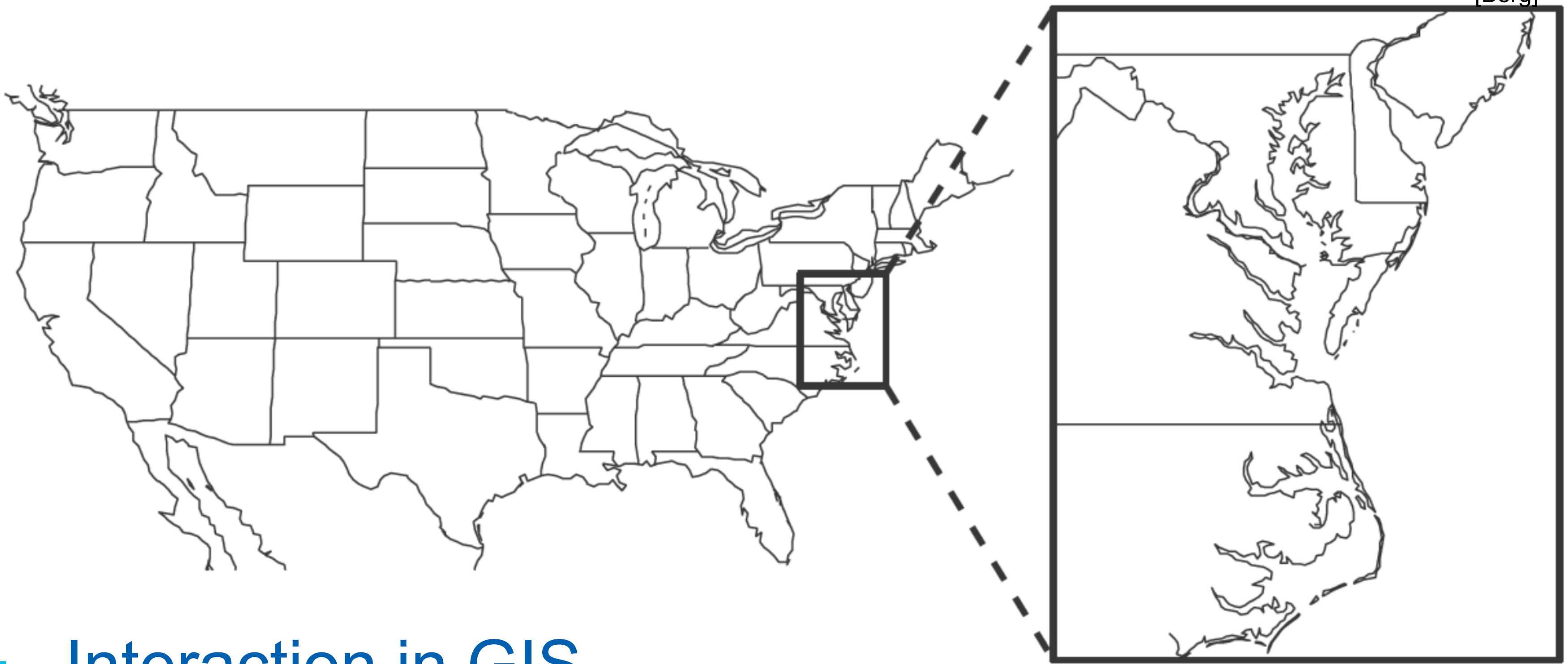
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Mount]

Version from 27.11.2014

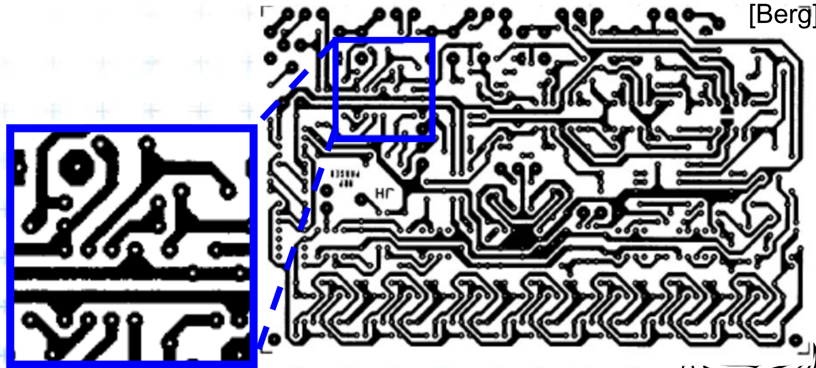
Windowing queries - examples

[Berg]



- Interaction in GIS
  – Select subset by outlining
  – Zoom in and re-center
- Circuit board inspection,…

[Berg]

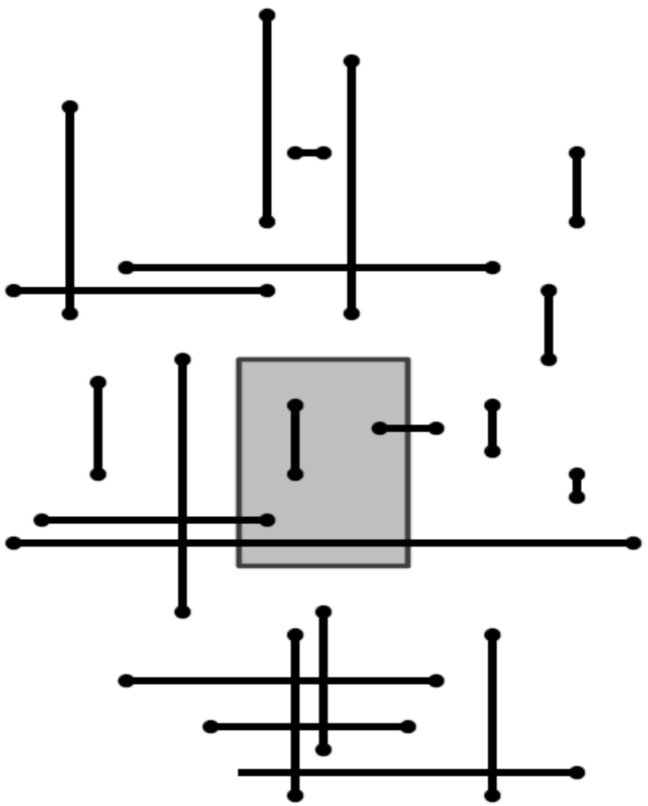


[Vakken]

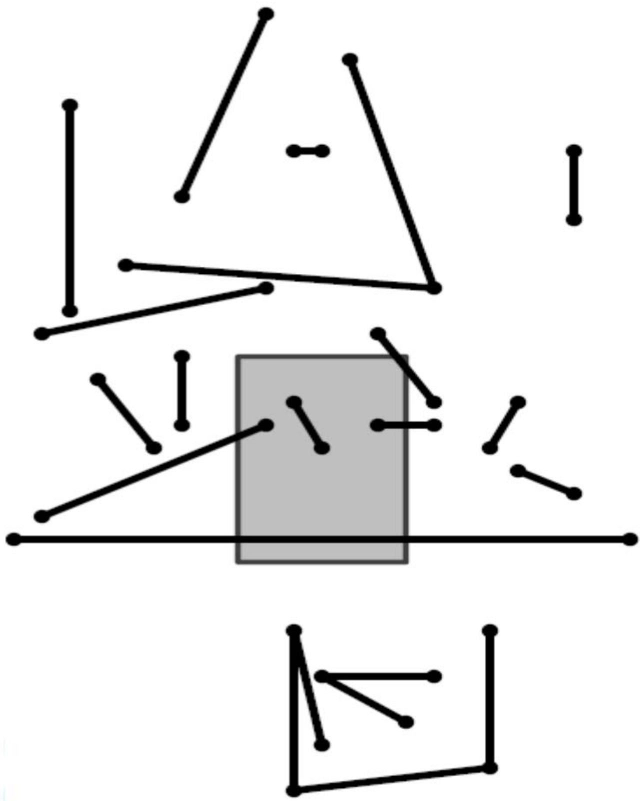
Windowing versus range queries

- Range queries (see range trees in Lecture 03)
  - Points
  - Often in higher dimensions
- Windowing queries
  - Line segments, curves, …
  - Usually in low dimension (2D, 3D)
- The goal for both:
  Preprocess the data into a data structure
  - so that the objects intersected by the query rectangle can be reported efficiently

Windowing queries on line segments



1. Axis parallel line segments



2. Arbitrary line segments (non-crossing)

[Vakken]

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Talk overview

1. Windowing of axis parallel line segments in 2D (3 variants of *interval tree - IT*)
   i. Line stabbing (*IT* with *sorted lists*) lecture 9 - intersections
   ii. Line segment stabbing (*IT* with *range trees*)
   iii. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   – *segment tree*

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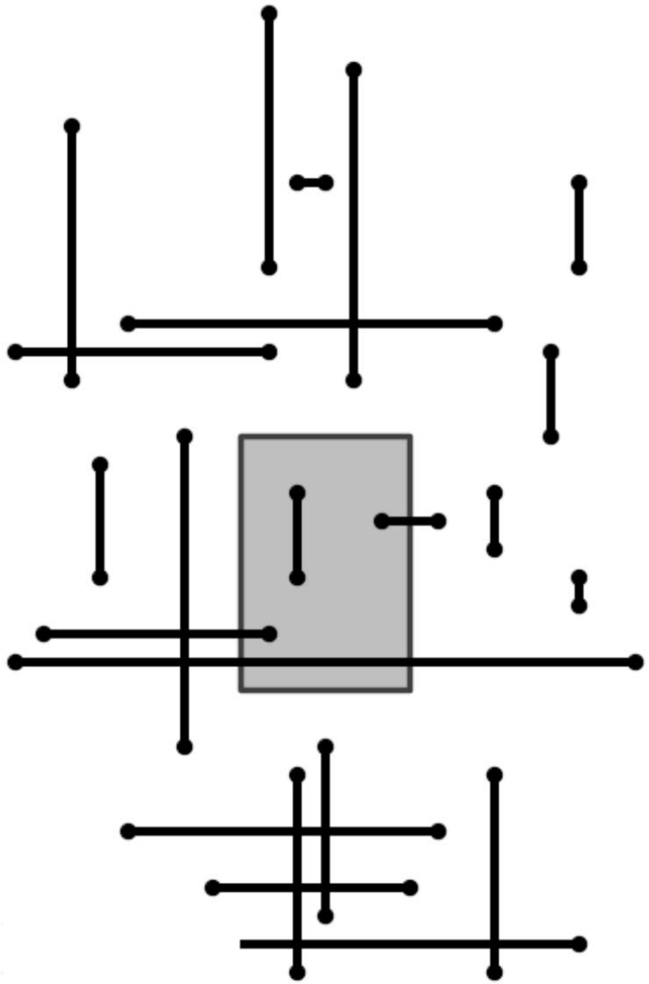
1. Windowing of axis parallel line segments

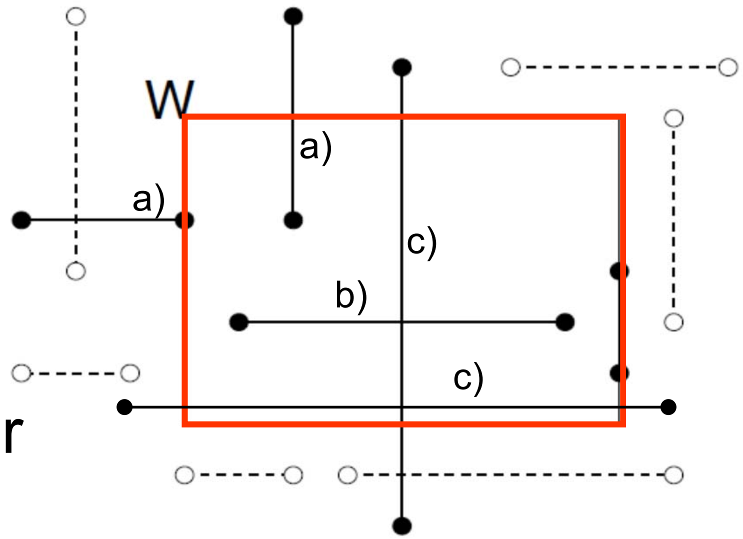


[Vakken]

