DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

# INTERSECTIONS OF LINE SEGMENTS AND POLYGONS


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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start


**Based on [Berg], [Mount], [Kukral], and [Drtina]**


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# Talk overview

- Intersections of line segments (Bentley-Ottmann)
  - Motivation
  - Sweep line algorithm recapitulation
  - Sweep line intersections of line segments
- Intersection of polygons or planar subdivisions
  - See assignment [21] or [Berg, Section 2.3]
- Intersection of axis parallel rectangles
  - See assignment [26]

# Geometric intersections – what are they for?

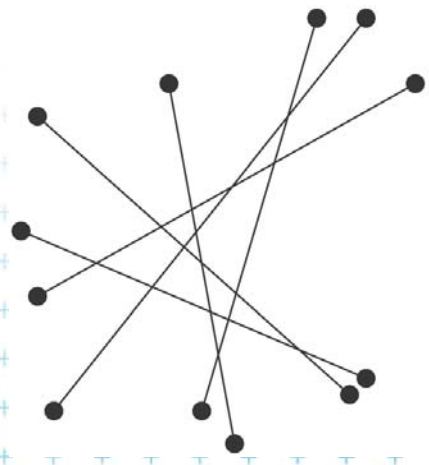
One of the most basic problems in computational geometry

- Solid modeling
  - Intersection of object boundaries in CSG
- Overlay of subdivisions, e.g. layers in GIS
  - Bridges on intersections of roads and rivers
  - Maintenance responsibilities (road network X county boundaries)
- Robotics
  - Collision detection and collision avoidance
- Computer graphics
  - Rendering via ray shooting (intersection of the ray with objects)
- …

# Line segment intersection

- Intersection of complex shapes is often reduced to simpler and simpler intersection problems
- Line segment intersection is the most basic intersection algorithm
- Problem statement:

  Given *n* line segments in the plane, report all points where a pair of line segments intersect.
- Problem complexity
  - Worst case – $I = O(n^2)$ intersections
  - Practical case – only some intersections
  - Use an output sensitive algorithm
    - $O(n \log n + I)$ optimal randomized algorithm
    - $O(n \log n + I \log n)$ sweep line algorithm - %



[Berg]

# Plane sweep line algorithm recapitulation

- Horizontal line (sweep line, *scan line*) $\ell$ moves top-down (or vertical line: left to right) over the set of objects
- The move is not continuous, but $\ell$ jumps from one event point to another
  - Event points are in priority queue or sorted list (~y)
  - The (left) top-most event point is removed first
  - New event points may be created (usually as interaction of neighbors on the sweep line) and inserted into the queue
- Scan-line status
  - Stores information about the objects intersected by $\ell$
  - It is updated while stopping on event point

# Line segment intersection - Sweep line alg.

- Avoid testing of pairs of segments far apart
- Compute intersections of neighbors on the sweep line only
- O($n$ log $n$ + $I$ log $n$ ) time in O($n$) memory
  - 2$n$ steps for end points,
  - $I$ steps for intersections,
  - log $n$ search the status tree

- Ignore “nasty cases” (most of them will be solved later on)
  - No segment is parallel to the sweep line
  - Segments intersect in one point and do not overlap
  - No three segments meet in a common point

# Line segment intersections

- *Status* = ordered sequence of segments intersecting the sweep line $\ell$
- *Events* (waiting in the priority queue)

  = points, where the algorithm actually does something

  – Segment *end-points*

    • known at algorithm start

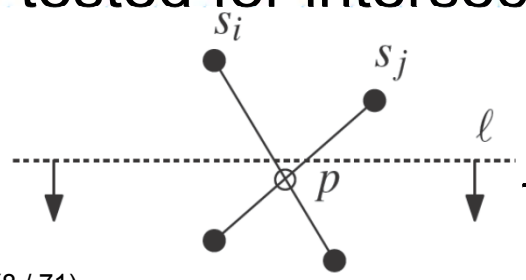
  – Segment *intersections* between neighboring segments along SL

    • Discovered as the sweep executes

# Detecting intersections

- Intersection events must be detected and inserted to the event queue before they occur
- Given two segments *a, b* intersecting in a point *p*, there must be a placement of sweep line $\ell$ prior to *p*, such that segments *a, b* are adjacent along $\ell$ (only adjacent will be tested for intersection)
  - segments *a, b* are not adjacent when the alg. starts
  - segments *a, b* are adjacent just before *p*

  => there must be an event point when *a,b* become adjacent and therefore are tested for intersection

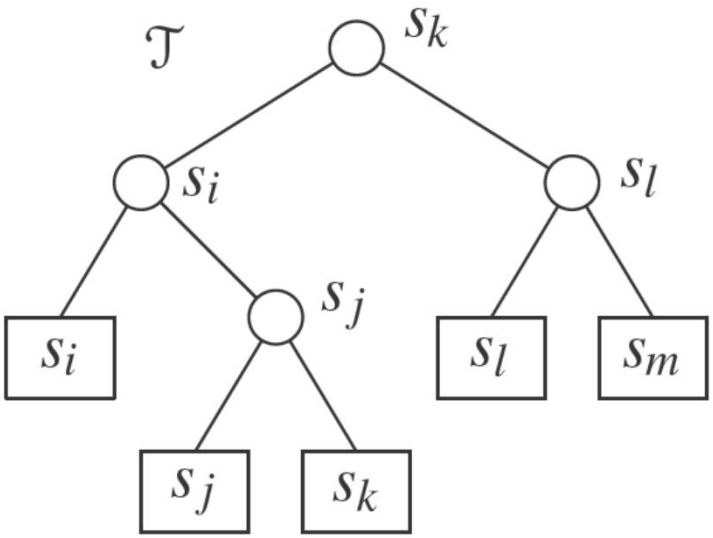


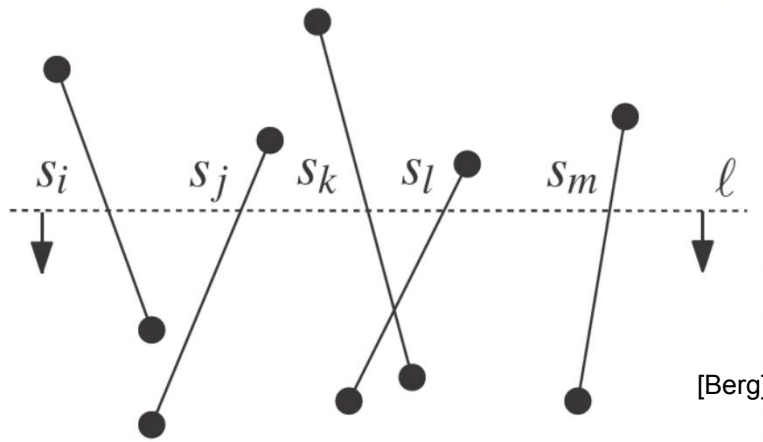

[Berg]

# Data structures

Sweep line $\ell$ status = order of segments along $\ell$

- Balanced binary search tree of segments
- Coords of intersections with $\ell$ vary as $\ell$ moves
  => store pointers to line segments in tree nodes
  - Position of $\ell$ is plugged in the *y=mx+b* to get the x-key

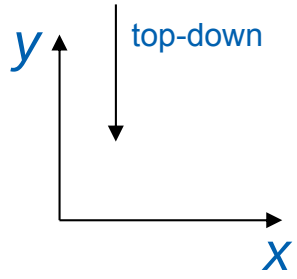



[Berg]

# Data structures

Event queue (postupový plán, časový plán)




- Define: Order $\prec$ (top-down, lexicographic)

  $p \prec q$ **iff** $p_y > q_y$ **or** $p_y = q_y$ and $p_x < q_x$

  top-down, left-right approach

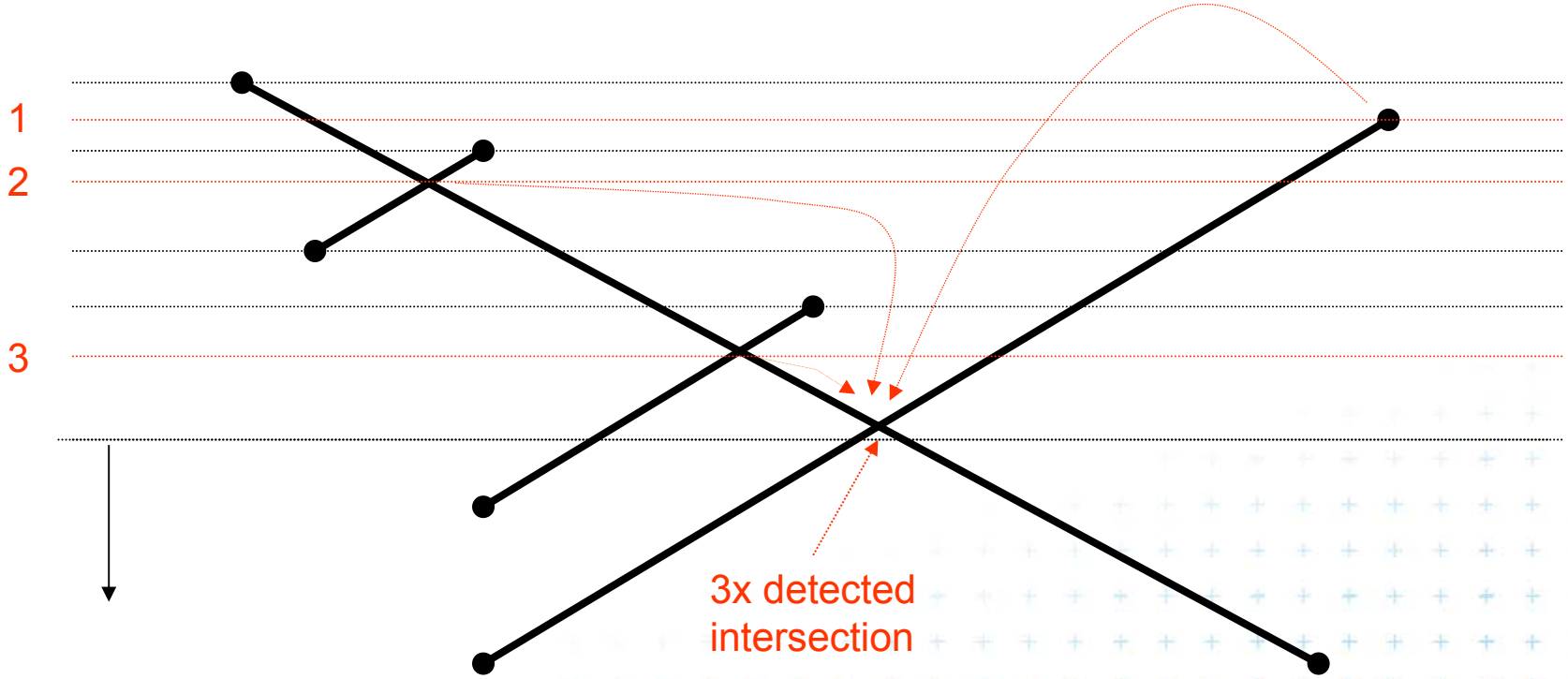
  (points on $\ell$ treated left to right)

- Operations
  - Insertion of computed intersection points
  - Fetching the next event

    (highest *y* below $\ell$ or the leftmost right of e)
  - Test, if the segment is already present in the queue

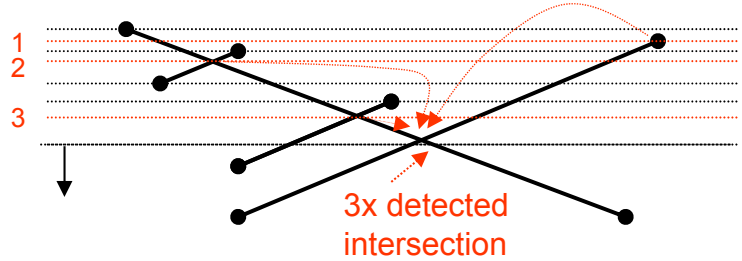
    (Locate and delete intersection event in the queue)

# Problem with duplicities of intersections



# Data structures

## Event queue data structure




- Heap
  - Problem: can not check duplicated intersection events (reinvented more than once)
  - Intersections processed twice or even more
  - Memory complexity up to $O(n^2)$
- Ordered dictionary (balanced binary tree)
  - Can check duplicated events (adds just constant factor)
  - Nothing inserted twice
  - If non-neighbor intersections are deleted
    i.e., if only intersections of neighbors along $\ell$ are stored
    then memory complexity just $O(n)$

# Line segment intersection algorithm

**FindIntersections(*S*)**

*Input:* A set *S* of line segments in the plane

*Output:* The set of intersection points + pointers to segments in each

1. init an empty event queue *Q* and insert the segment endpoints
2. init an empty status structure *T*
3. **while** Q in not empty
4. remove next event *p* from *Q*
5. handleEventPoint(*p*)