1. Windowing of axis parallel line segments

Window query

- Given
  – a set of orthogonal line segments S (preprocessed),
  – and orthogonal query rectangle $W = [\, x : x' \,] \circ [\, y : y' \,]$
- Count or report all the line segments of S that intersect W
- Such segments have
  a) 1 endpoint in
  b) 2 end points in – Included
  c) no end point in – Cross over

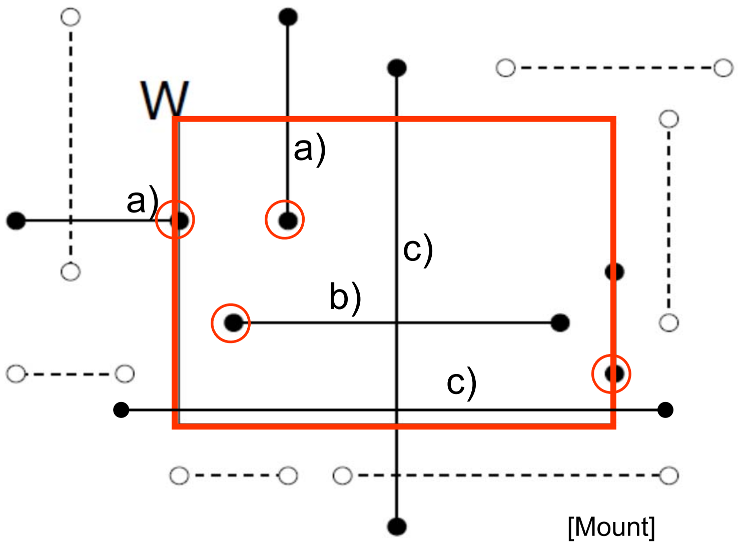


[Mount]

Line segments with 1 or 2 points inside

a) 1 point inside

- Use a range tree (Lesson 3)
- O(n log n) storage
- O($\log^2 n + k$) query time or
- O($\log n + k$) with fractional cascading



b) 2 points inside – as a) 1 point inside

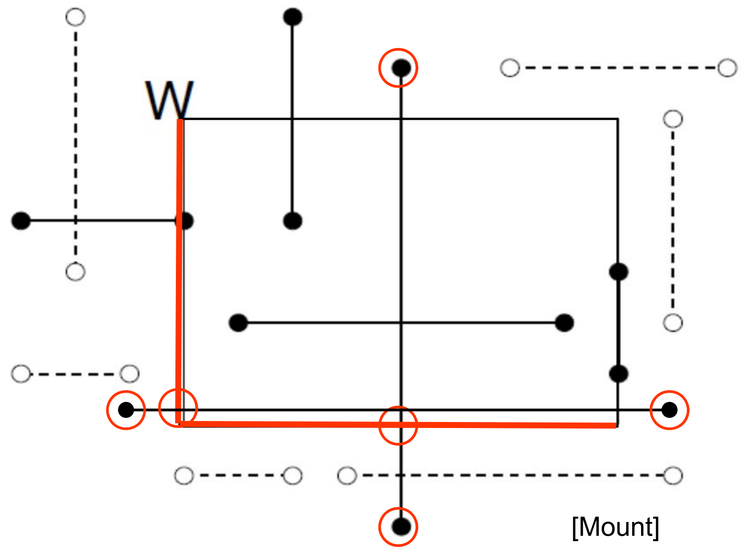
- Avoid reporting twice

1. Mark segment when reported (clear after the query)
2. When end point found, check the other end-point. Report only the leftmost or bottom endpoint

Line segments that cross over the window

c) No points inside

- not detected using a range tree
- Cross the boundary twice or contain one boundary edge
- It is enough to detect segments intersected by the left and bottom boundary edges (not having end point inside)
- For left boundary: Report the segments intersecting vertical query *line segment* (1/ii.)
- Let's discuss vertical query *line* first (1/i.)
- Bottom boundary is rotated 90°



[Mount]

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   ii. Line segment stabbing (*IT* with *range trees*)
   iii. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   – *segment tree*

i. Segment intersected by vertical line->1D

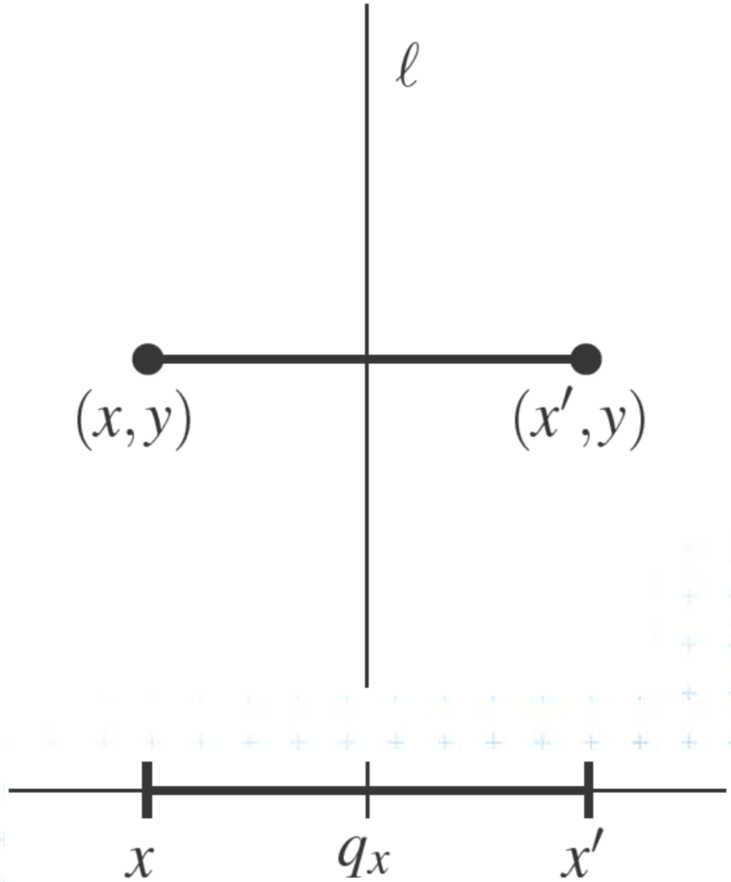
- Query line ℓ := ($x=q_x$)

  Report the segments stabbed by a vertical line

  = 1 dimensional problem

  (ignore y coordinate)

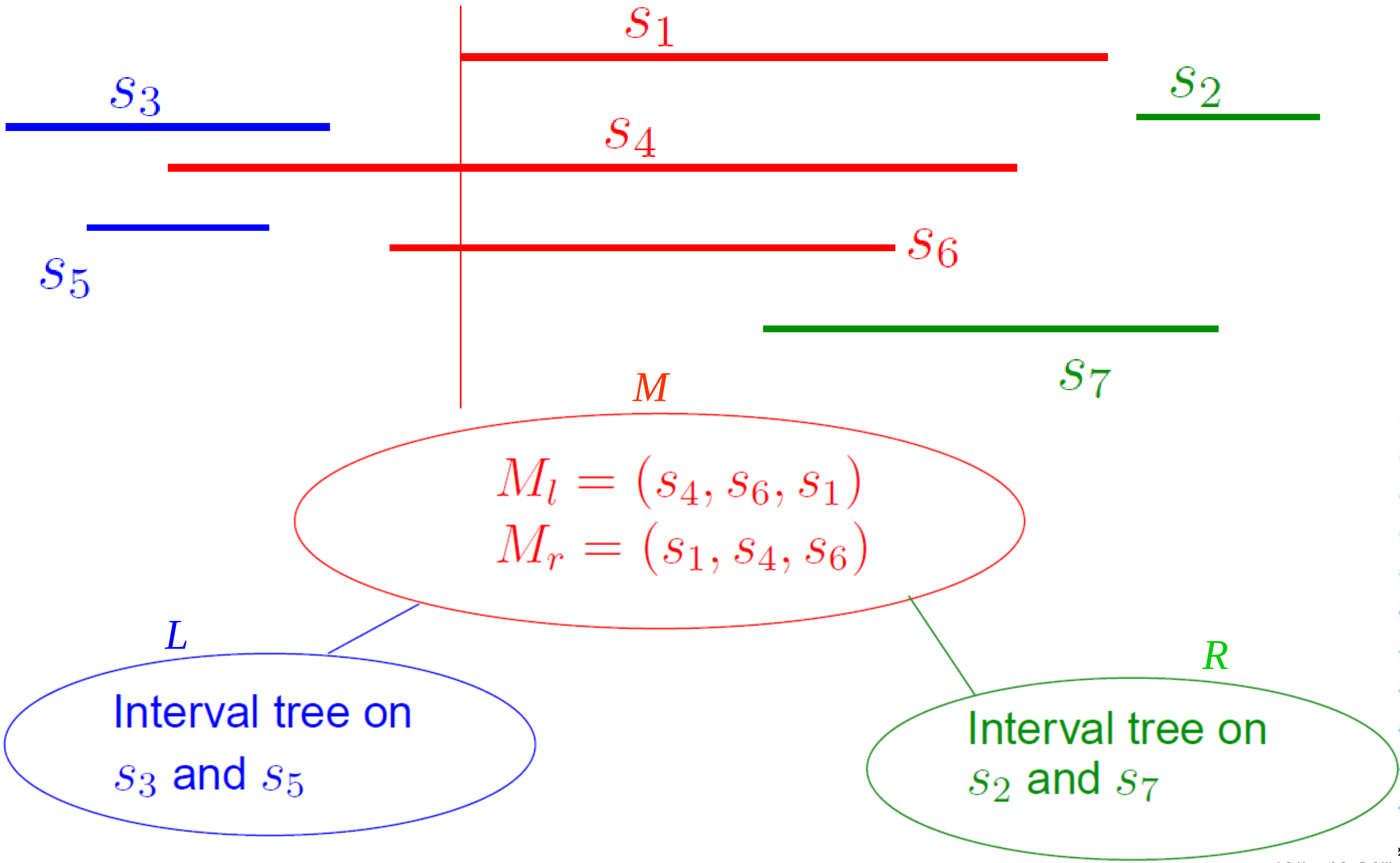
=> Report the interval containing query point q_x



[Mount]