Upper endpoint

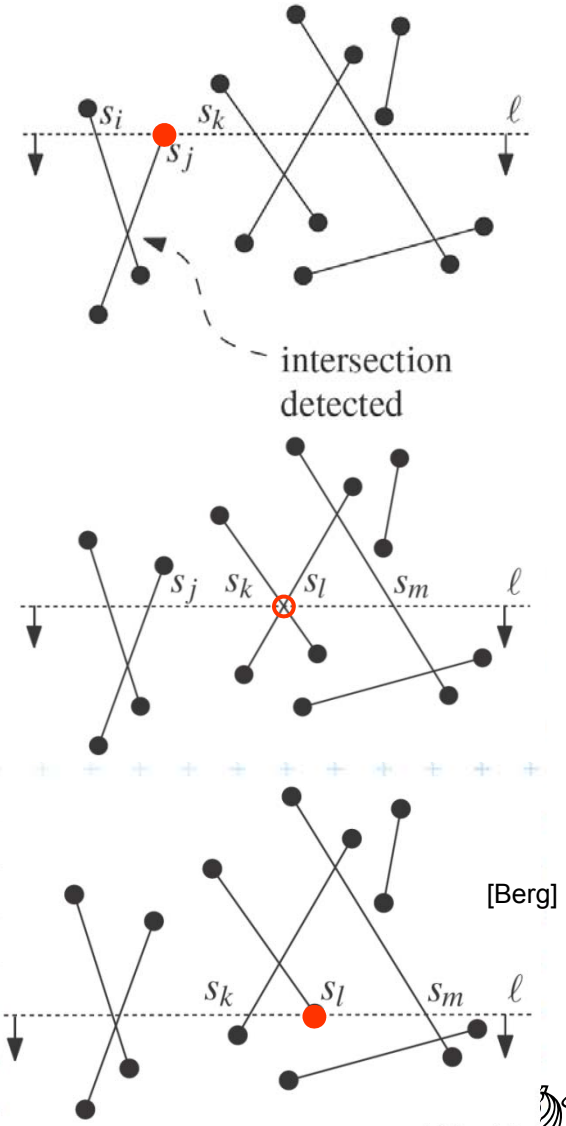
Intersection

Lower endpoint

Note: Upper-end-point events store info about the segment

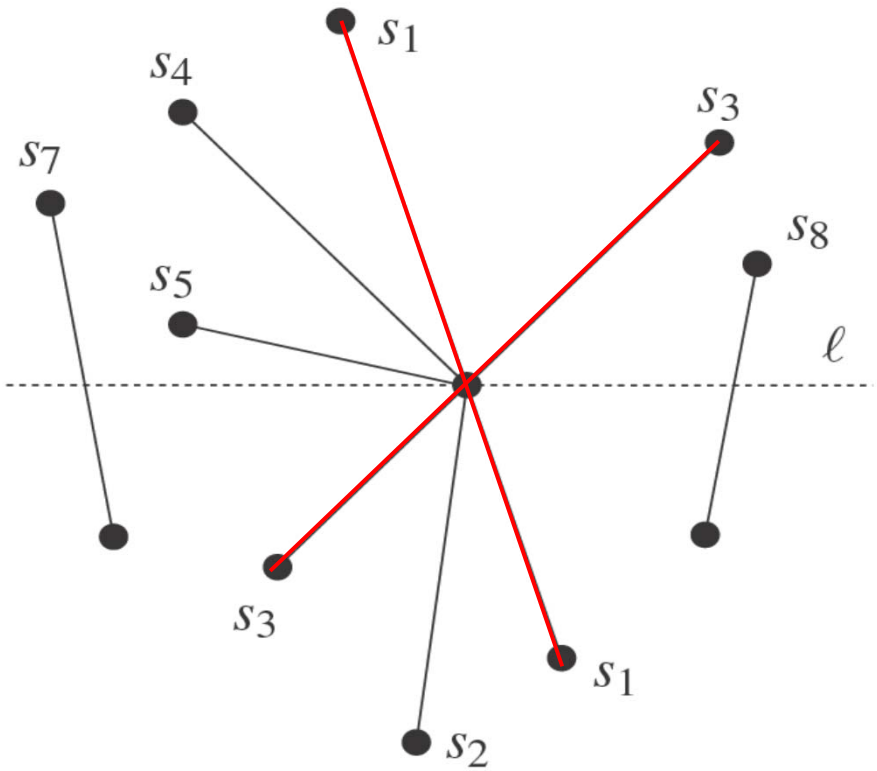
# handleEventPoint() principle

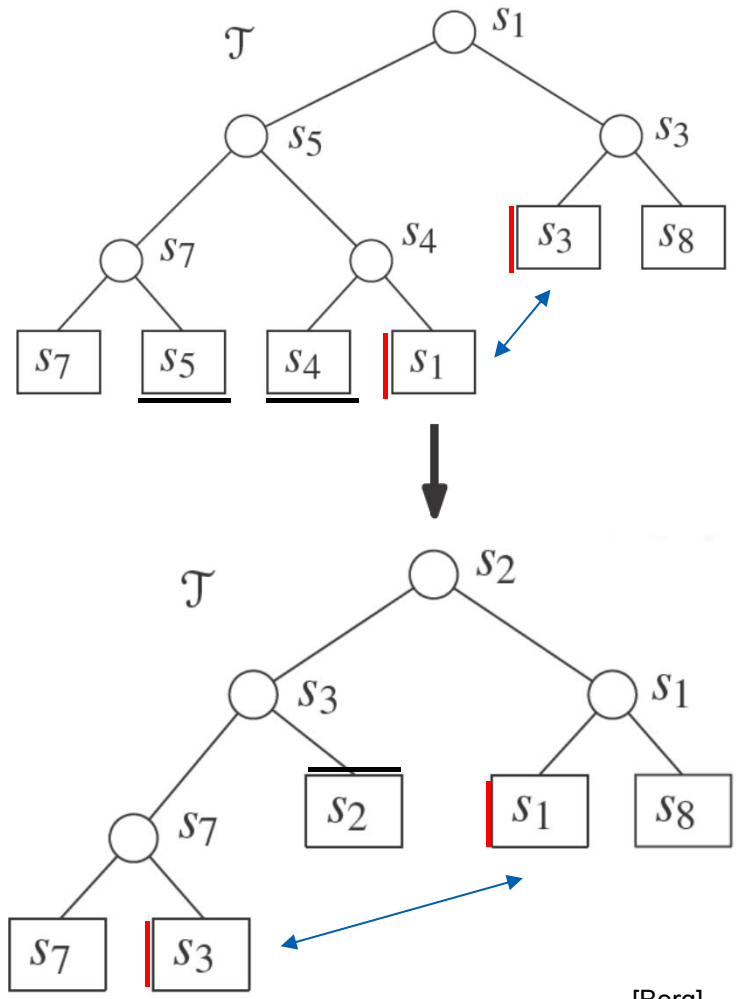
- Upper endpoint $U(p)$
  - insert $p$ (on $s_j$) to status $T$
  - add intersections with left and right neighbors to $Q$
- Intersection $C(p)$
  - switch order of segments in $T$
  - add intersections with nearest left and nearest right neighbor to $Q$
- Lower endpoint $L(p)$
  - remove $p$ (on $s_l$) from $T$
  - add intersections of left and right neighbors to $Q$




[Berg]

# More than two segments incident



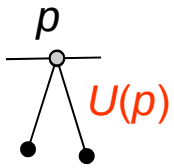

$U(p) = \{s_2\}$ start here

$C(p) = \{s_1, s_3\}$ cross on $\ell$

$L(p) = \{s_4, s_5\}$ end here

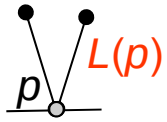
[Berg]

# Handle Events [Berg, page 25]

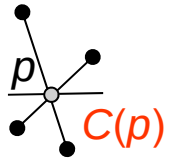



**handleEventPoint(p)**

1. Let $U(p)$ = set of segments whose Upper endpoint is $p$.
   These segmets are stored with the event point $p$ (will be added to $T$)
2. Search $T$ for all segments $S(p)$ that contain $p$ (are adjacent in $T$):
   Let $L(p) \subset S(p)$ = segments whose Lower endpoint is $p$
   Let $C(p) \subset S(p)$ = segments that Contain $p$ in interior



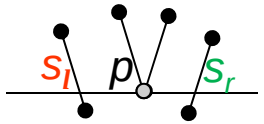

3. **if**( $L(p) \cup U(p) \cup C(p)$ contains more than one segment )
4. report $p$ as intersection ∘ together with $L(p)$, $U(p)$, $C(p)$




5. Delete the segments in $L(p) \cup C(p)$ from $T$
6. Insert the segments in $U(p) \cup C(p)$ into $T$

} Reverse order of $C(p)$ in $T$

(order as below $\ell$, horizontal segment as the last)




7. **if**( $U(p) \cup C(p) = \emptyset$ ) then findNewEvent($s_l$, $s_r$, $p$) // left & right neighbors
8. **else** s' = leftmost segment of $U(p) \cup C(p)$; findNewEvent($s_l$, $s'$, $p$)
   s'' = rightmost segment of $U(p) \cup C(p)$; findNewEvent($s''$, $s_r$, $p$)

# Detection of new intersections

**findNewEvent($s_l$, $s_r$, $p$)** ***// with handling of horizontal segments***

*Input:* two segments (left & right from $p$ in $T$) and a current event point $p$

*Output:* updated event queue $Q$ with new intersection

1. **if** [ ( $s_l$ and $s_r$ intersect below the sweep line $\ell$ ) // line 7. above

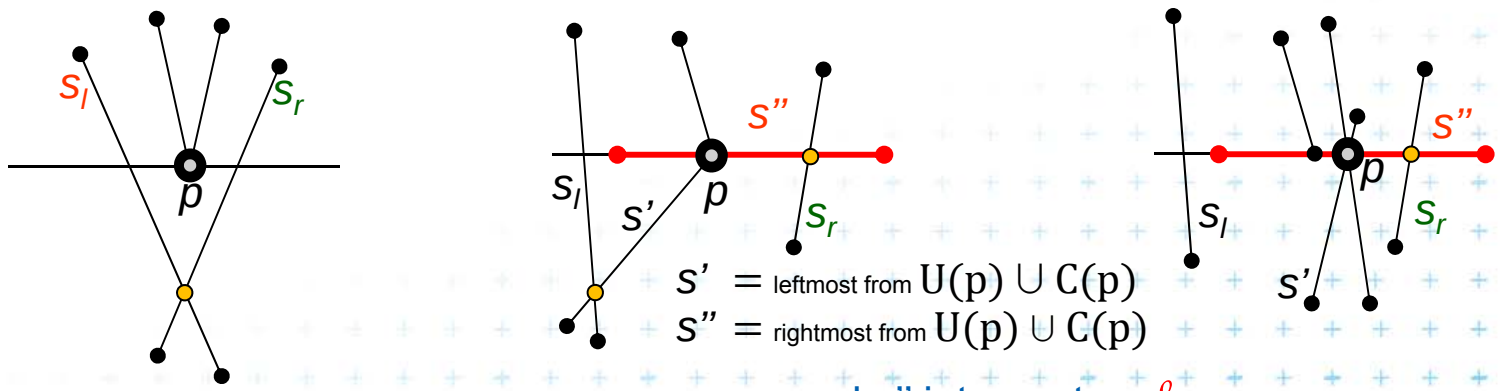
   or ($s_r$ intersect $s''$ on $\ell$ and to the right of $p$ ) ] // Non-overlapping horizontal segm.

   and( the intersection ∘ is not present in $Q$ )

2. **then**

   insert intersection ∘ as a new event into $Q$

- ○ Intersection - line 4
- ● Intersection - line 7,8




$s_l$ and $s_r$ intersect below

$s_r$ and $s''$ intersect on $\ell$,
$s''$ is horizontal and to the right of $p$

# Line segment intersections

- Memory O($I$) = O($n^2$) with duplicities in Q
  or O(n ) with duplicities in Q deleted
- Operational complexity
  - $n + I$ stops
  - log $n$ each

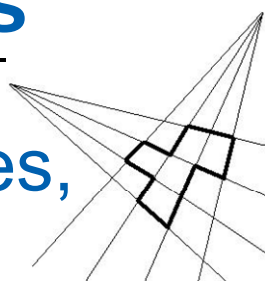
  => O( $I + n$) log $n$ total
- The algorithm is by Bentley-Ottmann

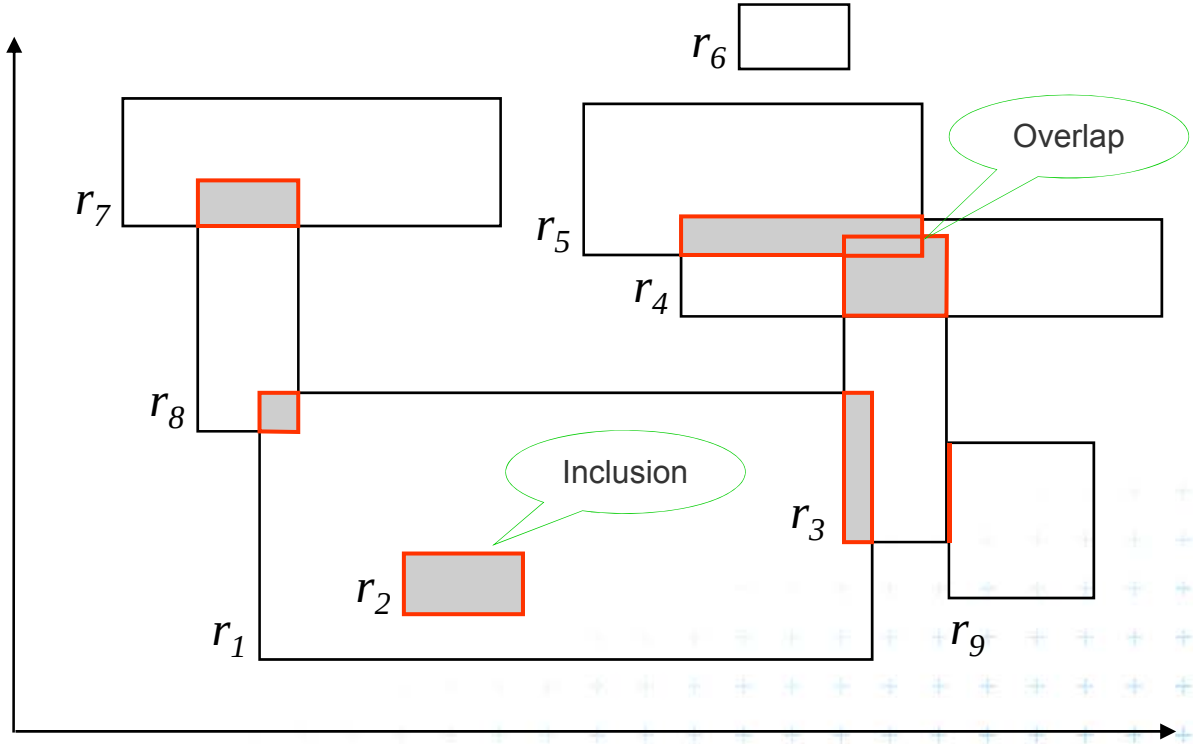
  Bentley, J. L.; Ottmann, T. A. (1979), "Algorithms for reporting and counting geometric intersections", *IEEE Transactions on Computers* **C-28** (9): 643-647, doi:10.1109/TC.1979.1675432 .

  See also http://wapedia.mobi/en/Bentley%E2%80%93Ottmann_algorithm

# Intersection of axis parallel rectangles

- Given the collection of *n isothetic* rectangles, report all intersecting parts






Alternate sides belong to two pencils of lines
(trsy přímek)
(often used with points in infinity = axis parallel)
2D => 2 pencils

[?]

Answer: $(r_1, r_2)$ $(r_1, r_3)$ $(r_1, r_8)$ $(r_3, r_4)$ $(r_3, r_5)$ $(r_3, r_9)$ $(r_4, r_5)$ $(r_7, r_8)$

# Brute force intersection

**Brute force algorithm**

*Input:* set *S* of axis parallel rectangles

*Output:* pairs of intersected rectangles

1. For every pair $(r_i, r_j)$ of rectangles $\in S$, $i \neq j$
2. if $(r_i \cap r_j \neq \emptyset)$ then
3. report $(r_i, r_j)$

**Analysis**

Preprocessing: None.