Interval tree principle

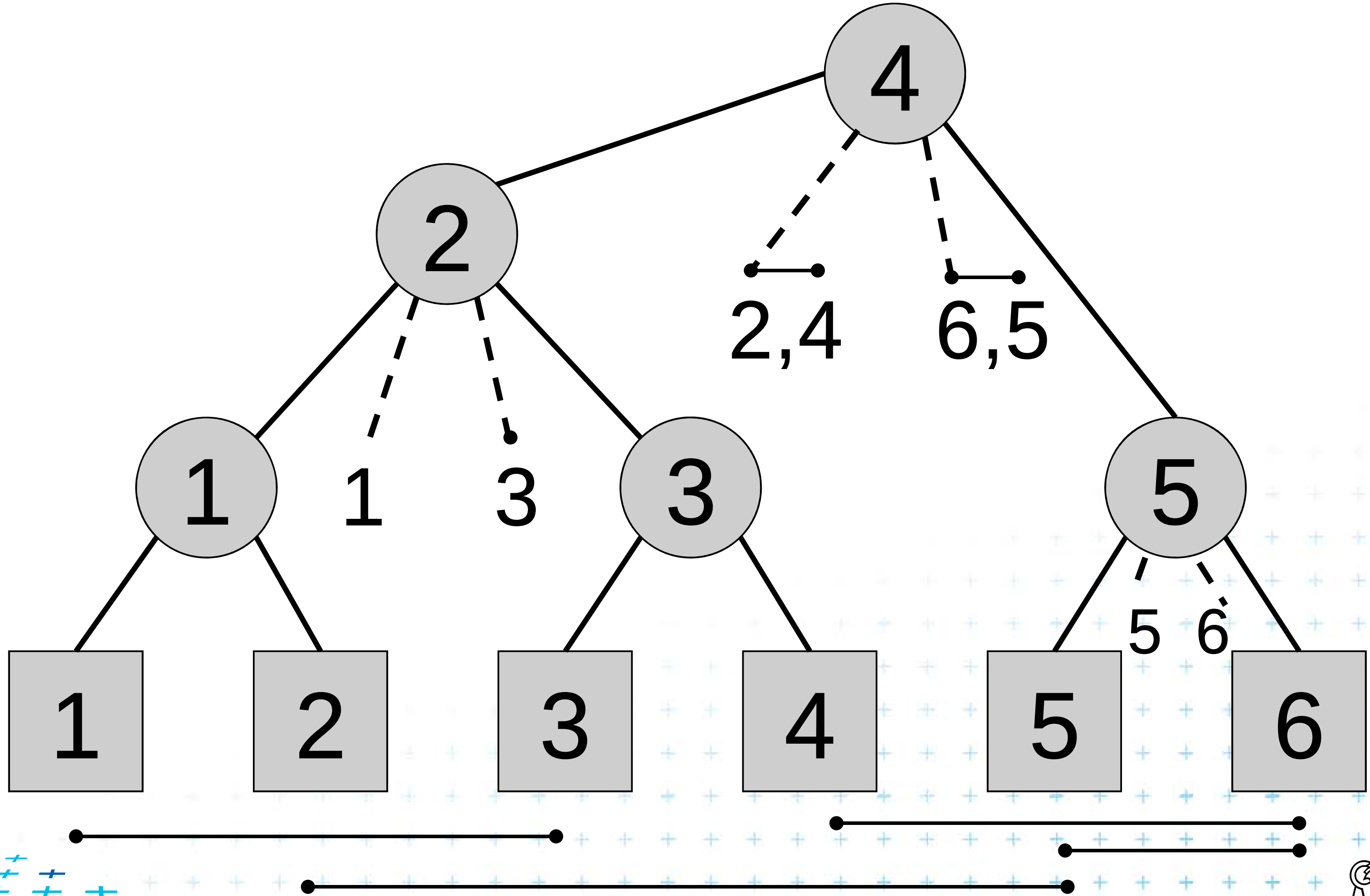
(see lecture 9 - intersections)



[Vigneron]

Static interval tree [Edelsbrunner80]

Tree over sorted segment end-points

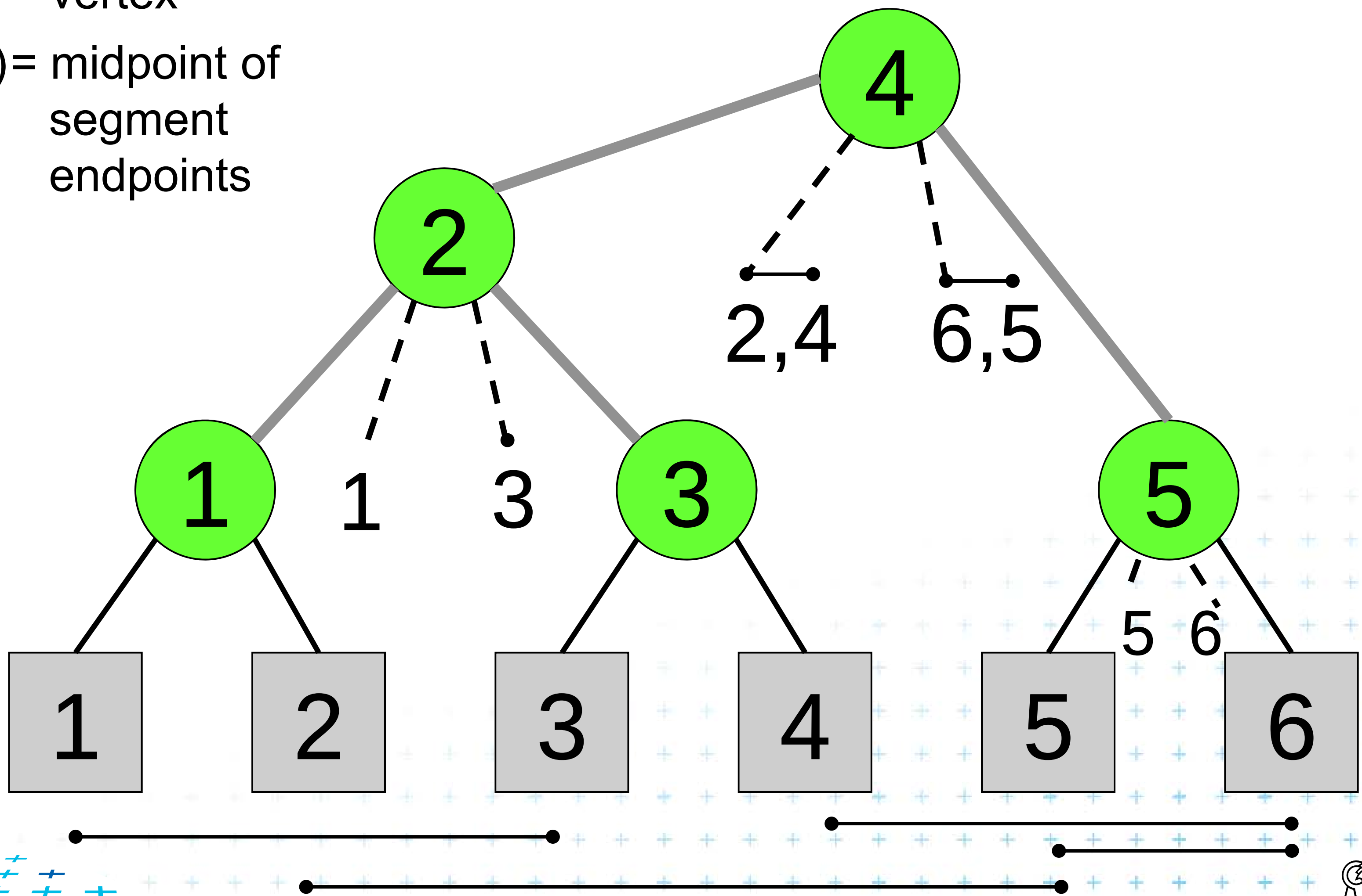


[Kukral]

Primary structure – static tree for endpoints

v = vertex

d(v)= midpoint of segment endpoints



[Kukral]

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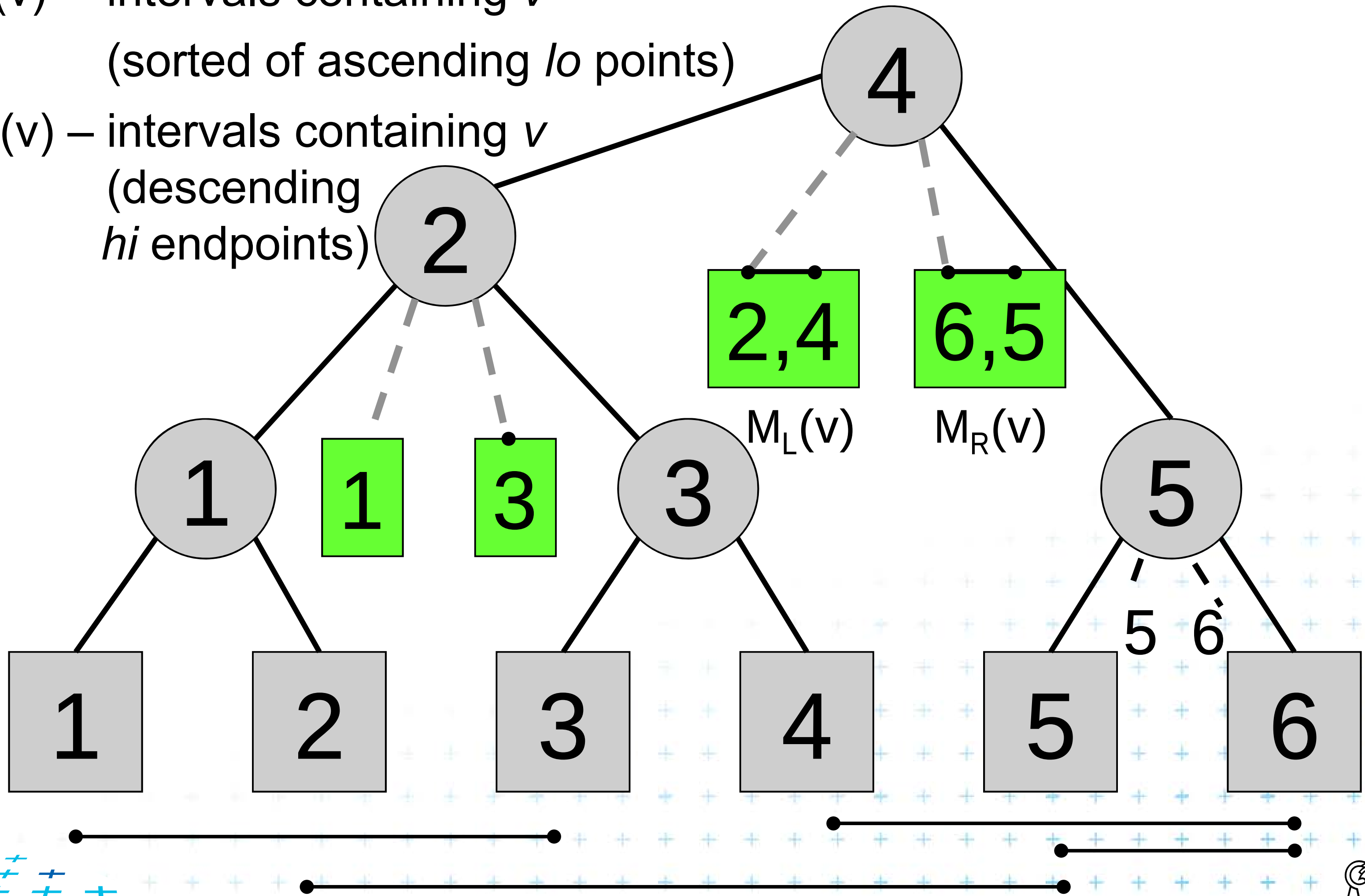
Secondary lists – sorted segments in M

ML(v) – intervals containing *v*

(sorted of ascending *lo* points)

MR(v) – intervals containing *v*

(descending *hi* endpoints)



[Kukral]

Interval tree construction (all intervals at once)

ConstructIntervalTree(*S*) **// Intervals all active – no active lists**

Input: Set S of intervals on the real line – on *x-axis*

Output: The root of an interval tree for *S*

```
if (|S| == 0) return null                               // no more
else
    xMed = median endpoint of intervals in S            // median endpoint
    L = { [xlo, xhi] in S | xhi < xMed }                // left of median
    R = { [xlo, xhi] in S | xlo > xMed }                // right of median
    M = { [xlo, xhi] in S | xlo <= xMed <= xhi }        // contains median
    ML = sort M in increasing order of xlo              // sort M
    MR = sort M in decreasing order of xhi
    t = new IntTreeNode(xMed, ML, MR)                   // this node
    t.left  = ConstructIntervalTree(L)                  // left subtree
    t.right = ConstructIntervalTree(R)                  // right subtree
    return t
```

[Mount]

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Line stabbing query for an interval tree

```
Stab( t, xq)
Input:    IntTreeNode t, Scalar xq
Output:  prints the intersected intervals
if (t == null) return                        // fell out of tree
if (xq < t.xMed)                             // left of median?
    for (i = 0; i < t.ML.length; i++)        // traverse ML
            if (t.ML[i].lo <= xq) print(t.ML[i])   // ..report if in range
            else break                       // ..else done
    stab(t.left, xq)                         // recurse on left
else  // (xq – t.xMed)                       // right of or equal to
median
    for (i = 0; i < t.MR.length; i++) {      // traverse MR
            if (t.MR[i].hi – xq) print(t.MR[i])    // ..report if in range
            else break                       // ..else done
    stab(t.right, xq)                        // recurse on right
```