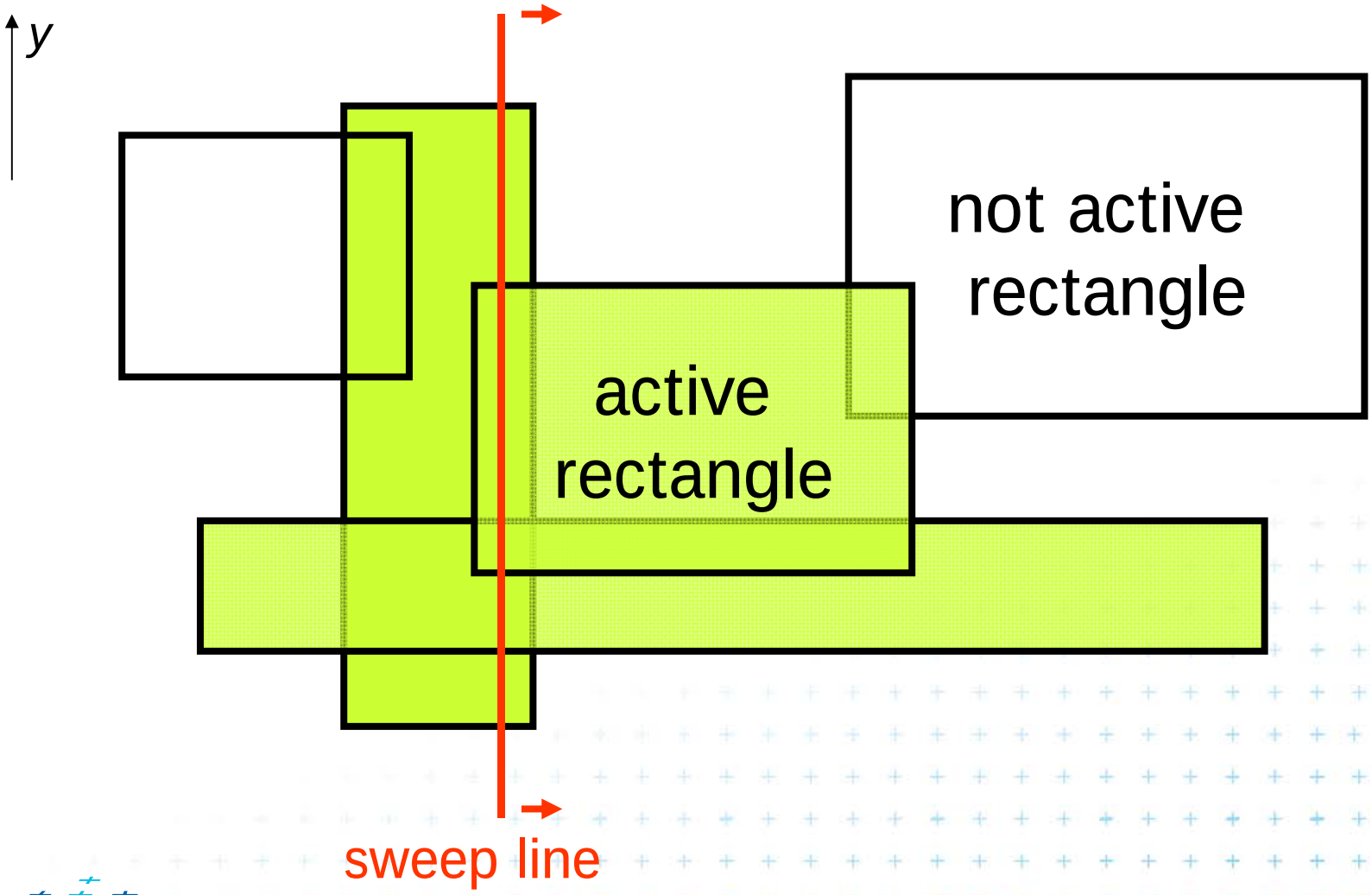
Query: $O(N^2)$ $\binom{N}{2} = \frac{N(N-1)}{2} \in O(N^2)$.

Storage: $O(N)$

# Plane sweep intersection algorithm

- Vertical sweep line moves from left to right
- Stops at every x-coordinate of a rectangle (either at its left side or at its right side).
- active rectangles – a set
  = rectangles currently intersecting the sweep line
  - left side event of a rectangle
    => the rectangle is added to the active set.
  - right side
    => the rectangle is deleted from the active set.
- The active set used to detect rectangle intersection

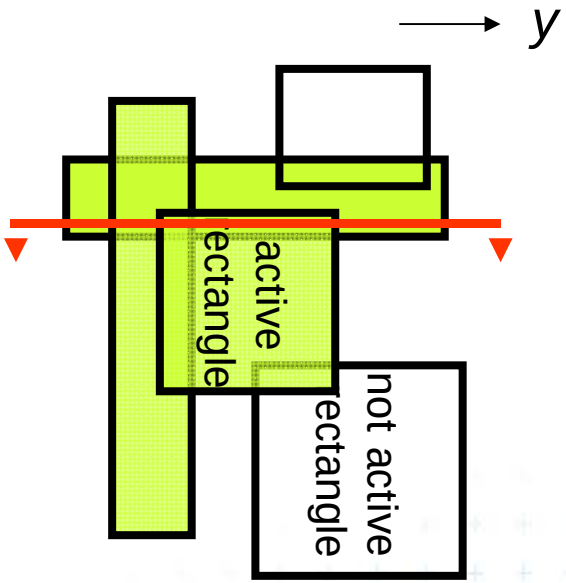
# Example rectangles and sweep line




[Drtina]

# Interval tree as sweep line status structure

- Vertical sweep-line => Only *y*-coordinates along it
- Turn our view in slides 90° right
- Sweep line (*y*-axis) will be drawn as horizontal



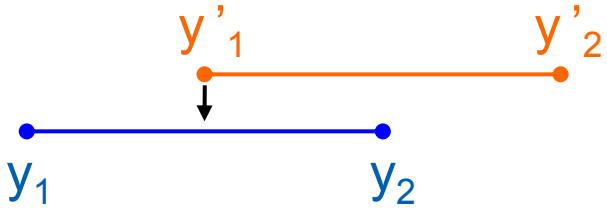


sweep line [Drtina]

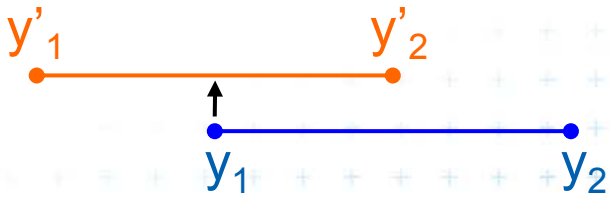
# Intersection test – between pair of intervals

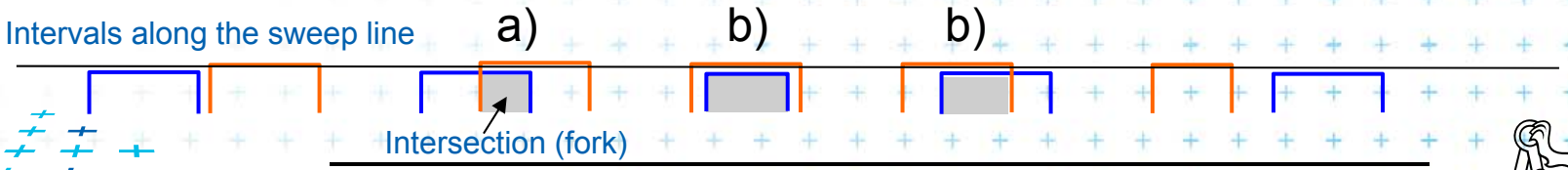
- Given two intervals $R = [y_1, y_2]$ and $R' = [y'_1, y'_2]$ the condition $R \cap R'$ is equivalent to one of these mutually exclusive conditions:

a) $y_1 \leq y'_1 \leq y_2$



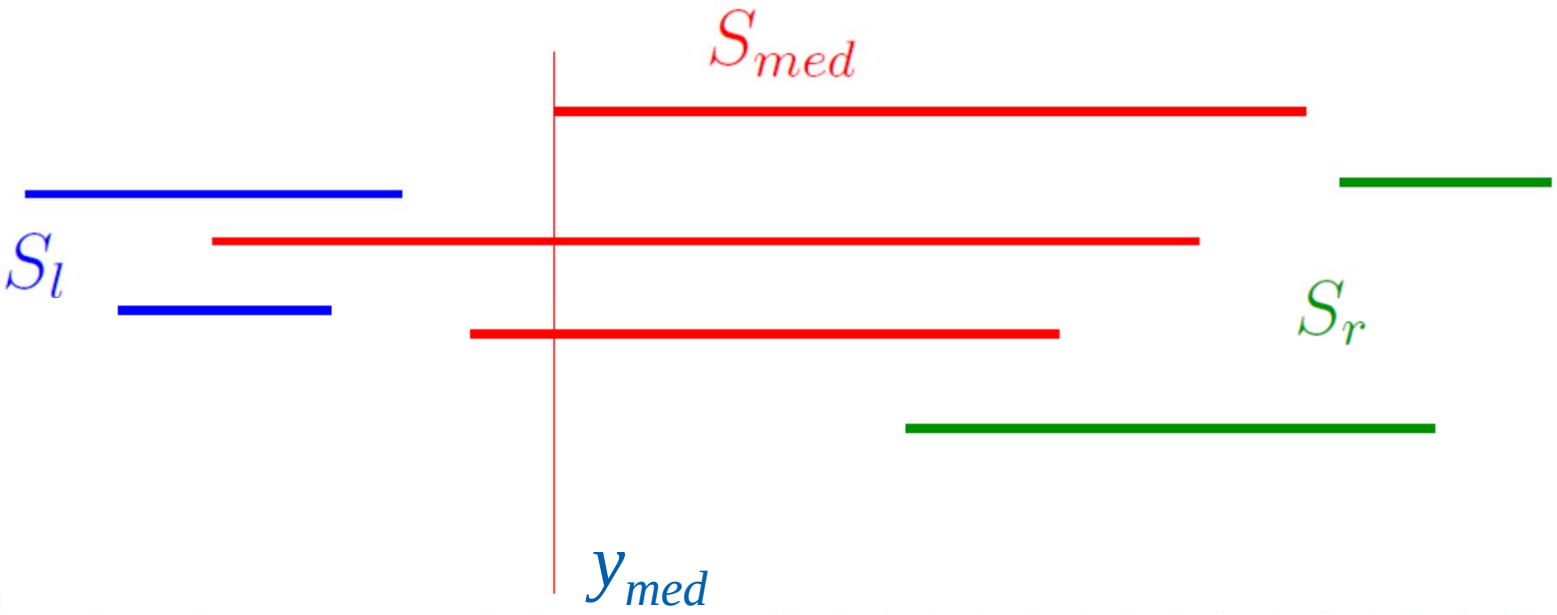

b) $y'_1 \leq y_1 \leq y'_2$



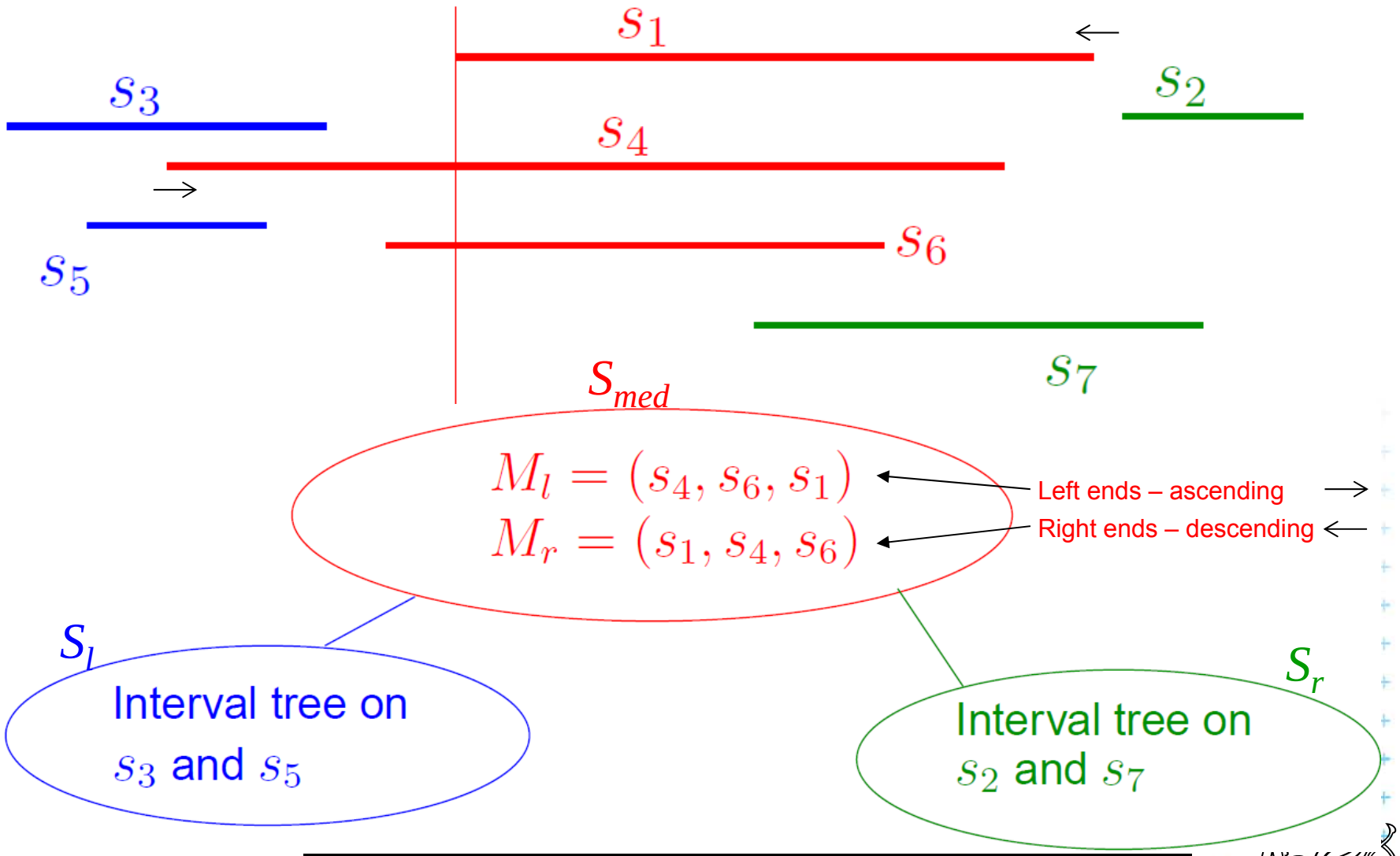
# Static interval tree – stores all end points

- Let $v = y_{med}$ be the median of end-points of segments
- $S_l$ : segments of S that are completely to the left of $y_{med}$
- $S_{med}$ : segments of S that contain $y_{med}$
- $S_r$ : segments of S that are completely to the right of $y_{med}$




[Vigneron]

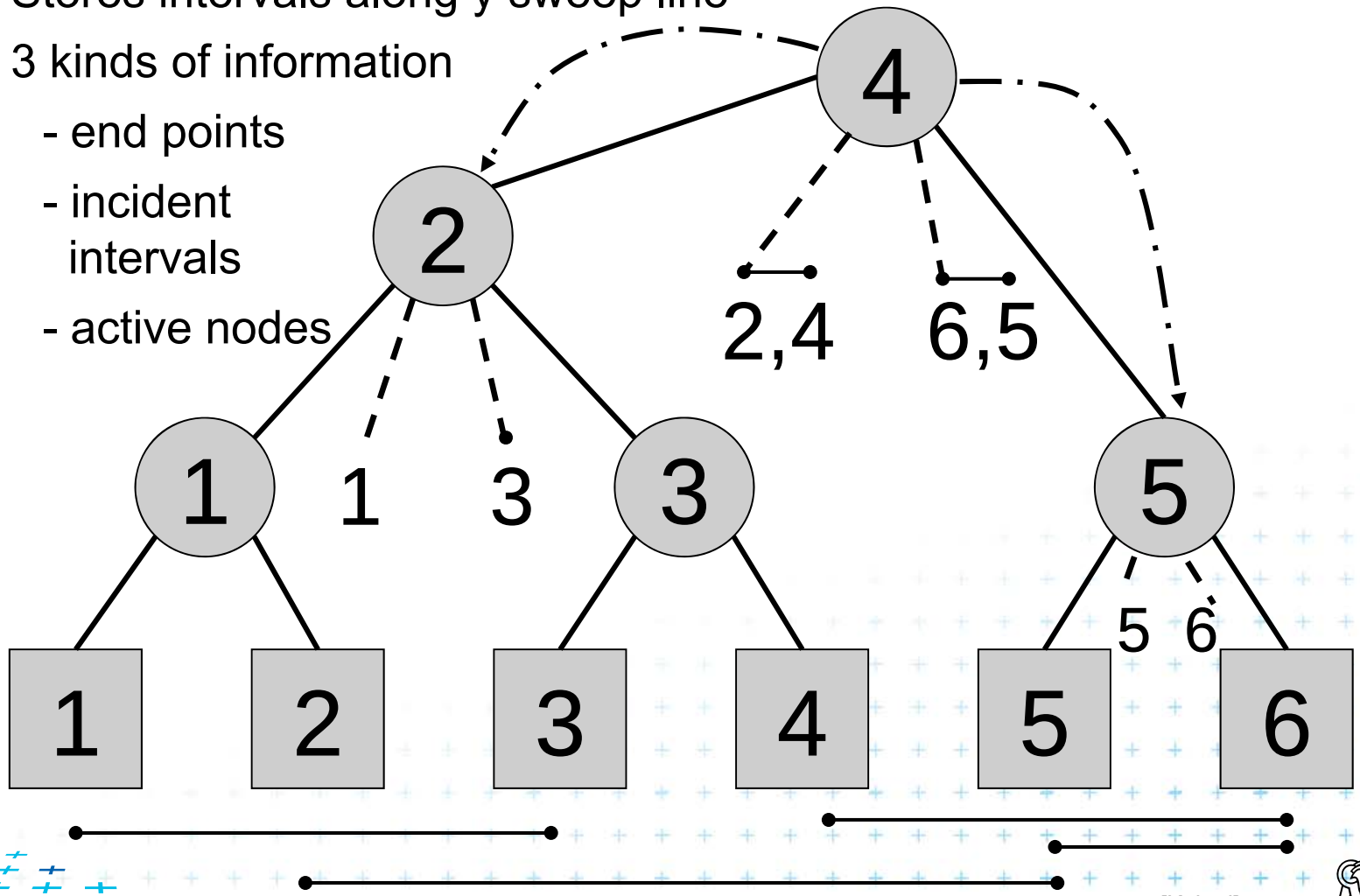
# Static interval tree – Example




[Vigneron]

# Static interval tree [Edelsbrunner80]

- Stores intervals along y sweep line
- 3 kinds of information
  - end points
  - incident intervals
  - active nodes

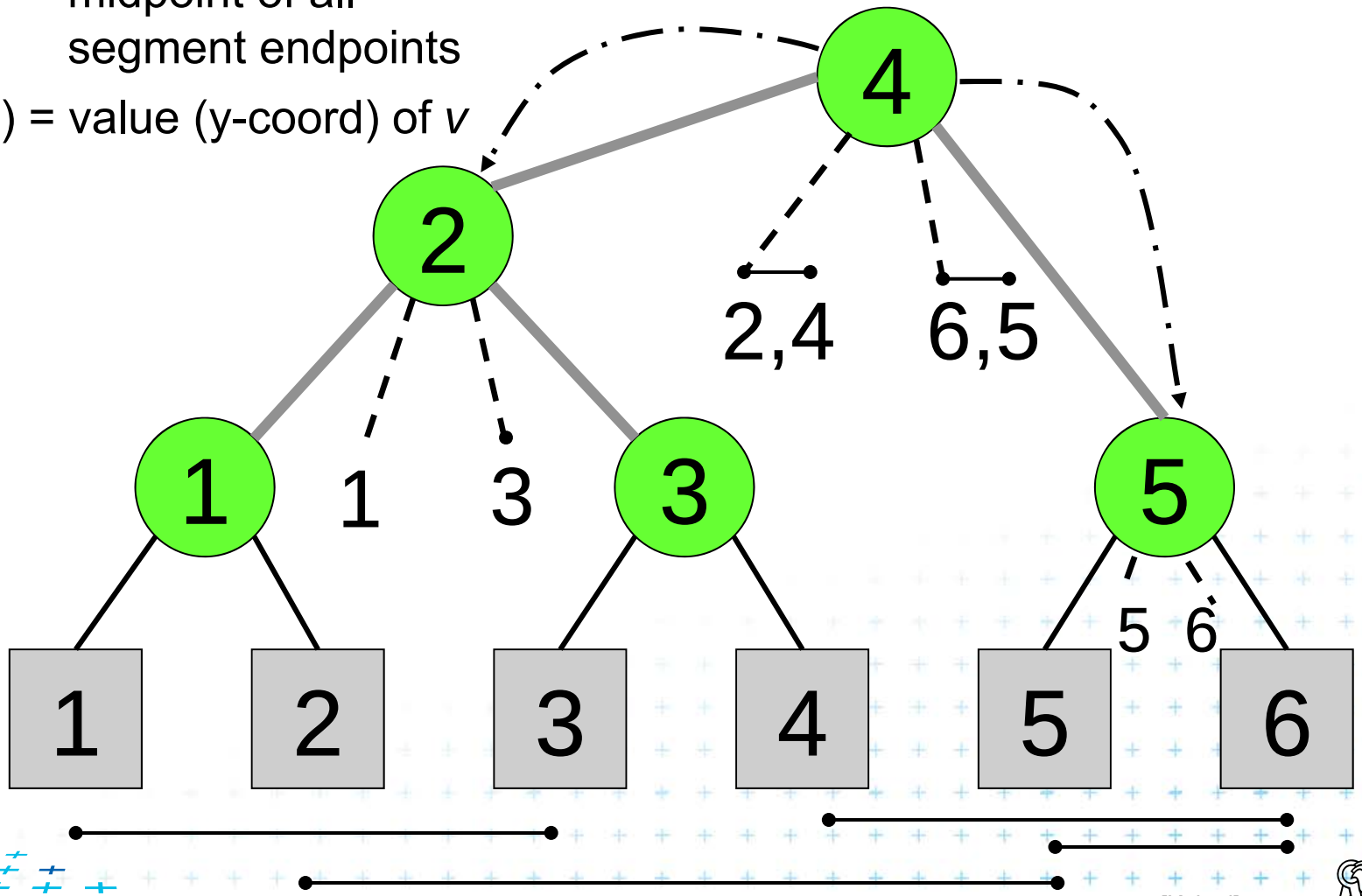



[Kukral]

# Primary structure – static tree for endpoints

v = midpoint of all segment endpoints

H(v) = value (y-coord) of *v*

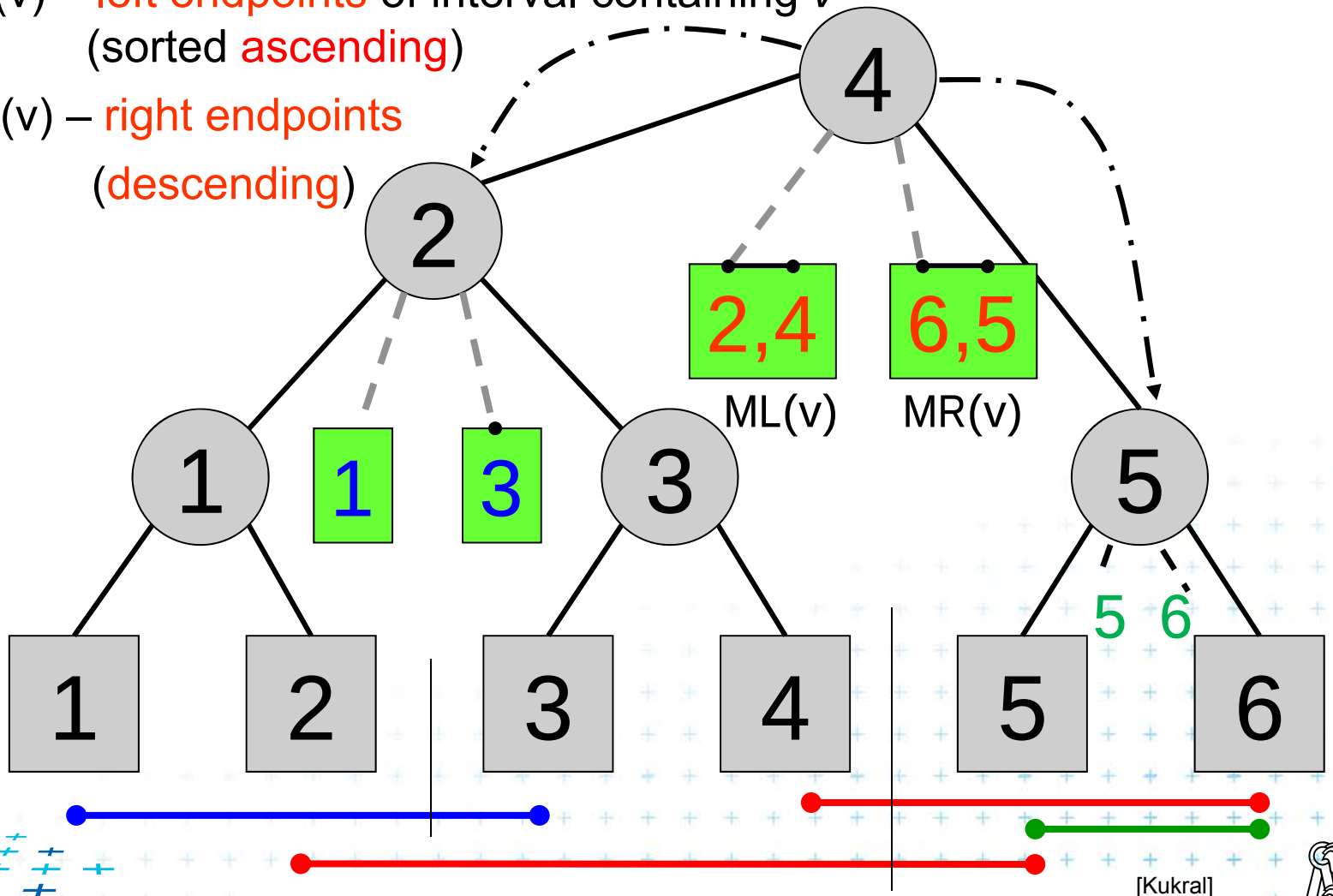



[Kukral]

# Secondary lists of incident interval end-pts.

ML(v) – left endpoints of interval containing *v* (sorted ascending)

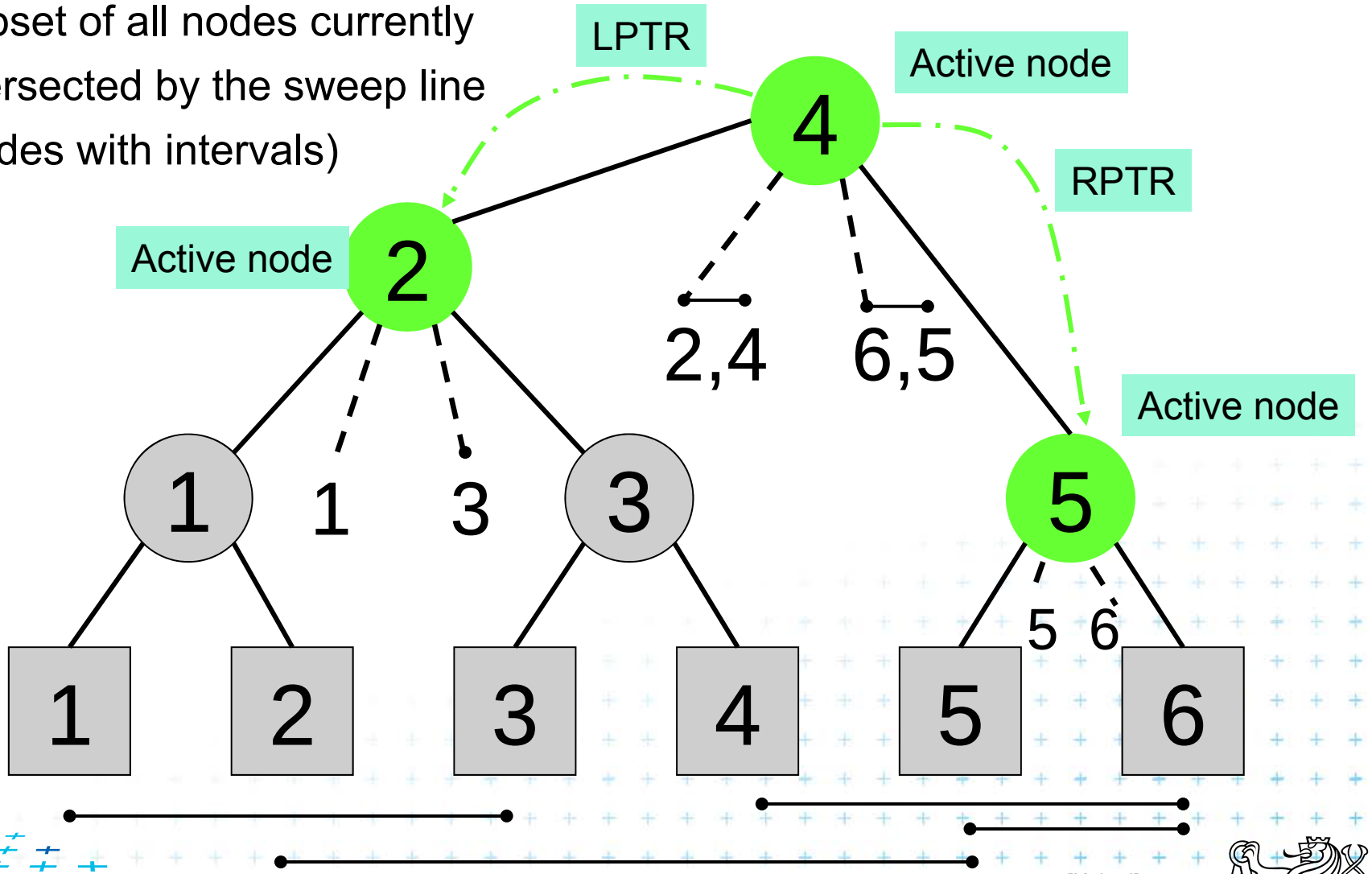
MR(v) – right endpoints (descending)




[Kukral]

# Active nodes – intersected by the sweep line

Subset of all nodes currently intersected by the sweep line (nodes with intervals)




[Kukral]

# Query = sweep and report intersections

**RectangleIntersections( *S* )**
*Input:* Set *S* of rectangles
*Output:* Intersected rectangle pairs

1. Preprocess( S ) // create the interval tree *T* (for y-coords)
   // and event queue *Q* (for x-coords)
2. **while** ( $Q \neq \emptyset$ ) do
3. Get next entry $(x_i, y_{il}, y_{ir}, t)$ from *Q* // $t \in \{ left \mid right \}$
4. **if** ( $t$ = left ) // left edge
5. a) QueryInterval ( $y_{il}, y_{ir}$, root($T$)) // report intersections
6. b) InsertInterval ( $y_{il}, y_{ir}$, root($T$)) // insert new interval
7. **else** // right edge
8. c) DeleteInterval ( $y_{il}, y_{ir}$, root($T$) )

# Preprocessing

**Preprocess( S )**
*Input:* Set *S* of rectangles
*Output:* Primary structure of the interval tree *T* and the event queue *Q*

1. *T* = PrimaryTree(*S*) // Construct the static primary structure
   // of the interval tree -> sweep line STATUS *T*

2. // Init event queue Q with vertical rectangle edges in ascending order ~x
   // Put the left edges with the same $x$ ahead of right ones
3. for $i$ = 1 to $n$
4. insert( ( $x_{il}$ , $y_{il}$ , $y_{ir}$ , left ), Q) // left edges of *i-th* rectangle
5. insert( ( $x_{ir}$, $y_{il}$ , $y_{ir}$ , right ), Q) // right edges