Note: Small inefficiency for xq == t.xMed – recurse on right [Mount]

Complexity of line stabbing via interval tree

- Construction - O(n log n) time
  - Each step divides at maximum into two halves or less (minus elements of M) => tree height O(log n)
  - If presorted the endpoints in three lists L,R,M then median in O(1) and copy to new L,R,M in O(n)]
- Vertical line stabbing query - O(k + log n) time
  - One node processed in O(1 + k’), k’=reported intervals
  - v visited nodes in O(v + k), k=total reported intervals
  - v = tree height = O(log n)
- Storage - O(n)
  - Tree has O(n) nodes, each segment stored twice (two endpoints)

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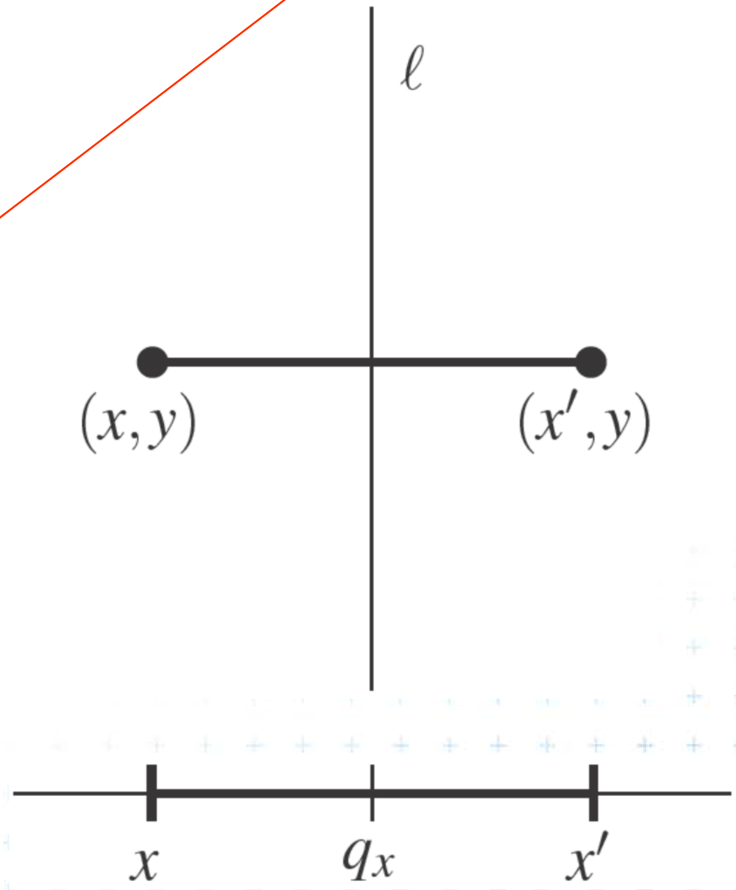
2. Windowing of line segments in general position
   – *segment tree*

i. Segment intersected by vertical line - 1D

- Query line ℓ := ($x = q_x$)
  Report the segments stabbed by a vertical line
  = 1 dimensional problem
  (ignore y coordinate)

=> Report the interval containing query point q_x

DS: Interval tree



[Berg]

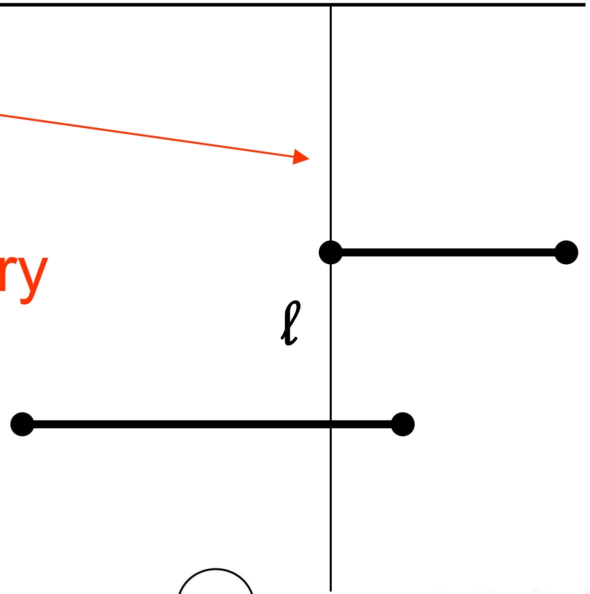
i. Segment intersected by vertical line - 2D

- Query line $\ell := q_x \circ$ [–– : –]

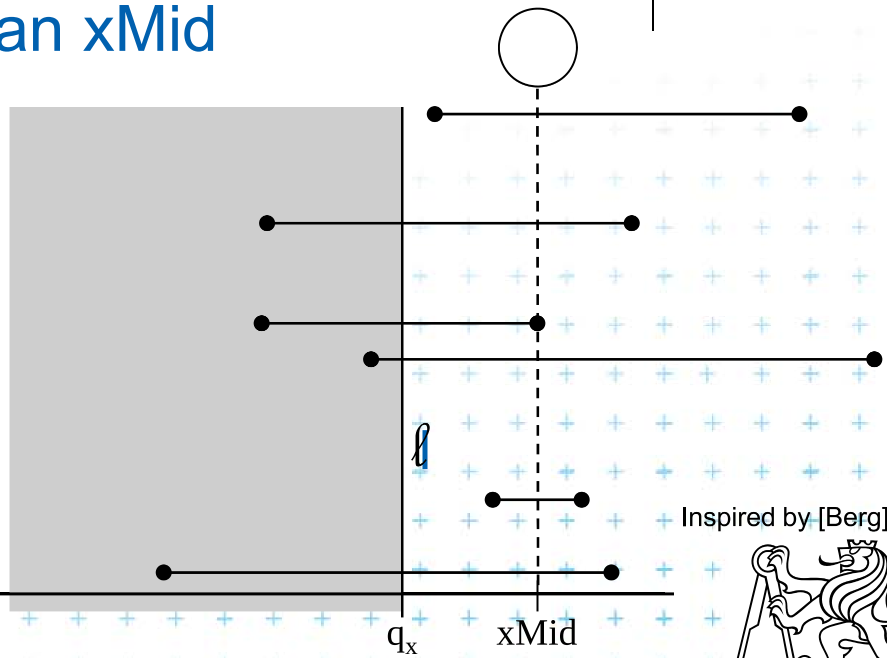
  ;

- Horizontal segment of M stabs the query line ℓ **iff** its left endpoint lies in halph-space

  (–– : q_x] ∘ [- – : –]



- In IT node with stored median xMid report all segments from M
  - whose left point lies in (–– : q_x] if ℓ lies left from xMid
  - whose right point lies in (q_x : +–] if ℓ lies right from xMid



Inspired by [Berg]

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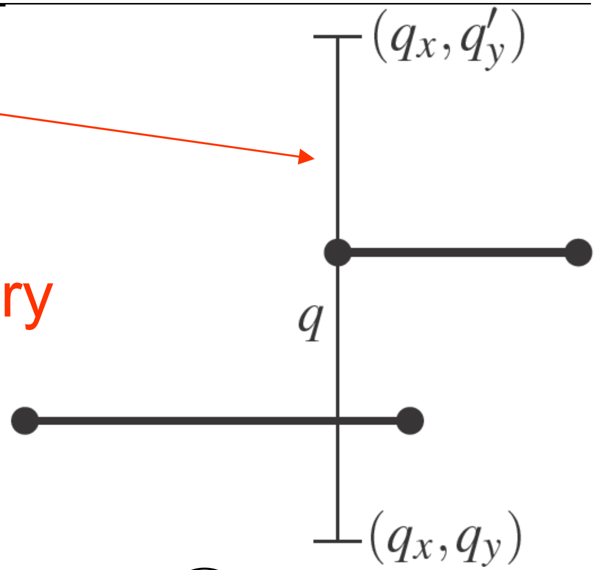
ii. Segment intersected by vertical line segment

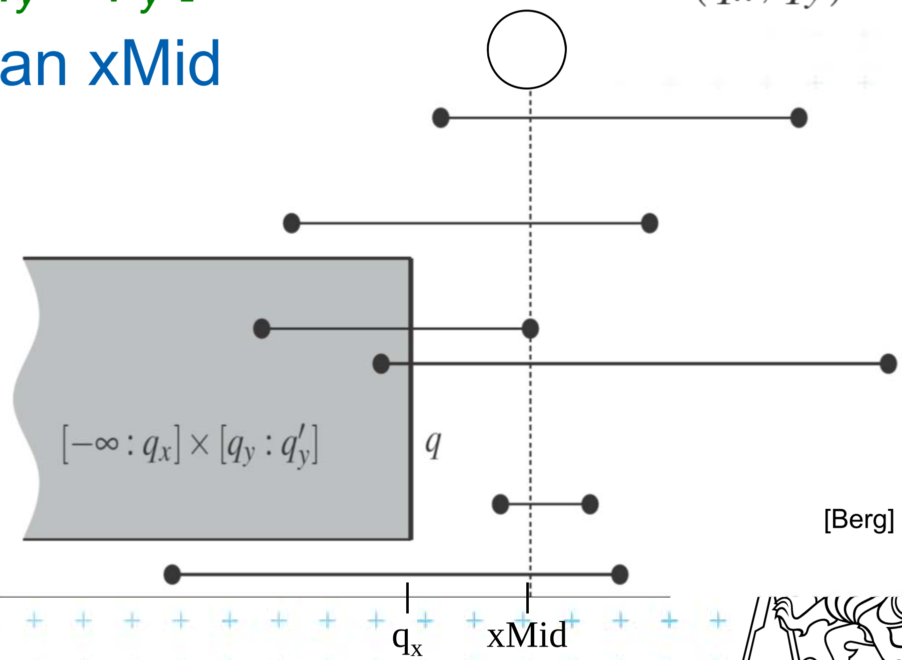
- Query segment $q := q_x \circ [q_y : q'_y]$



- Horizontal segment of *M* stabs the query segment *q* **iff** its left endpoint lies in semi-infinite rectangular region

  $(—— : q_x] \circ [q_y ; q'_y]$

- In IT node with stored median xMid report all segments
  - whose left point lies in $(—— : q_x] \circ [q_y ; q'_y]$ if *q* lies left from xMid
  - whose right point lies in $(q_x : +—] \circ [q_y ; q'_y]$ if *q* lies right from xMid



[Berg]