# Interval tree – primary structure construction

**PrimaryTree(*S*)** ***//* only the y-tree structure, without intervals**

*Input:* Set *S* of rectangles

*Output:* Primary structure of an interval tree *T*

1. $S_y$ = Sort endpoints of all segments in *S* according to *y*-coordinate
2. *T* = BST( $S_y$ )
3. **return** *T*

**BST( $S_y$ )**

1. **if**( $|S_y| = 0$ ) **return** null
2. *yMed* = *median of* $S_y$
3. *L* = endpoints $p_y \leq yMed$
4. R = endpoints $p_y > yMed$
5. *t* = *new* IntervalTreeNode( *yMed* )
6. *t.left* = BST(*L*)
7. *t.right* = BST(*R*)
8. **return** *t*

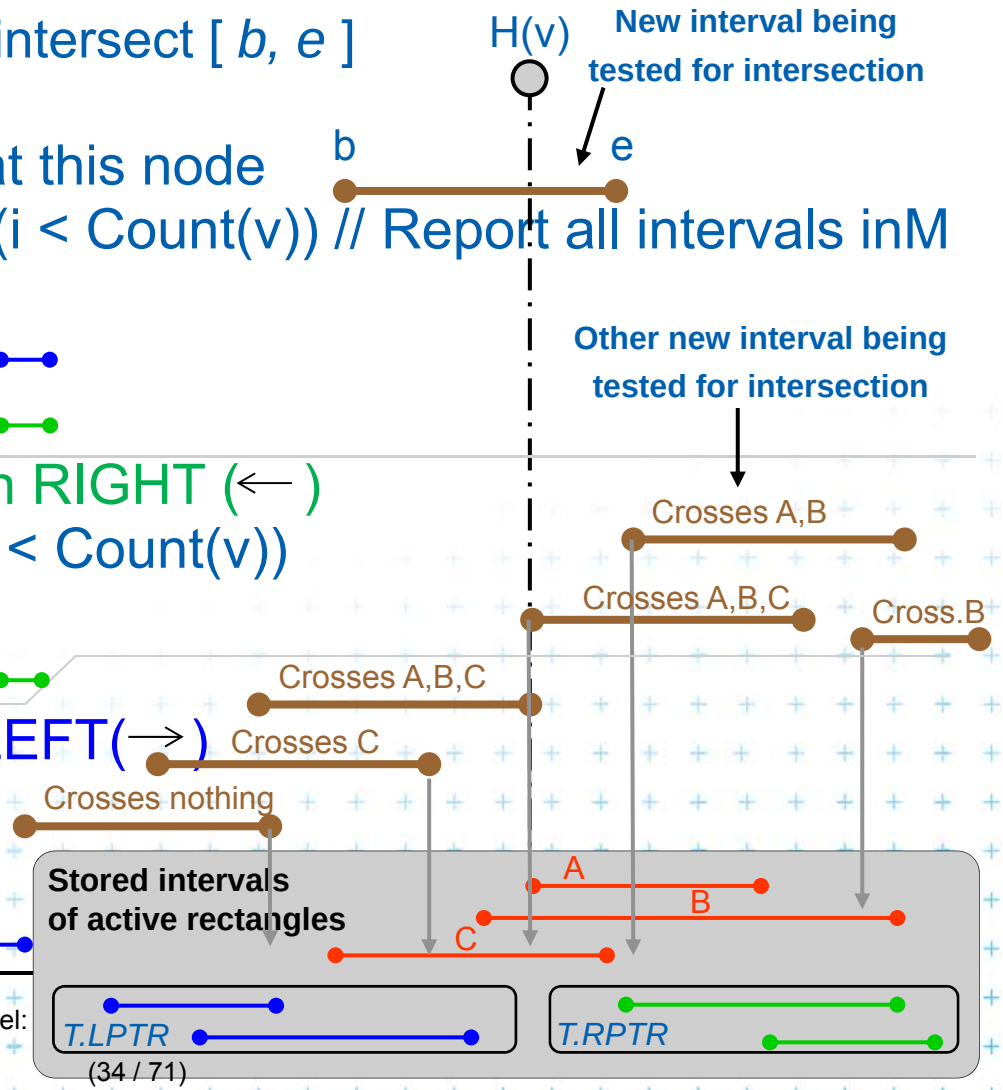
# Interval tree – search the intersections

**QueryInterval ( *b, e, T* )**

*Input:* Interval of the edge and current tree *T*

*Output:* Report the rectangles that intersect [ *b, e* ]

1. **if**( *T* = null ) **return**
2. i=0; **if**( b < H(v) < e ) // forks at this node
3. **while** ~~( *MR*(v).[i] >= b ) &&~~ (i < Count(v)) // Report all intervals inM
4. ReportIntersection; i++
5. QueryInterval( *b,e,T.LPTR* )
6. QueryInterval( *b,e,T.RPTR* )
7. **else if** (H(v) ≤ b < e) // search RIGHT (←)
8. **while** (*MR*(v).[i] >= b) && (i < Count(v))
9. ReportIntersection; i++
10. QueryInterval( *b,e,T.RPTR* )
11. **else** // b < e ≤ H(v) //search LEFT(→)
12. **while** (*ML*(v).[i] <= e)
13. ReportIntersection; i++
14. QueryInterval( *b,e,T.LPTR* )



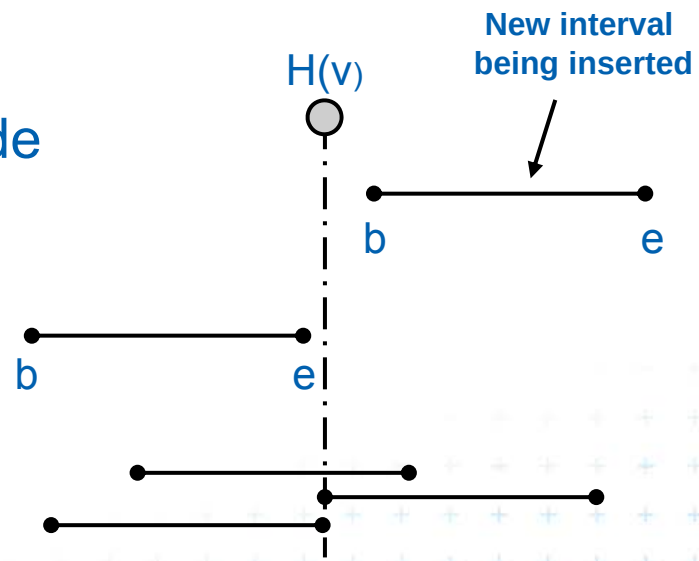
# Interval tree - interval insertion

**InsertInterval ( *b, e, T* )**

*Input:* Interval [*b*,*e*] and interval tree *T*

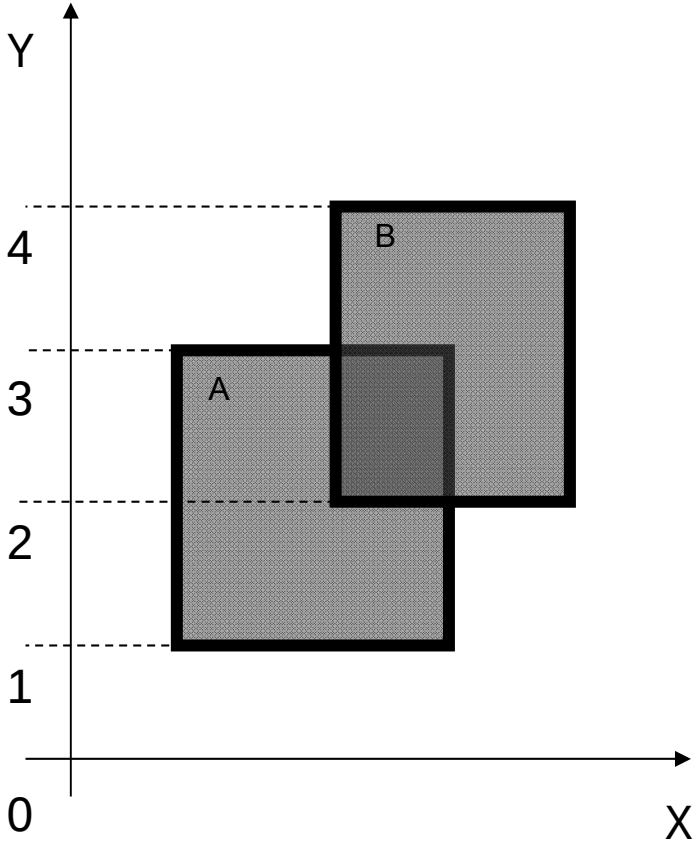
*Output:* *T* after insertion of the interval

```
v = root(T )
while( v != null )        // find the fork node
    if (H(v) < b < e)
        v = v.right       // continue right
    else if (b < e < H(v))
        v = v.left        // continue left
    else  // b ≤ H(v) ≤ e  // insert interval
        set v node to active
        connect LPTR resp. RPTR to its parent
        insert [b,e] into list ML(v) – sorted in ascending order of b's
        insert [b,e] into list MR(v) – sorted in descending order of e's
        break
endwhile
return T
```

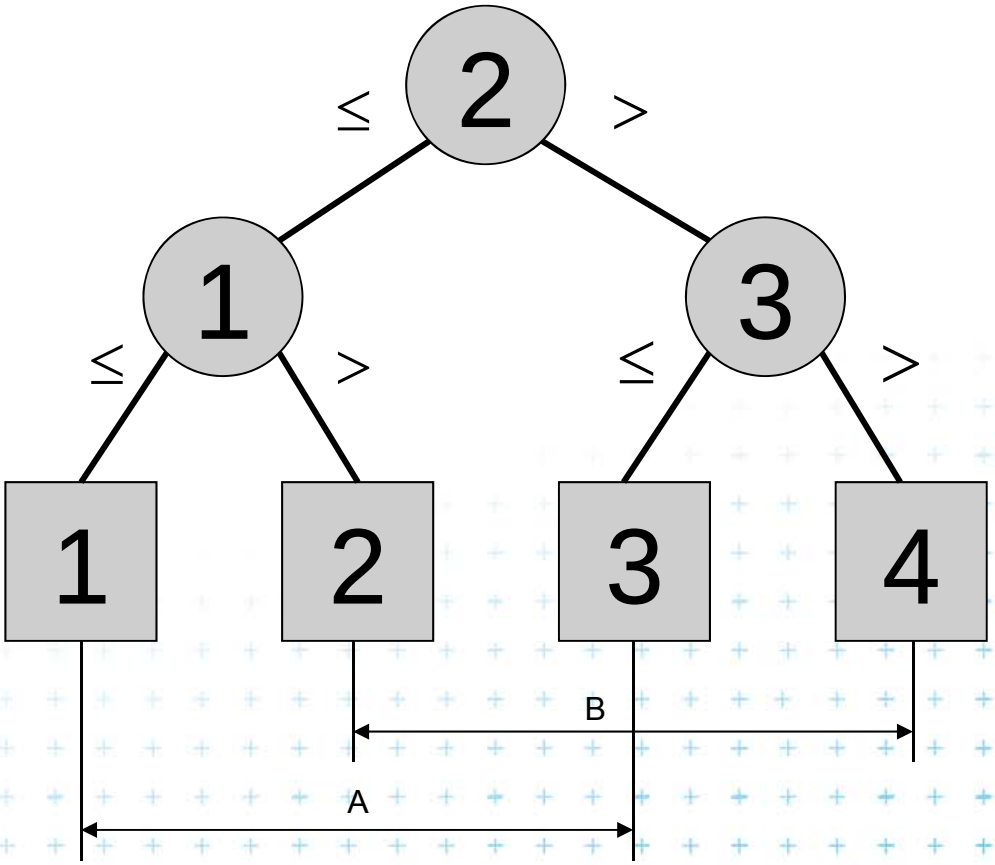


# Example 1

# Example 1 – static tree on endpoints

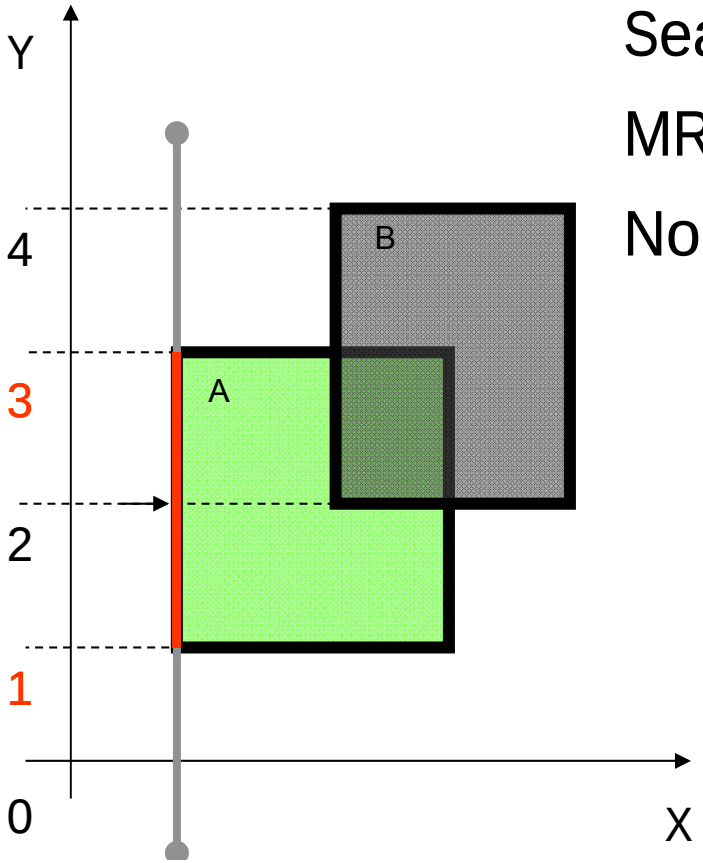



H(v) – value of node *v*




[Drtina]