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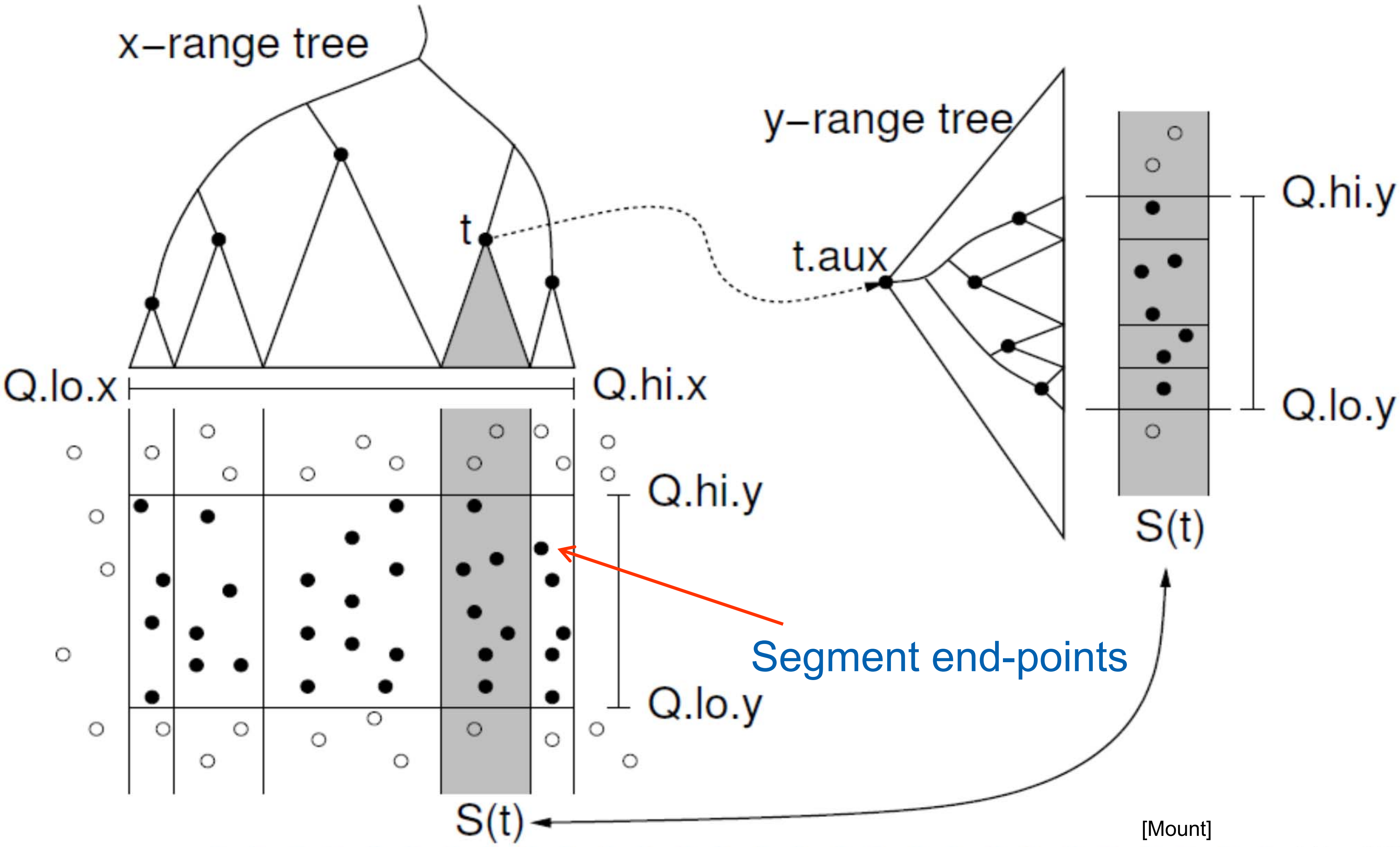
Data structure for endpoints

- Storage of ML and MR
  - Sorted lists not enough for line segments
  - Use two range trees
- Instead O(n) sequential search in ML and MR perform O(log n) search in range tree with fractional cascading

2D range tree (without fractional casc. - see more in Lecture 3)



[Mount]

Complexity of line segment stabbing

- Construction - O(n log n) time
  - Each step divides at maximum into two halves L,R or less (minus elements of M) => tree height O(log n)
  - If the range trees are efficiently build in O(n)
- Vertical line segment stab. q. - O(k + log^2 n) time
  - 2D range tree search with FC
  - One node processed in O(log n + k'), k'=reported inter.
  - v-visited nodes in O(v log n + k), k=total reported inter.
  - v = interval tree height = O(log n)
  - O(k + log^2 n) time - range tree with fractional cascading
  - O(k + log^3 n) time - range tree without fractional casc.
- Storage - O(n log n)
  - Dominated by the range trees

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iii. Priority search trees

[McCreight85]

- Priority search trees – in case c) on slide 8
  – Exploit the fact that query rectangle in range queries is unbounded
  – Can be used as secondary data structures for both left and right endpoints (ML and MR) of segments (intervals) in nodes of interval tree
  – Improve the storage to O(*n*) for horizontal segment intersection with window edge (Range tree has O(*n* log *n*))
- For cases a) and b) - O(*n* log *n*) remains
  – we need range trees for windowing segment endpoints

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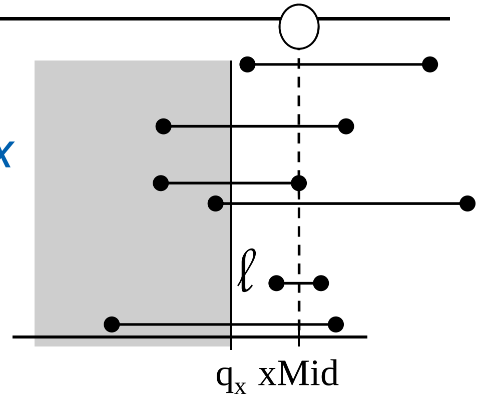
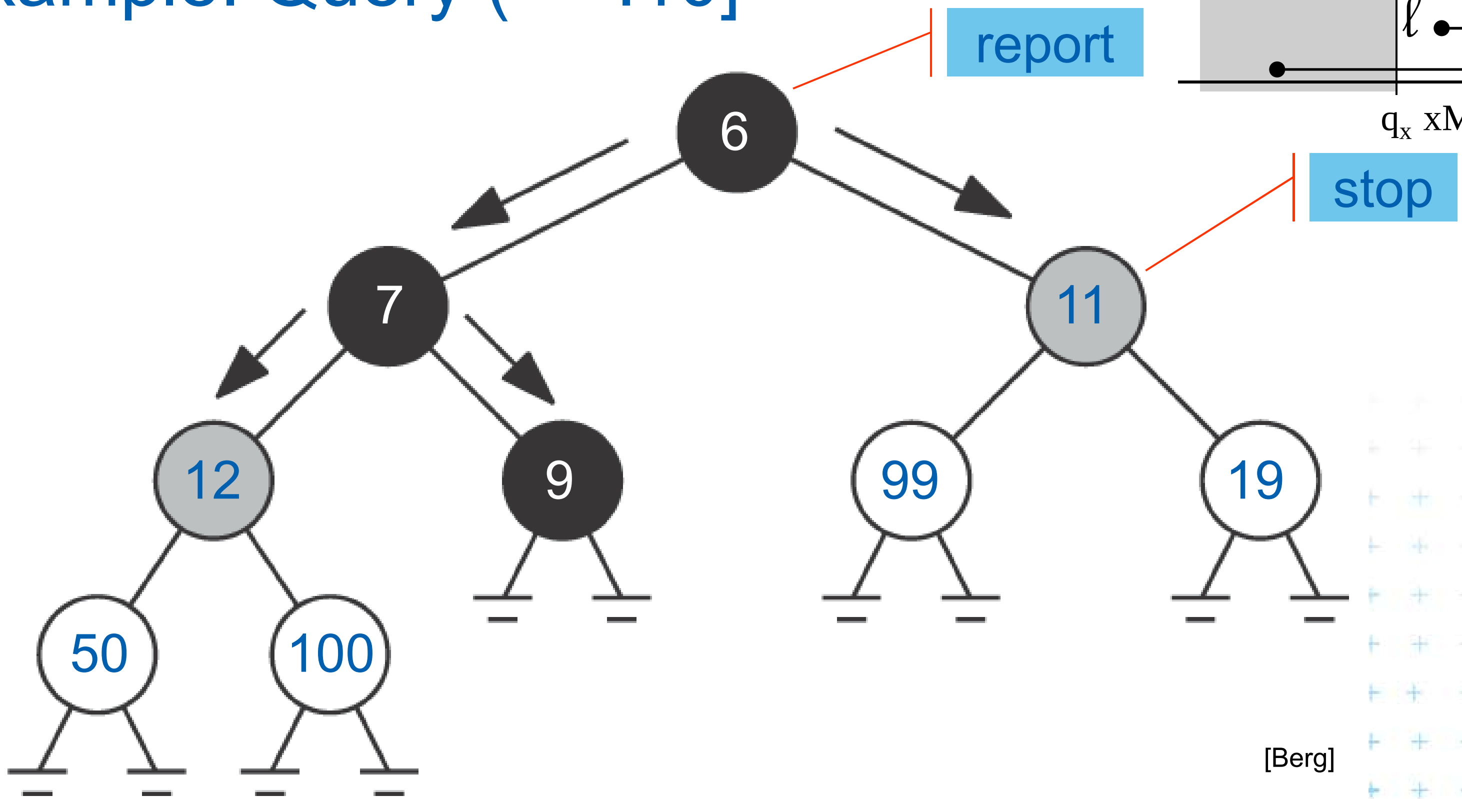
Rectangular range queries variants

- Let $P = \{ p_1, p_2, \ldots, p_n \}$ is set of points in plane
- Goal: rectangular range queries of the form
  (— : q_x] ° [q_y ; q'_y]
- In 1D: search for nodes v with v_x µ (— : q_x]
  - range tree   O(log n + k) time
  - ordered list   O(1 + k) time
    (start in the leftmost, stop on v with v_x>q_x)
  - use heap   O(1 + k) time
    (traverse all children, stop when v_x>q_x)
- In 2D – use heap for points with x µ (— : q_x]
  + integrate information about y-coordinate

Heap for 1D unbounded range queries

- Traverse all children, stop when $v_x > q_x$
- Example: Query (— :10]





[Berg]

Priority search tree (PST)

- Heap in 2D can incorporate info about both *x,y*
  - BST on *y*-coordinate (horizontal slabs) ~ range tree
  - Heap on *x*-coordinate (minimum *x* from slab along *x*)
- If *P* is empty, PST is empty leaf
- else
  - p_{min} = point with smallest x-coordinate in *P* --- a heap root
  - y_{med} = *y*-coord. median of points $P \setminus \{p_{min}\}$ --- BST root
  - P_{below} := { *p* µ $P \setminus \{p_{min}\}$: p_y ¸ y_{med}}
  - P_{above} := { *p* µ $P \setminus \{p_{min}\}$: $p_y > y_{med}$}
- Point p_{min} and scalar y_{med} are stored in the PST root
- The left subtree is PST of P_{below}
- The right subtree is PST of P_{above}

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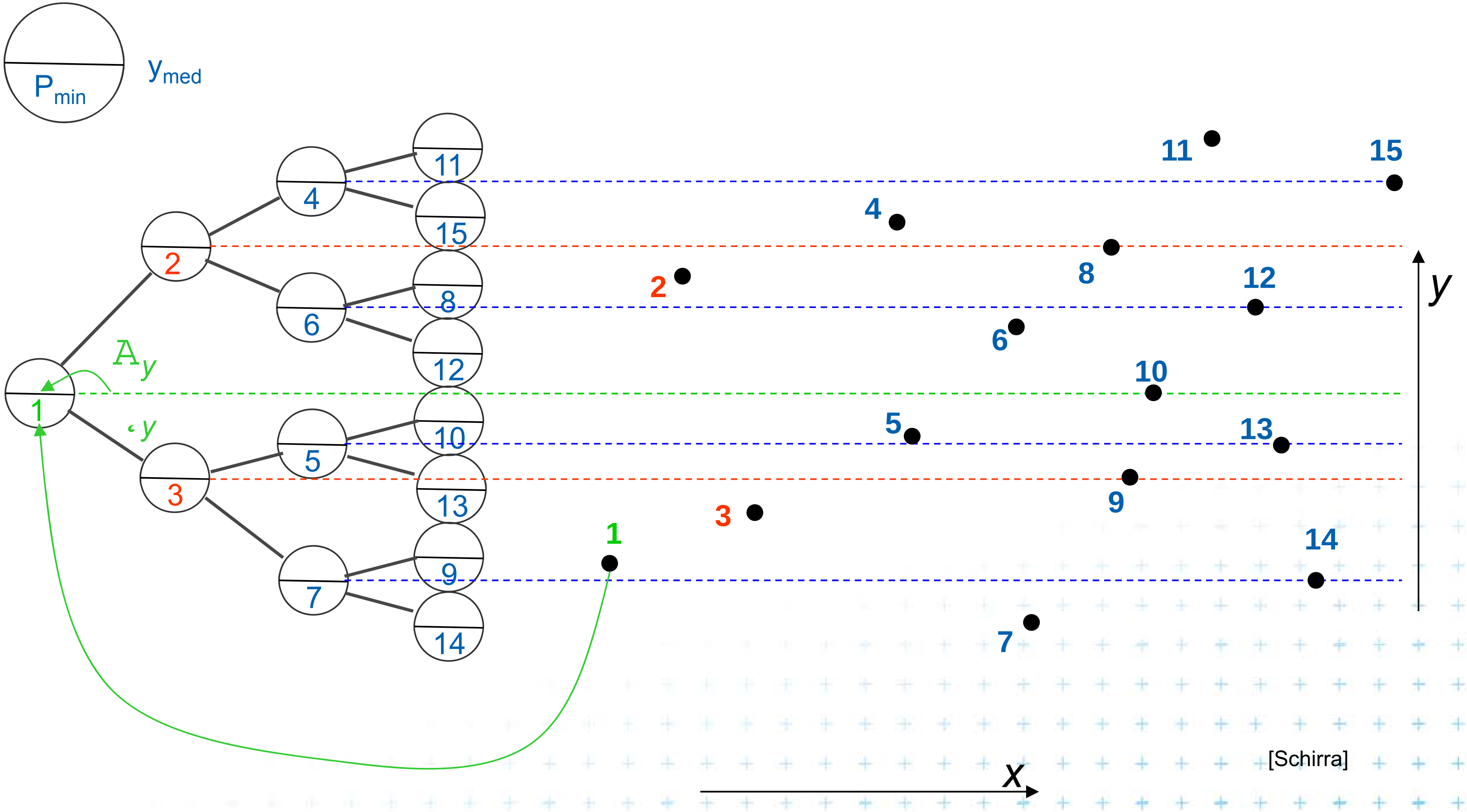
Priority search tree construction example



[Schirra]

Priority search tree definition

PrioritySearchTree(*P*)

Input: set *P* of points in plane

Output: priority search tree *T*

1. if $P = \emptyset$ then PST is an empty leaf
2. else
3. p_{min} = point with smallest x-coordinate in P // heap on x root
4. y_{med} = y-coord. median of points $P \setminus \{p_{min}\}$ // BST on y root
5. Split points $P \setminus \{p_{min}\}$ into two subsets – according to y_{med}
6. $P_{below} := \{ p \in P \setminus \{p_{min}\} : p_y \leq y_{med} \}$
7. $P_{above} := \{ p \in P \setminus \{p_{min}\} : p_y > y_{med} \}$
8. T = newTreeNode()
9. $T.p = p_{min}$ // point [x, y]
10. $T.y = y_{mid}$ // skalar
11. *T.left* = PrioritySearchTree(P_{below})
12. *T.rigft* = PrioritySearchTree(P_{above})

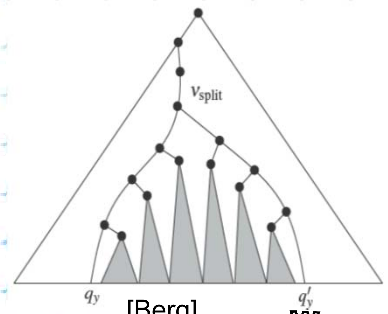
Notation in alg:
… p(v)
… y(v)
… lc(v)
… rc(v)

13. O(*n* log *n*) , but O(*n*) if presorted on *y*-coordinate and bottom up

Query Priority Search Tree

QueryPrioritySearchTree(T**, (–— :** q_x**] – [**q_y **;** q'_y **])**
Input: A priority search tree and a range, unbounded to the left
Output: All points lying in the range

1. Search with q_y and q'_y in T // BST on *y*-coordinate – select *y* range
   Let v_{split} be the node where the two search paths split (split node)
2. for each node v on the search path of q_y or q'_y // points along the paths
3. if $p(v)$ µ (–— : q_x] ∘ [q_y ; q'_y] then report $p(v)$ // starting in tree root
4. for each node v on the path of q_y in the left subtree of v_{split} // inner trees
5. if the search path goes left at v
6. ReportInSubtree($rc(v)$, q_x) // report right subtree
7. for each node v on the path of q'_y in right subtree of v_{split}
8. if the search path goes right at v
9. ReportInSubtree($lc(v)$, q_x) // rep. left subtree



[Berg]

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Reporting of subtrees between the paths

ReportInSubtree(*v*, q_x)

Input: The root *v* of a subtree of a priority search tree and a value q_x.

Output: All points in the subtree with *x*-coordinate at most q_x.

1. if *v* is not a leaf and *x*(*p*(*v*)) ˛ q_x // *x* µ (— : q_x] -- heap condition
2. Report *p*(*v*).
3. ReportInSubtree(*lc*(*v*), q_x)
4. ReportInSubtree(*rc*(*v*), q_x)

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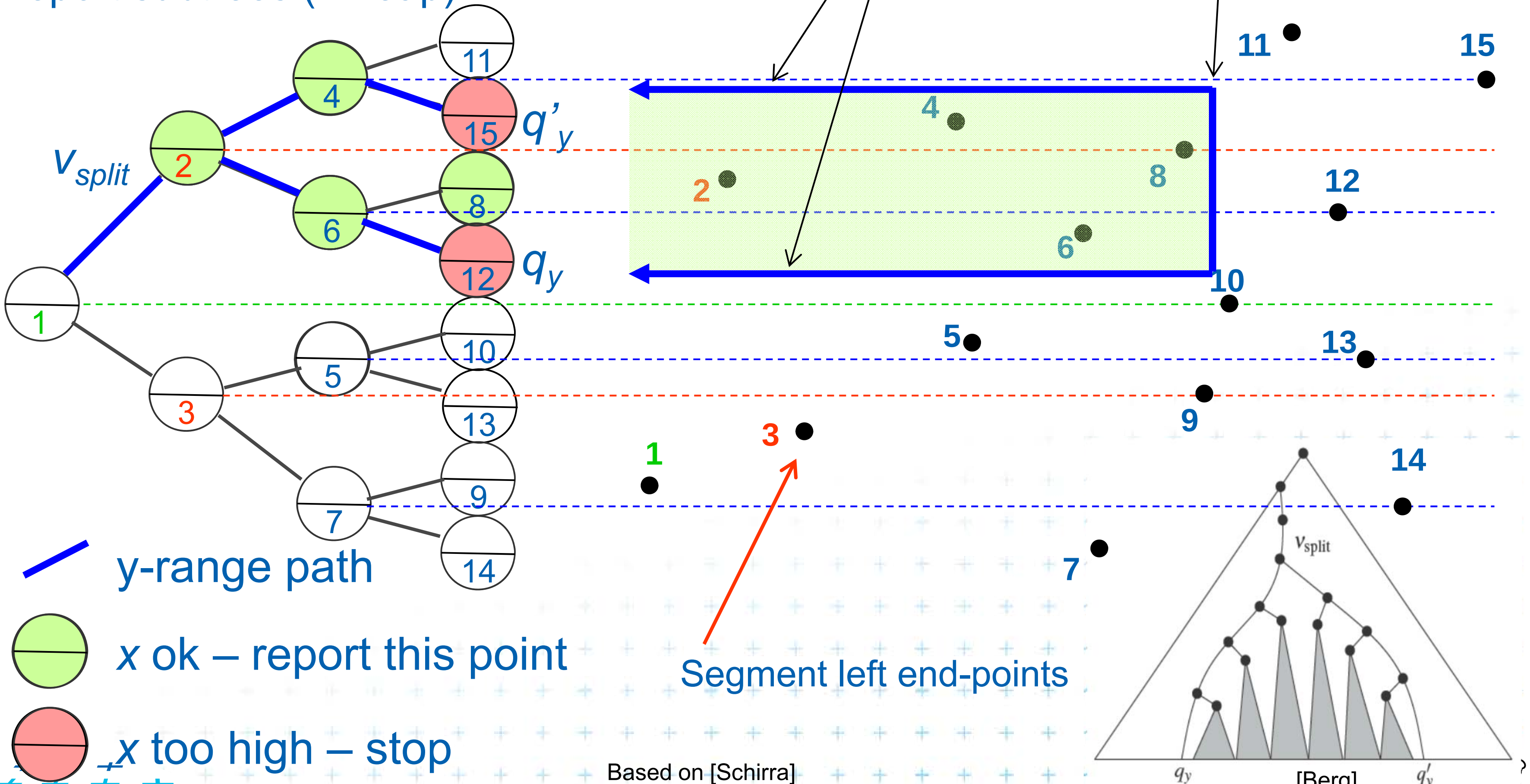
Priority search tree query

1. select *y* range (y-BVS~ 1D range tree)
2. report points on paths (x-heap)
3. report subtrees (x-heap)



Based on [Schirra] [Berg]

Priority search tree complexity

For set of *n* points in the plane

- Build   O(*n* log *n*)
- Storage   O(*n*)
- Query   O(*k* + log *n*)
  - points in query range $(—\ : q_x] \circ [q_y\ ;\ q'_y\ ])$
  - *k* is number of reported points

- Use PST as associated data structure for interval trees for storage of M

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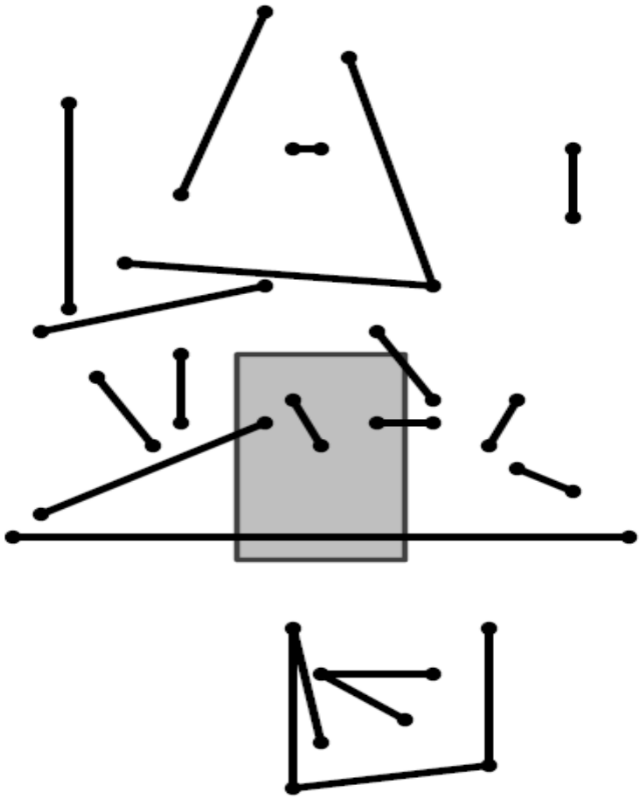
2. Windowing of line segments in general position



[Vakken]

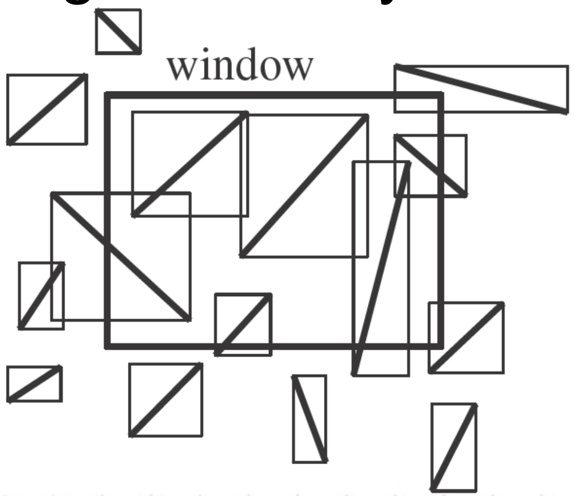
Windowing of arbitrary oriented line segments

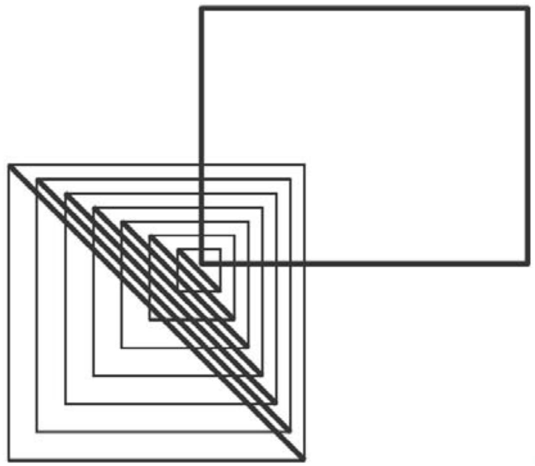
- Two cases of intersection

  a,b) Endpoint inside the query window => range tree

  c) Segment intersects side of query window => ???