# Interval insertion [1,3] a) Query Interval

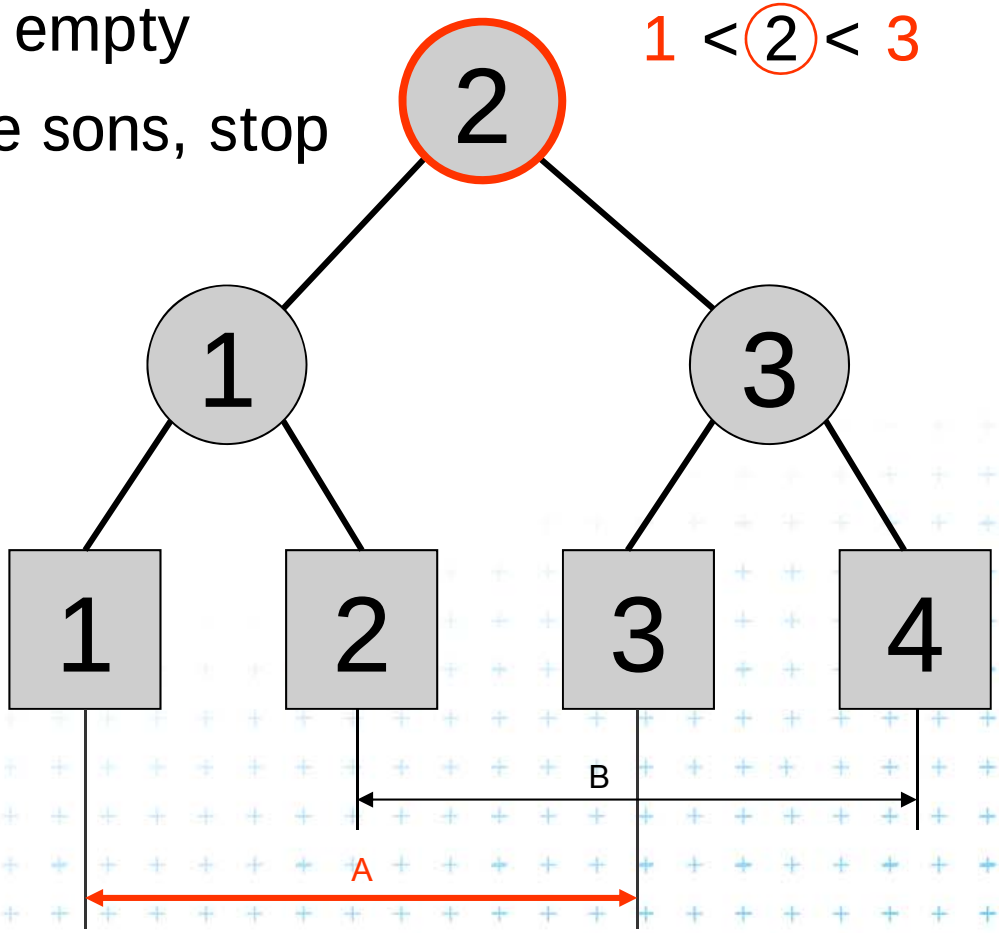



Search MR(v) or ML(v): ← b < H(v) < e

MR(v) is empty

No active sons, stop

1 < 2 < 3



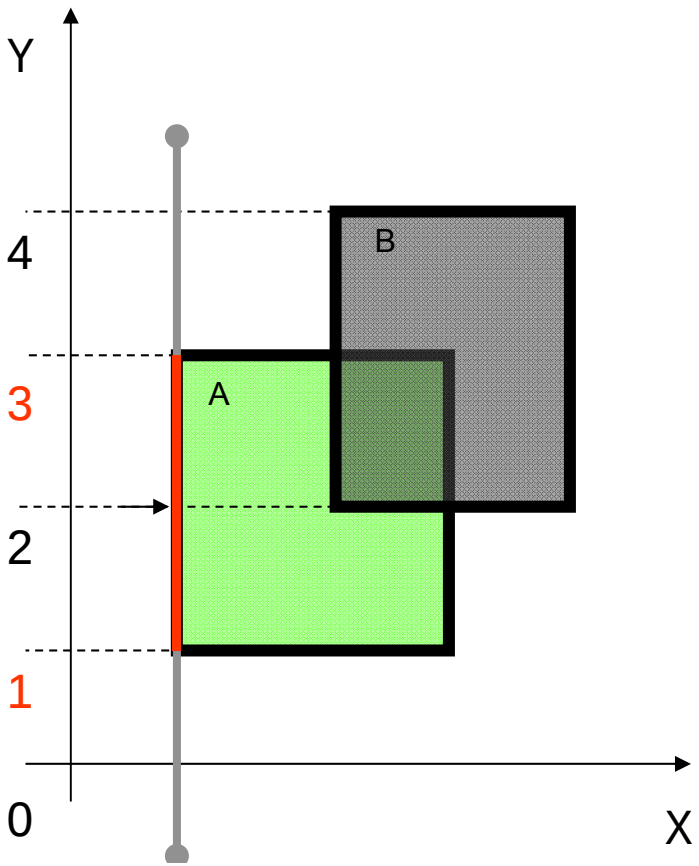

Active rectangle

Current node

Active node

[Drtina]

# Interval insertion [1,3] b) Insert Interval



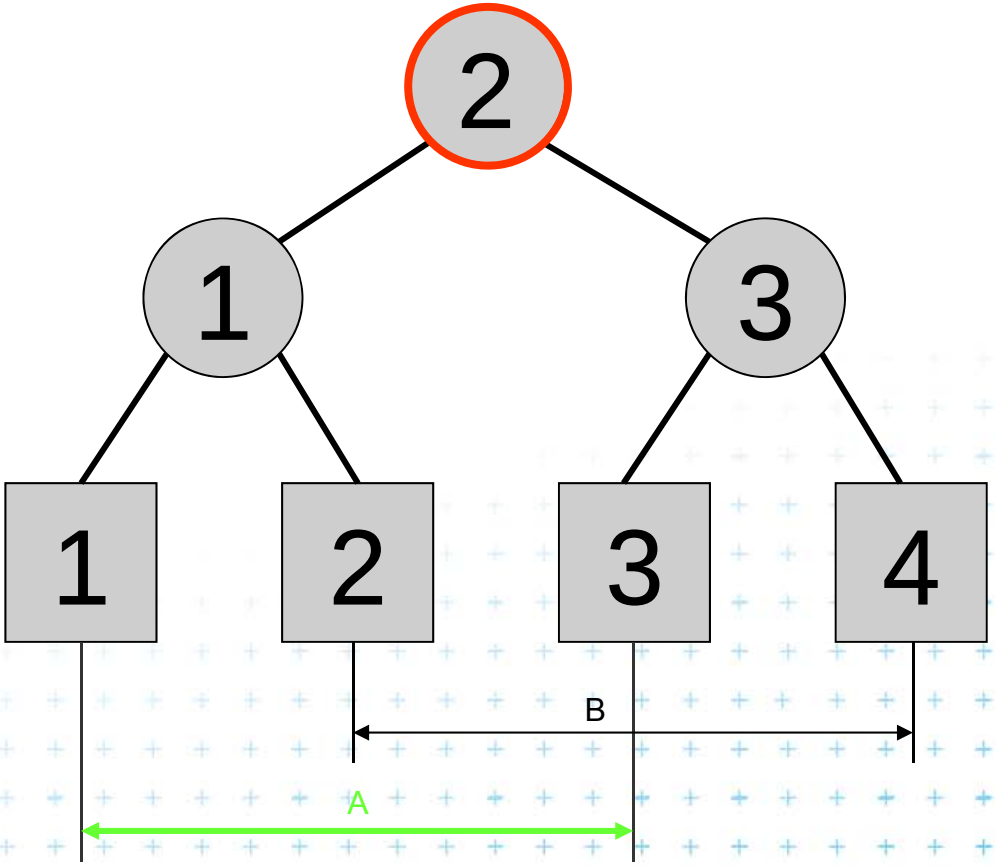

Active rectangle

Current node

Active node

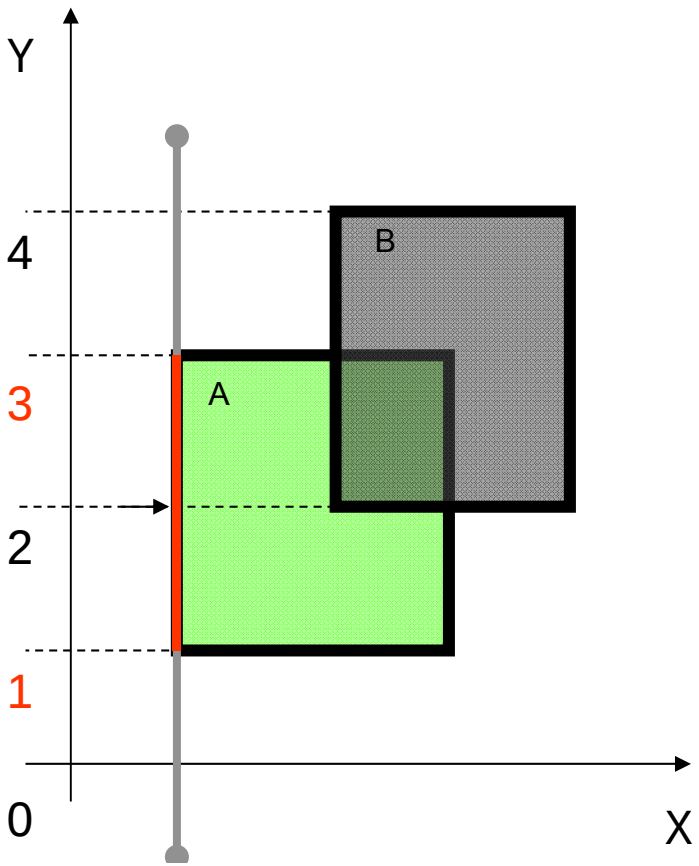
$b \le H(v) \le e$

? 1 ≤ 2 ≤ 3 ?




[Drtina]

# Interval insertion [1,3] b) Insert Interval




Active rectangle

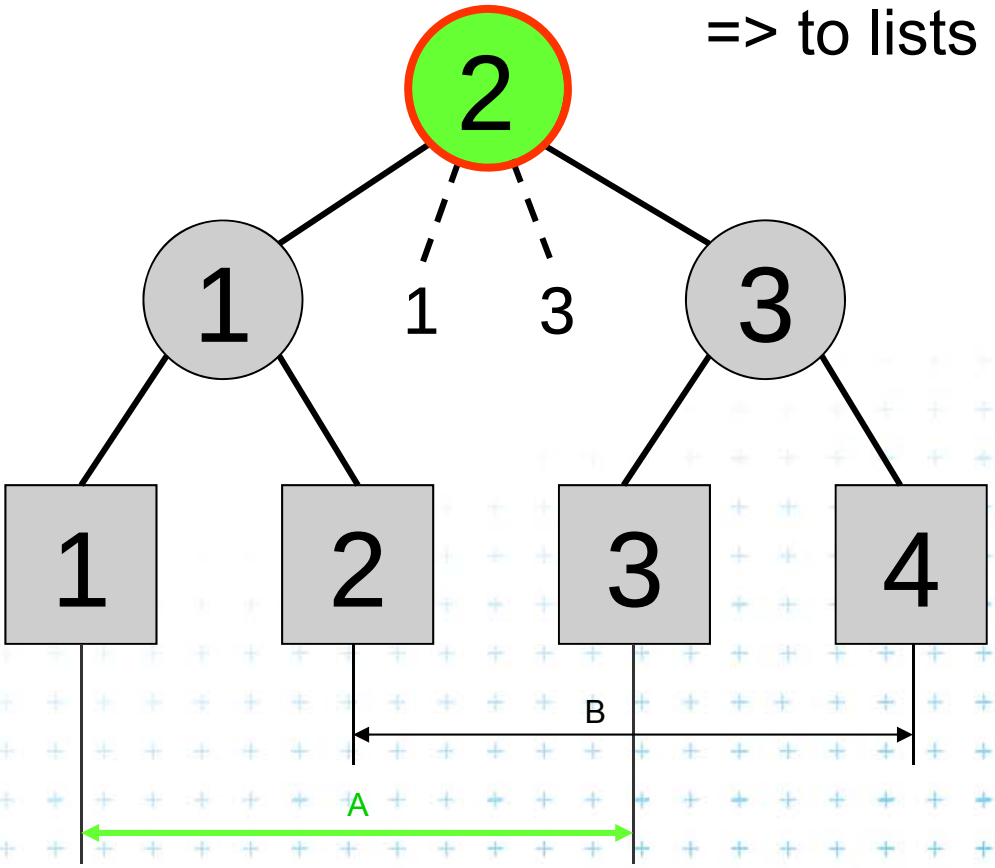
Current node

Active node

$b \le H(v) \le e$

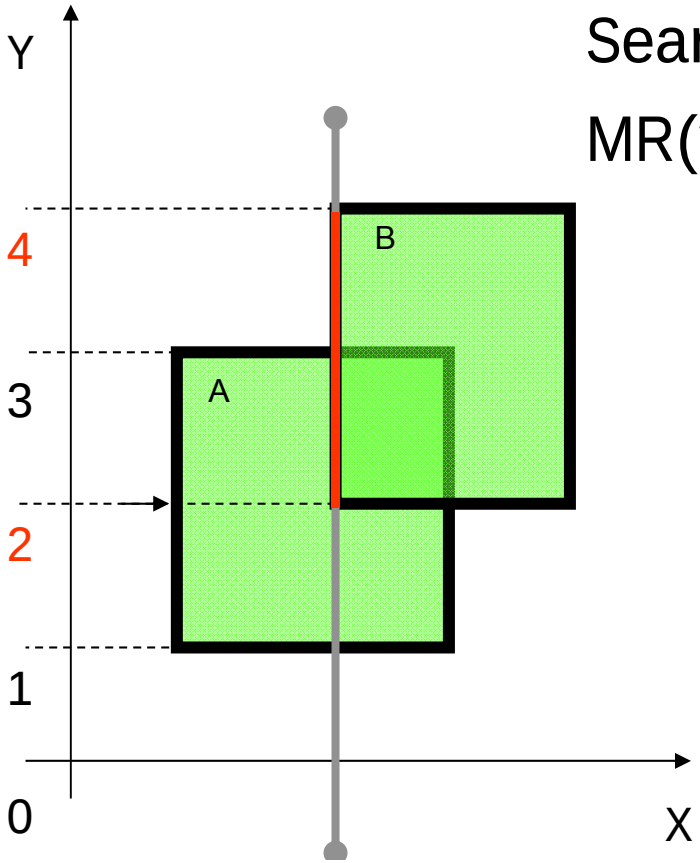
$1 \le 2 \le 3$

fork
=> to lists




[Drtina]

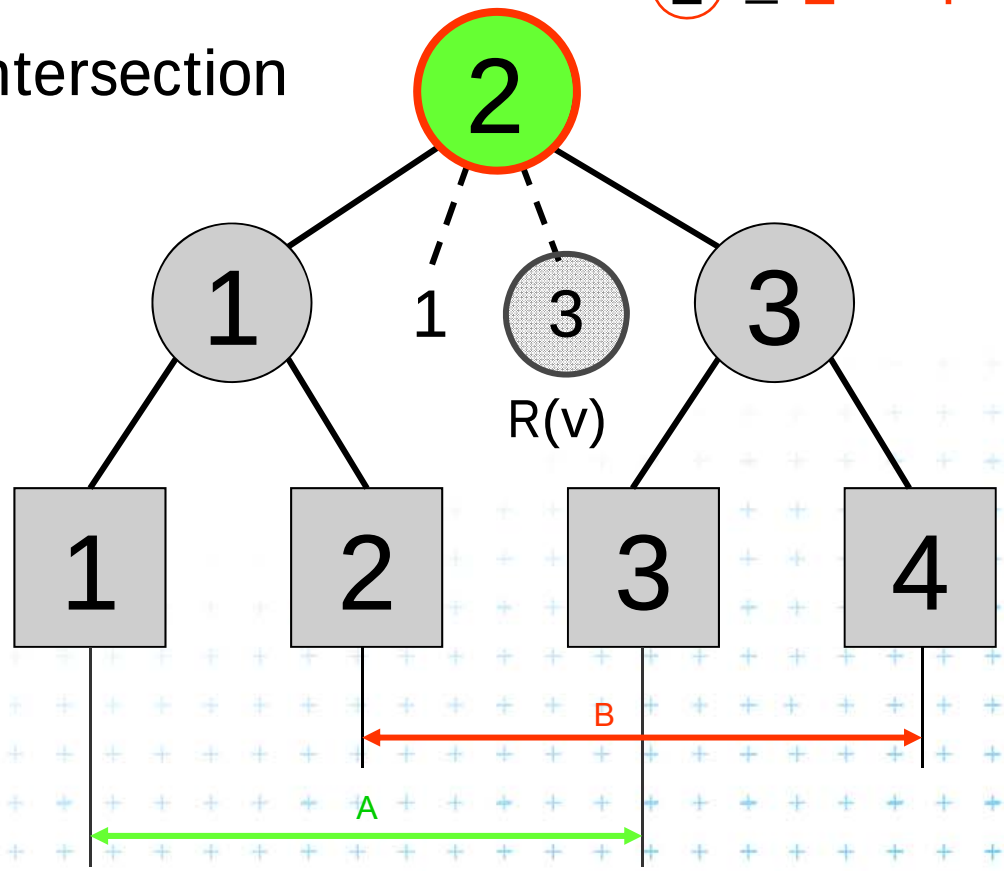
# Interval insertion [2,4] a) Query Interval