- Intersection with BBOX (segment bounding box)?
  - Intersection with 4n sides
  - But segments may not intersect the window –> query y





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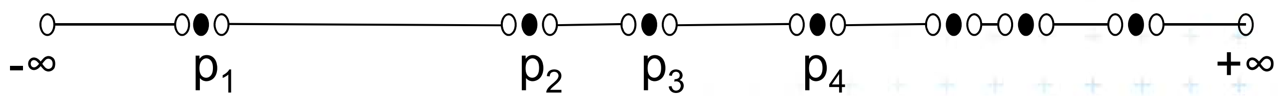
Segment tree

[Bentley, 1977]

- Exploits locus approach
  - Partition parameter space into regions of same answer
  - Localization of such region = knowing the answer
- For given set *S* of *n* intervals (segments) on real line
  - Finds *m* elementary intervals (induced by interval end-points)
  - Partitions 1D parameter space into these elementary intervals



$$(-\infty : p_1), [p_1 : p_1], (p_1 : p_2), [p_2 : p_2], \ldots,$$
$$(p_{m-1} : p_m), [p_m : p_m], (p_m : +\infty)$$

  - Stores intervals s_i with the elementary intervals
  - Reports the intervals s_i containing query point q_x.

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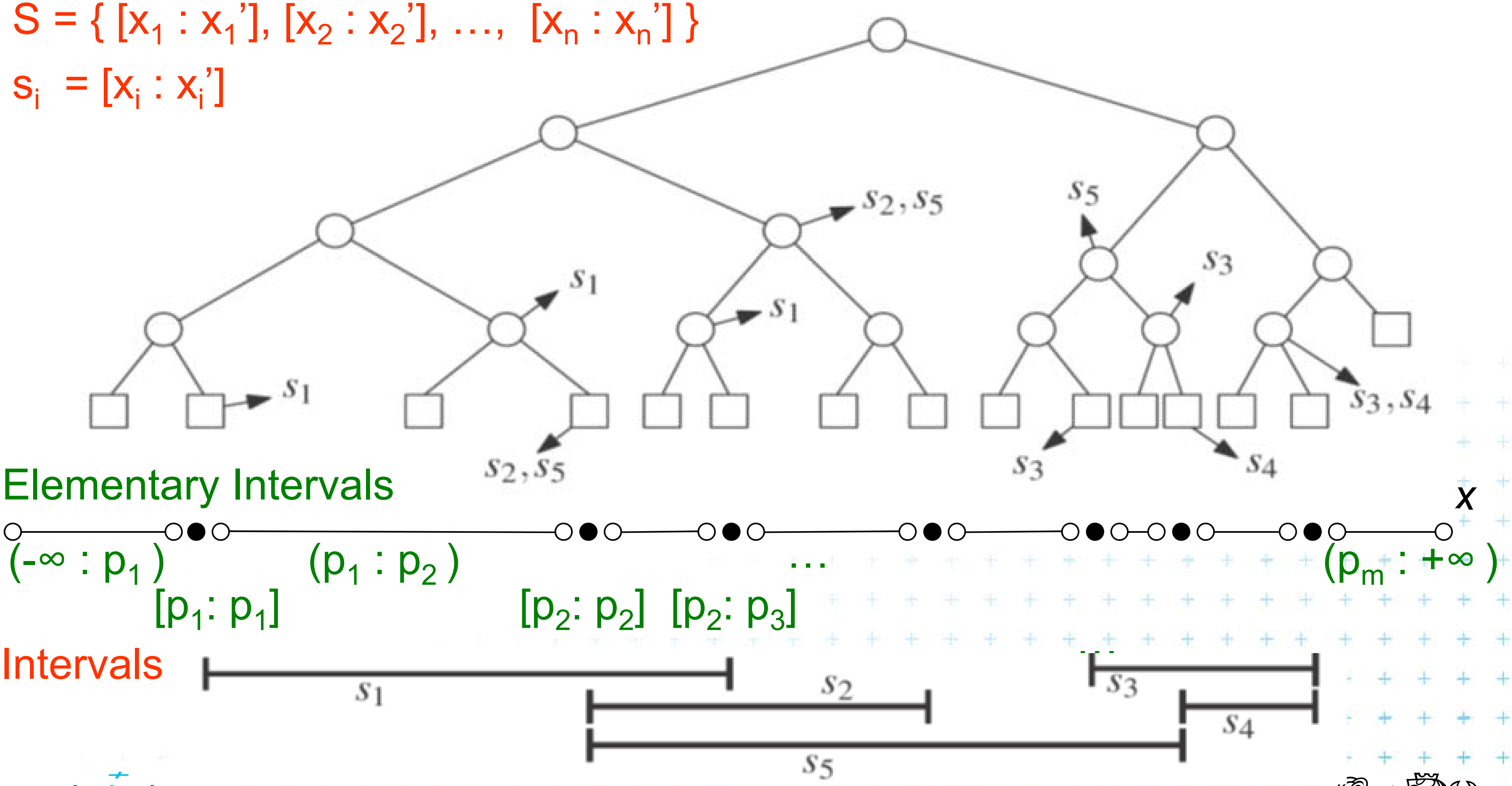
Segment tree example

Intervals

$S = \{ [x_1 : x_1'], [x_2 : x_2'], \ldots, [x_n : x_n'] \}$

$s_i = [x_i : x_i']$



[Berg]

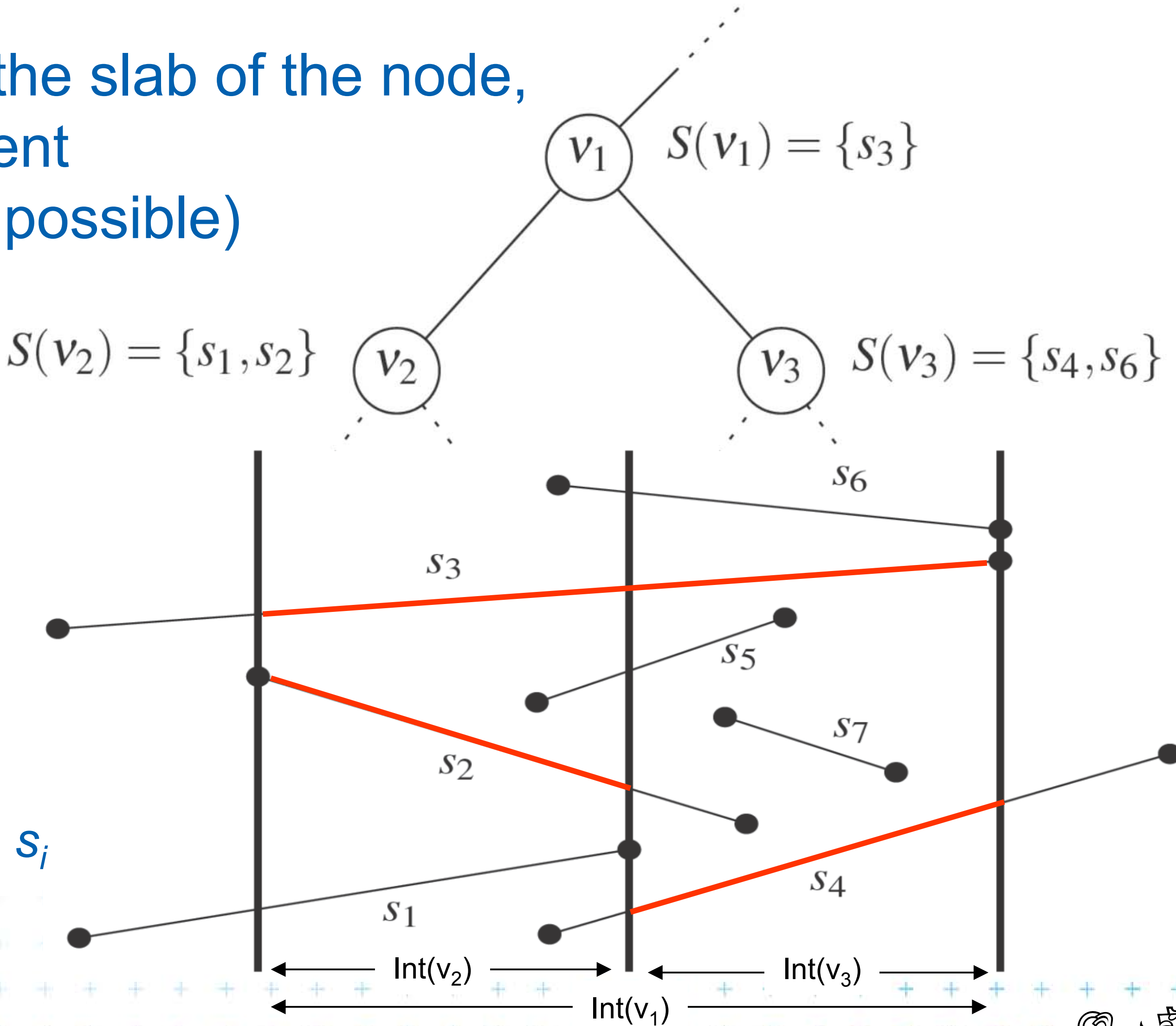
Segment tree definition

Segment tree

- Skeleton is a balanced binary tree *T*
- Leaves ~ elementary intervals Int(v)
- Internal nodes *v*
  ~ union of elementary intervals of its children
  - Store: 1. interval Int(v) = union of elementary intervals of its children
    2. canonical set *S*(*v*) of intervals $[x : x']$ µ *S* (segments s_i)
  - Holds Int(*v*) Å $[x : x']$ and Int(parent(*v*)] † $[x : x']$
    (node interval is not larger than a segment)
  - Intervals $[x : x']$ are stored as high as possible, such that Int(*v*) is completely contained in the segment

Segments span the slab

Segments span the slab of the node,
but not of its parent
(stored as up as possible)



Int(v_j) Å s_i

and

Int(parent(v_j)] † s_i

[Berg]

Query segment tree

QuerySegmentTree*(v*, q_x)
Input: The root of a (subtree of a) segment tree and a query point q_x
Output: All intervals in the tree containing q_x.

1. Report all the intervals s_i in $S(v)$. // current node
2. **if** *v* is not a leaf
3. if q_x µ Int(*lc*(*v*)) // go left
4. QuerySegmentTree(*lc*(*v*), q_x)
5. else // or go right
6. QuerySegmentTree(*rc*(*v*), q_x)

Query time O(log *n* + *k*), where *k* is the number of reported intervals
Height O(log *n*), O(1 + k_v) for node

Segment tree construction

ConstructSegmentTree(S)
Input: Set of intervals S - segments
Output: segment tree

1. Sort endpoints of segments in S -> get elemetary intervals …O(n log n)
2. Construct a binary search tree T on elementary intervals …O(n) (bottom up) and determine the interval Int(v) it represents
3. Compute the canonical subsets for the nodes (lists of their segments):
4. v = root(T)
5. for all segments $s_i = [x : x']$ µ S
6. InsertSegmentTree(v, $[x : x']$)

Segment tree construction – interval insertion

```
InsertSegmentTree( v, [x : x'] )
Input:   The root of (a subtree of) a segment tree and an interval.
Output:  The interval will be stored in the subtree.
if Int(v) Å [ x : x' ]                          // Int(v) contains s_i = [ x : x' ]
    store [ x : x' ] at v
else if Int( lc(v) ) ∩ [ x : x' ] û þ
        InsertSegmentTree( lc(v), [x : x' ] )
      if Int( rc(v) ) ∩ [ x : x' ] û þ
        InsertSegmentTree(rc(v), [x : x' ] )
```

One interval is stored at most twice in one level =>

Single interval insert $O(\log n)$, insert n intervals

Construction total $O(n \log n)$

Storage $O(n \log n)$

Tree height $O(\log n)$, name stored max 2x in one level

Storage total $O(n \log n)$ – see next slide

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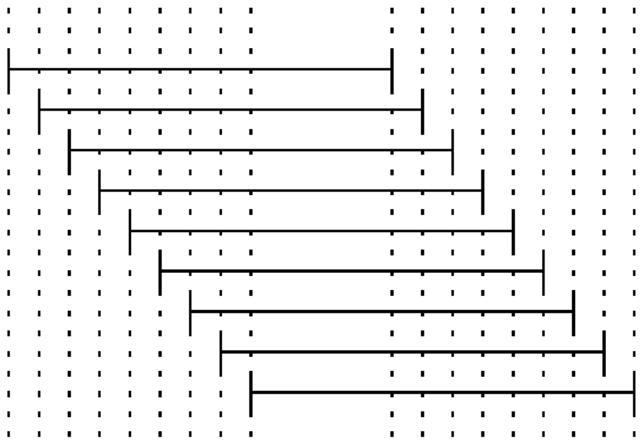
Space complexity - notes



Worst case – $O(n^2)$ segments in leaf

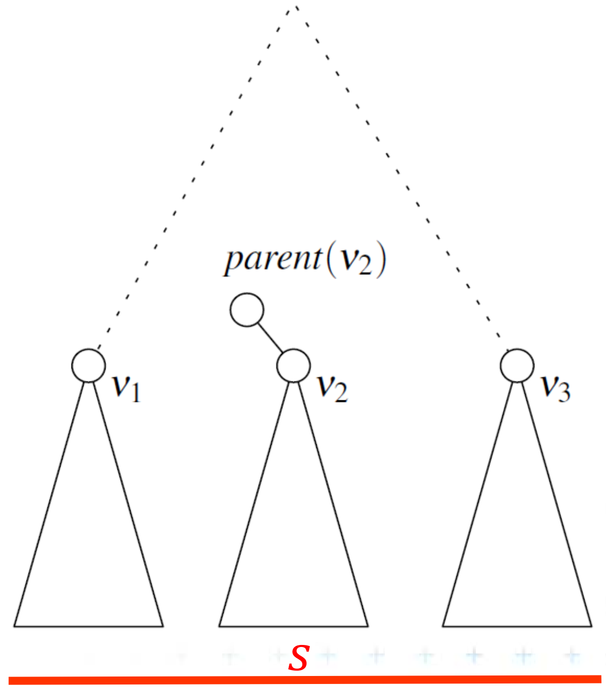
But

Store segments as high, as possible

Segment max 2 times in one level

$\max 4n + 1$ elementary intervals (leaves)
$\Rightarrow O(n)$ space for the tree

$\Rightarrow O(n \log n)$ space for interval names



s covered by v_1 and v_3

$\Rightarrow v_2$covered - $Int(v_2) \in s$

$\Rightarrow$ segment must be stored in parent(v_2)

Segment tree complexity

A segment tree for set S of n intervals in the plane,

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log n)$
  - Report all intervals that contain a query point
  - k is number of reported intervals

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Segment tree versus Interval tree

- Segment tree
  - O(*n* log *n*) storage x O(*n*) of Interval tree
  - But returns exactly the intersected segments s_i, interval tree must search the lists ML and/or MR
- Good for
  1. extensions (allows different structuring of intervals)
  2. stabbing counting queries
     – store number of intersected intervals in nodes
     – O(n) storage and O(log *n*) query time = optimal
  3. higher dimensions – multilevel segment trees
     (Interval and priority search trees do not exist in ^dims)

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Talk overview

1. Windowing of axis parallel line segments in 2D (variants of *interval tree - IT*)
   1. Line stabbing (*IT* with *sorted lists*)
   2. Line segment stabbing (*IT* with *range trees*)
   3. Line segment stabbing (*IT* with *priority search trees*)
2. Windowing of line segments in general position
   - *segment tree*
   - the algorithm

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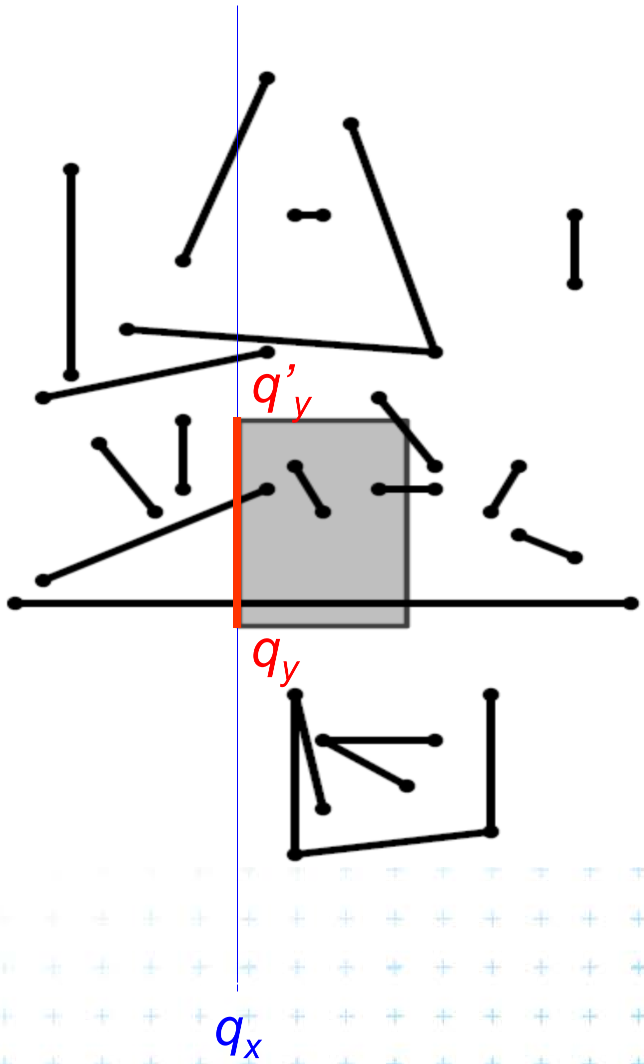
2. Windowing of line segments in general position

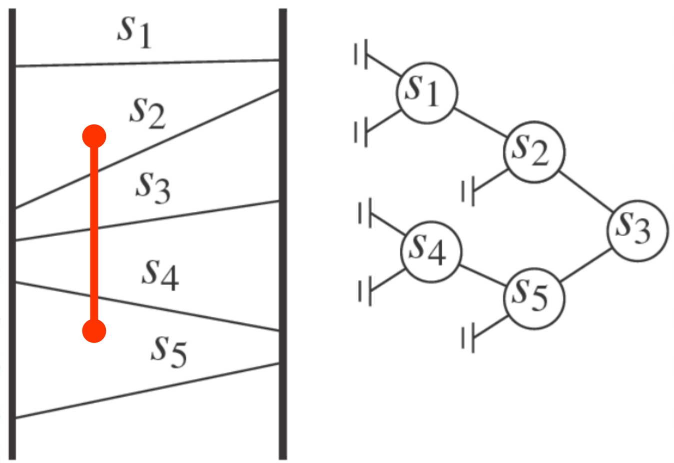


[Vakken]

Windowing of arbitrary oriented line segments

- Let S be a set of arbitrarily oriented line segments in the plane.
- Report the segments intersecting a vertical query segment $q := q_x \circ [q_y : q'_y]$
- Segment tree T on x intervals of segments in S
  - node v of T corresponds to vertical slab Int(v) ∘ (-— : –)
  - segments span the slab of the node, but not of its parent
  - segments do not intersect

  => segments in the slab (node) can be vertically ordered – BST



[Berg]

Segments between vertical segment endpoints

- Segments (in the slab) do not mutually intersect
  => segments can be vertically ordered and stored in BST

  – Each node v of the x segment tree has an associated y BST
  – BST $T(v)$ of node v stores the canonical subset $S(v)$ according to the vertical order
  – Intersected segments can be found by searching $T(v)$ in $O(k_v + \log n)$, k_v is the number of intersected segments

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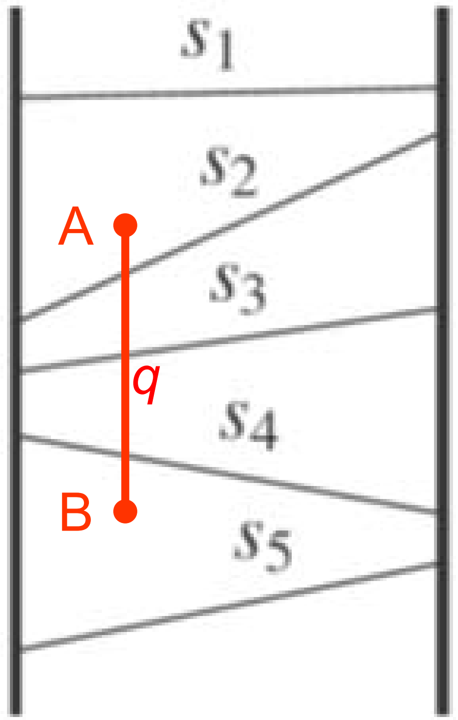
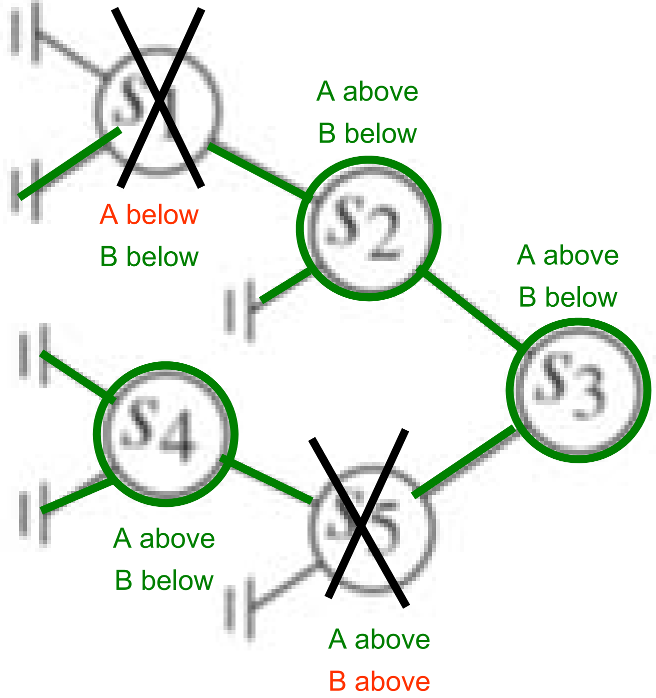
Segments between vertical segment endpoints

- Segment *s* is intersected by vert.query segment *q* iff
  - The lower endpoint (B) of *q* is below *s* and
  - The upper endpoint (A) of *q* is above *s*





[Berg]

Windowing of arbitrary oriented line segments complexity

Structure associated to node (BST) uses storage linear in the size of $S(v)$

- Build $O(n \log n)$
- Storage $O(n \log n)$
- Query $O(k + \log^2 n)$
  – Report all segments that contain a query point
  – k is number of reported segments

Windowing of line segments in 2D – conclusions

Construction: all variants O(n logn)

1. Axis parallel	Search	Memory
i. Line (*sorted lists*)	$O(k + \log n)$	$O(n)$
ii. Segment (*range trees*)	$O(k + \log^2 n)$	$O(n \log n)$
iii. Segment (*priority s. tr.*)	$O(k + \log n)$	$O(n)$

2. In general position

– *segment tree*	$O(k + \log n)$	$O(n \log n)$

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