Search MR(v) only: ← H(v) ≤ b < e

MR(v)[1] = 3 ≥ 2? 2 ≤ 2 < 4

=> intersection



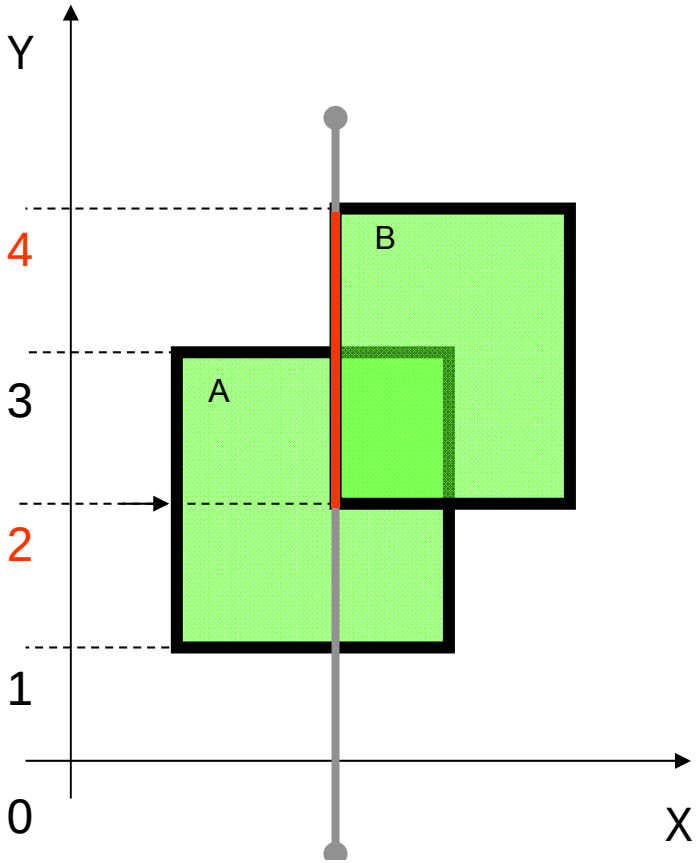

Active rectangle

Current node

Active node

[Drtina]

# Interval insertion [2,4] b) Insert Interval




Active rectangle

Current node

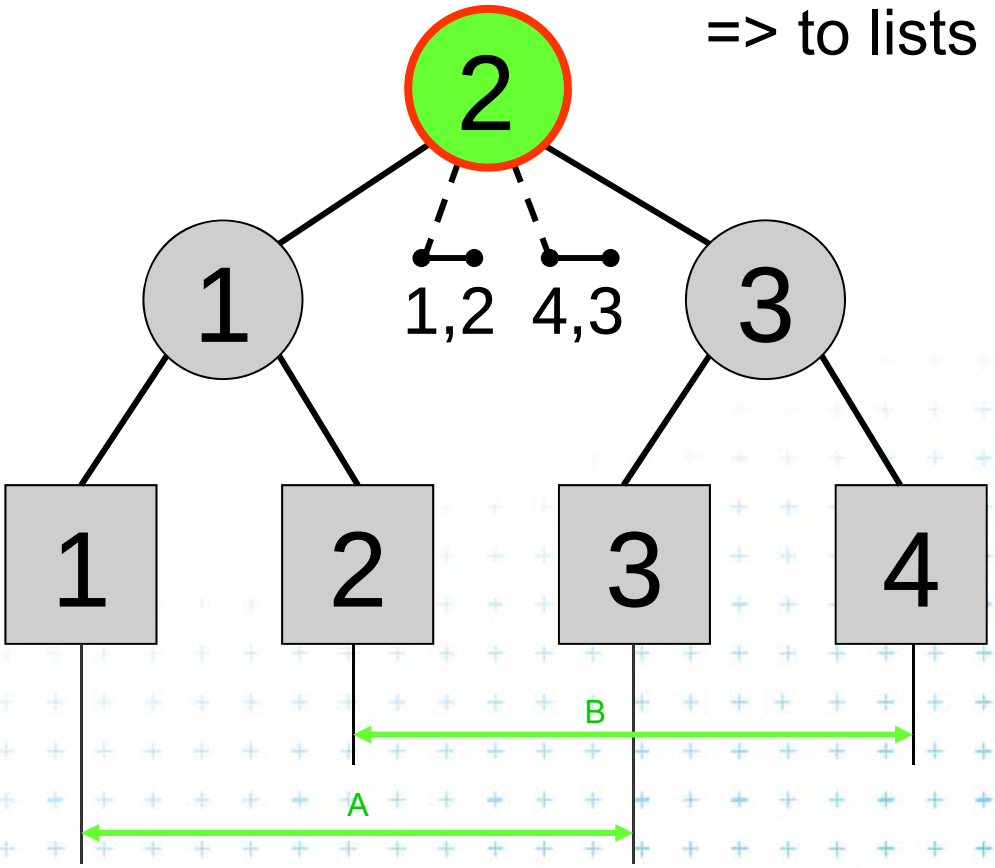
Active node

$b \le H(v) \le e$

$2 \le 2 \le 4$

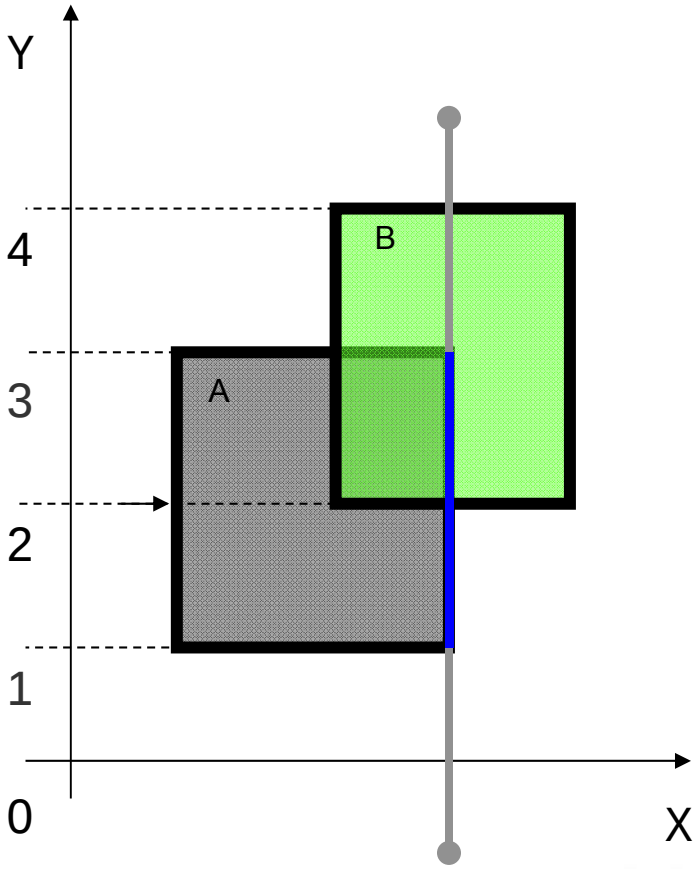
fork

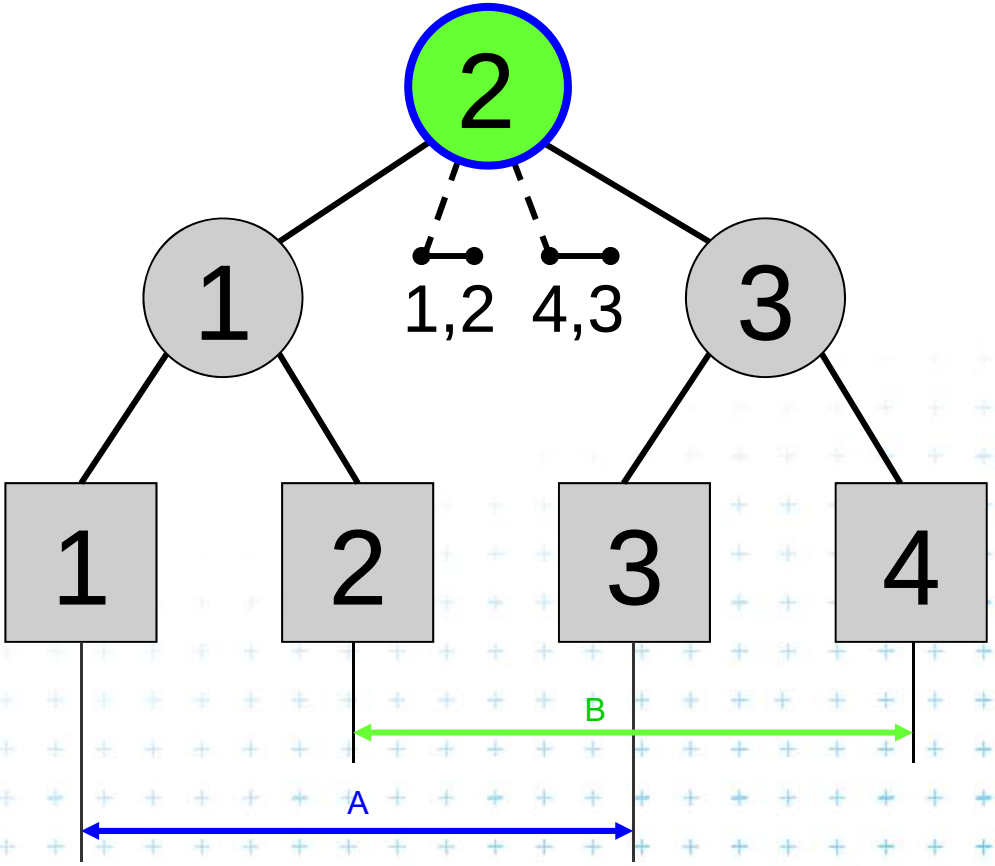
=> to lists




[Drtina]

# Interval delete [1,3]



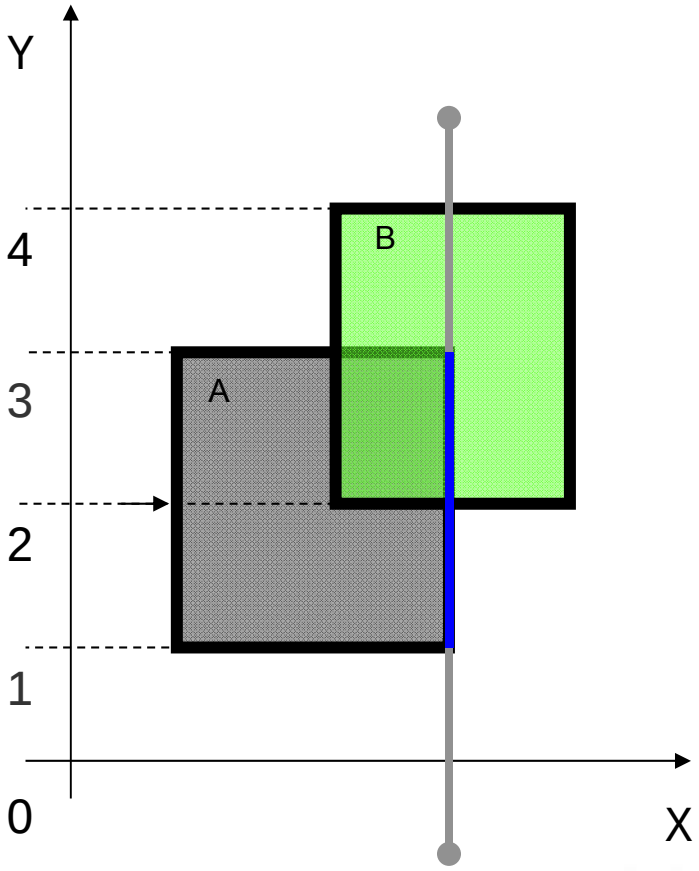

Active rectangle

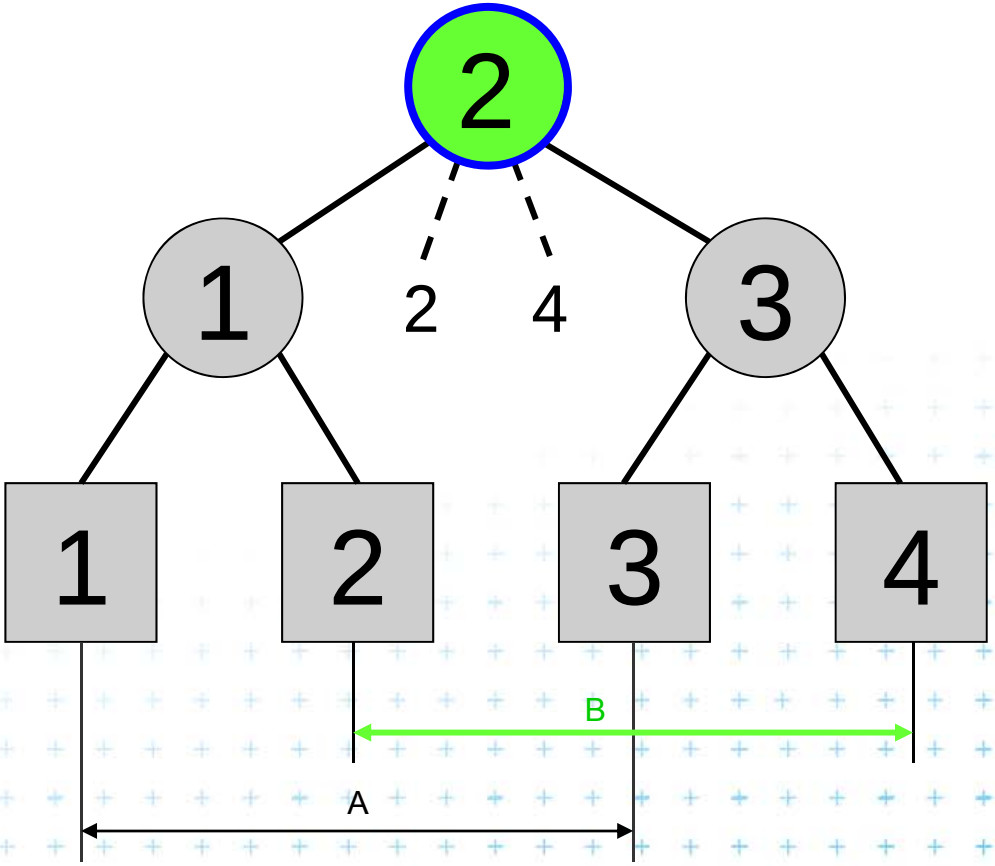
Current node

Active node

[Drtina]

# Interval delete [1,3]



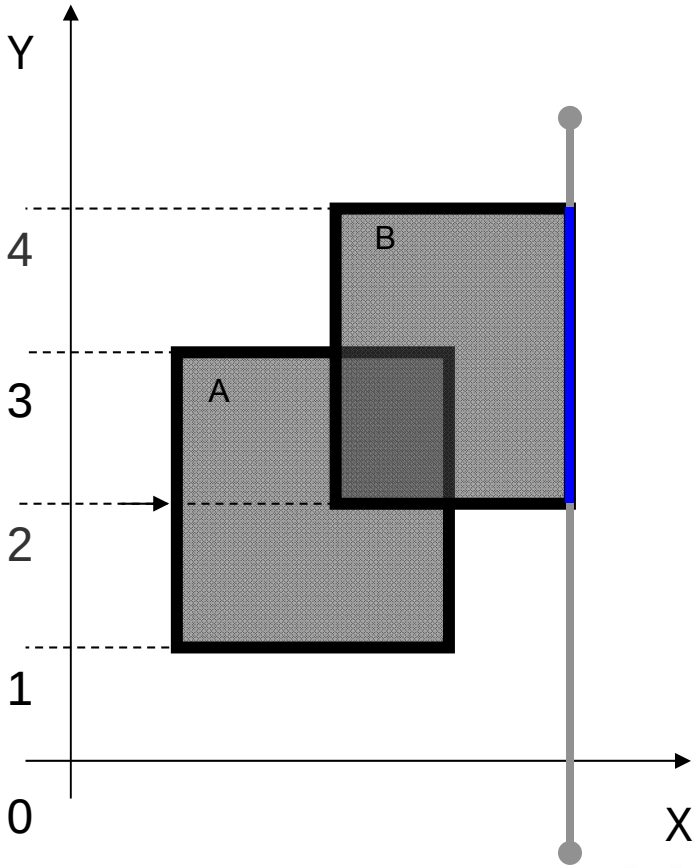

Active rectangle

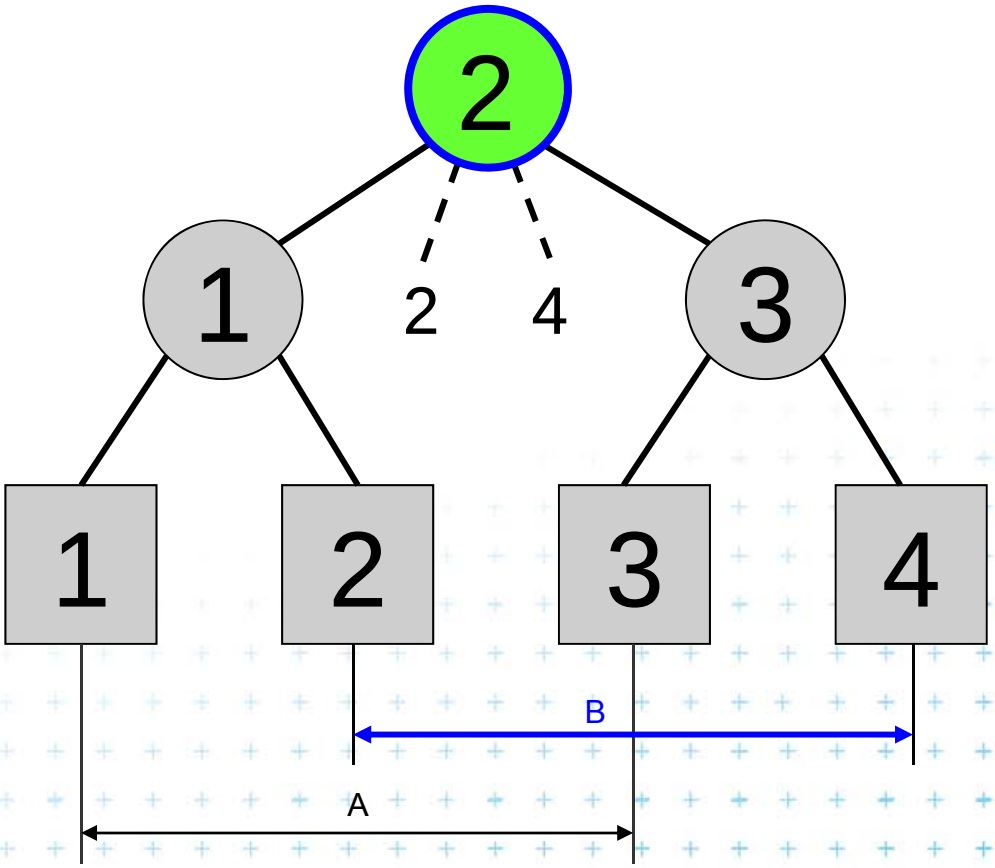
Current node

Active node

[Drtina]

# Interval delete [2,4]



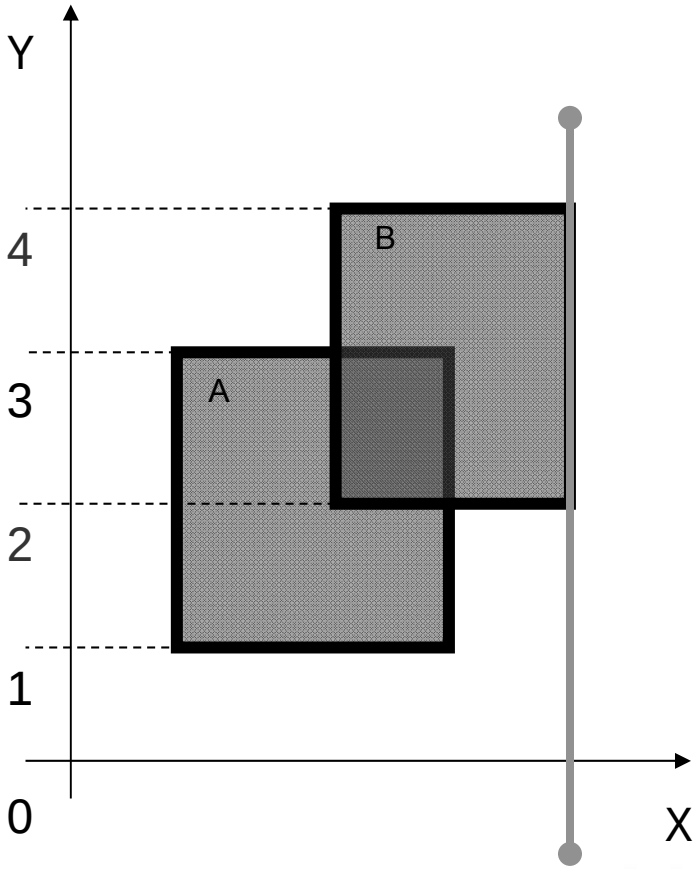

Active rectangle

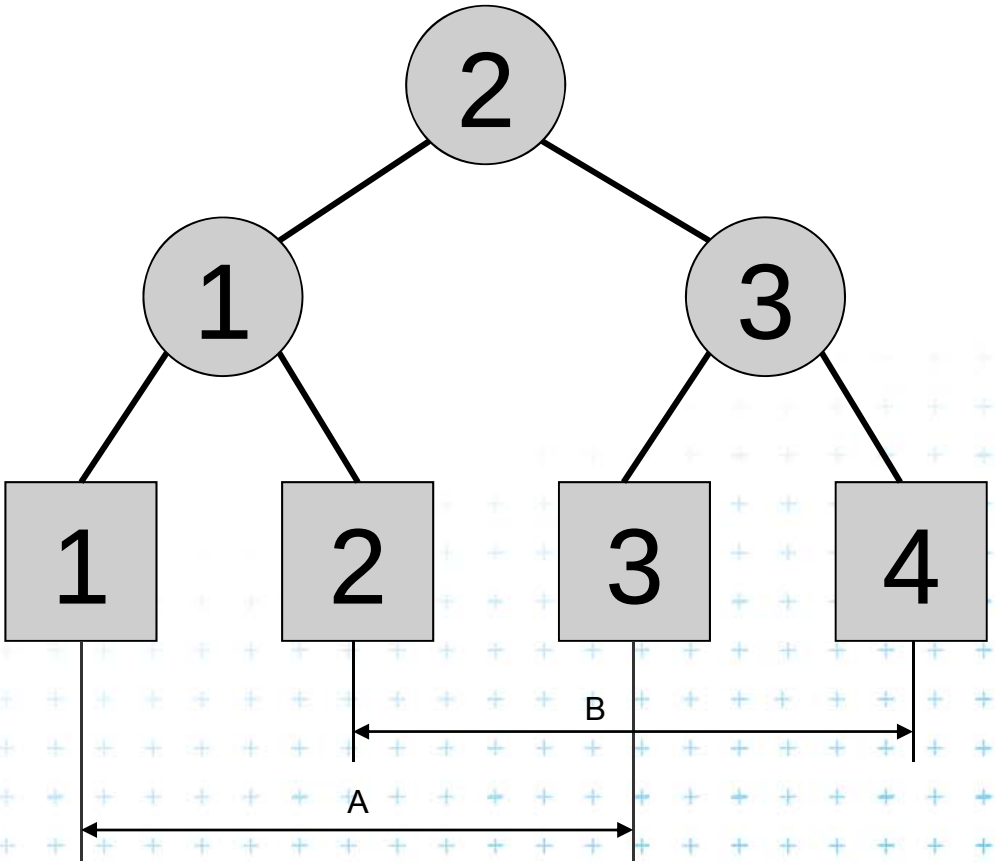
Current node

Active node

[Drtina]

# Interval delete [2,4]




[Drtina]

# Example 2

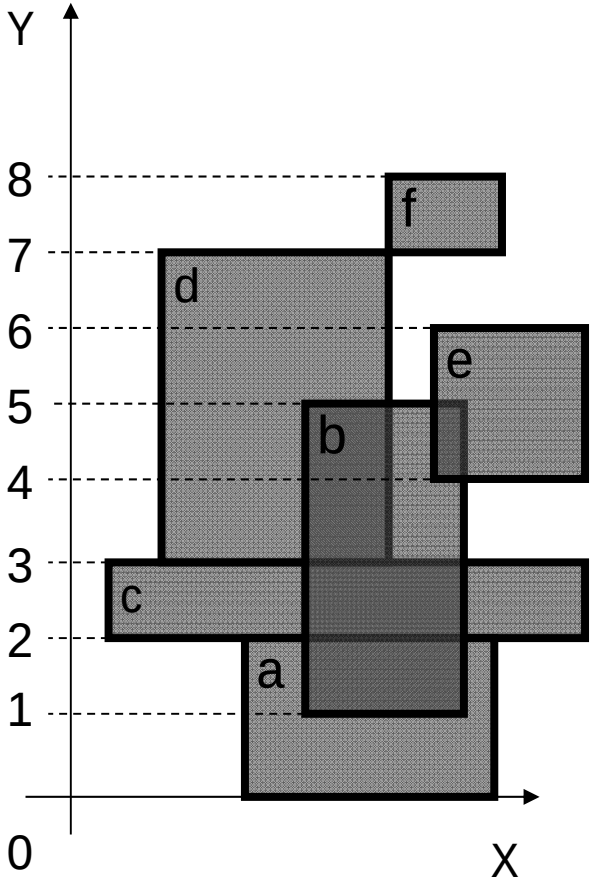
**RectangleIntersections( *S* )**
*Input:* Set *S* of rectangles
*Output:* Intersected rectangle pairs

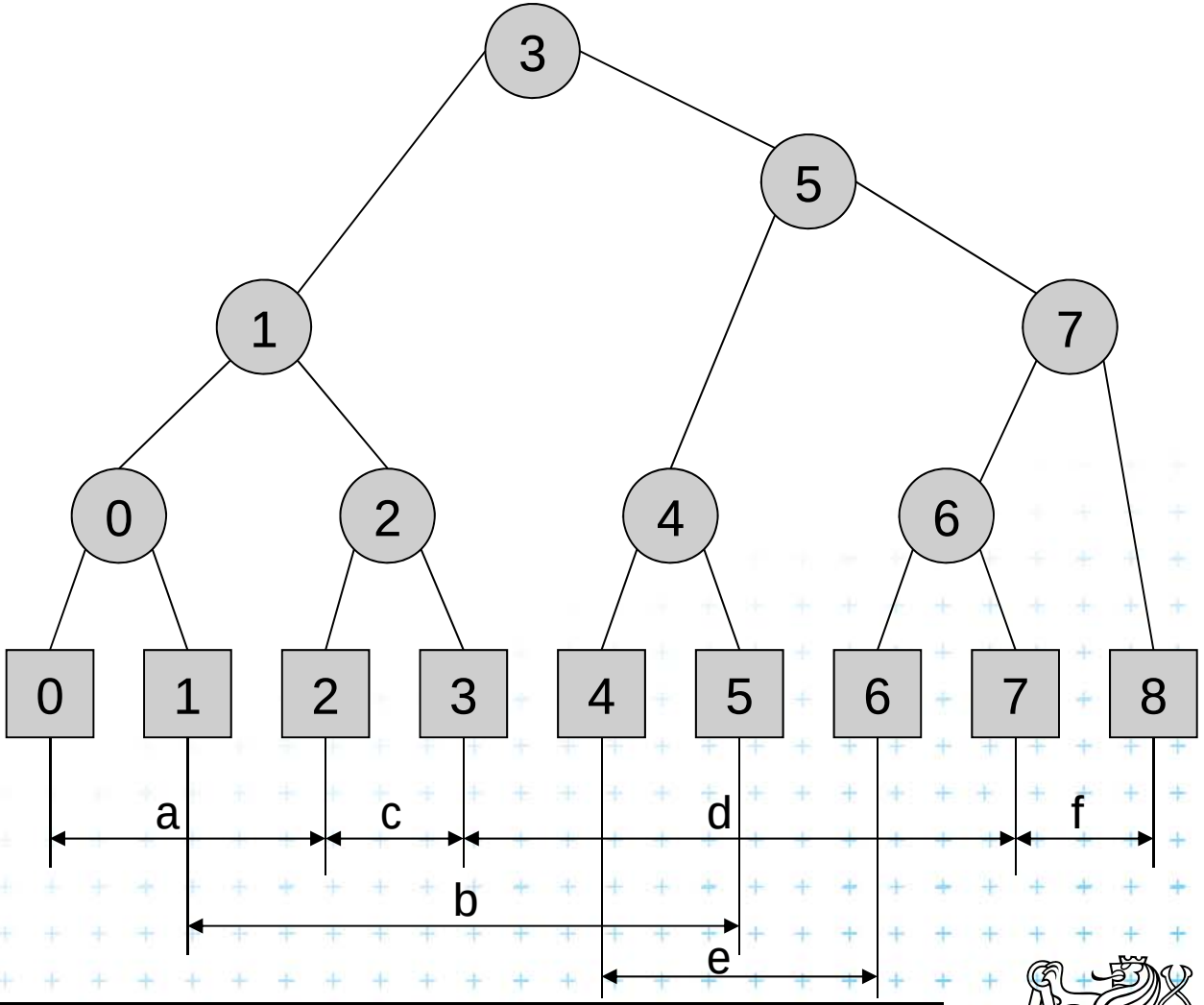
**// this is a copy of the slide before**
**// just to remember the algorithm**

1. Preprocess( S ) // create the interval tree *T* and event queue *Q*

2. **while** ( $Q \neq \emptyset$ ) do
3. Get next entry ($x_{il}$, $y_{il}$, $y_{ir}$, $t$) from *Q* // $t \in$ { *left* | *right* }
4. **if** ( $t$ = left ) // left edge
5. a) QueryInterval ( $y_{il}$, $y_{ir}$, root($T$)) // report intersections
6. b) InsertInterval ( $y_{il}$, $y_{ir}$, root($T$)) // insert new interval
7. **else** // right edge
8. c) DeleteInterval ( $y_{il}$, $y_{ir}$, root($T$) )

# Example 2




[Drtina]

# Insert [2,3] – empty => b) Insert Interval

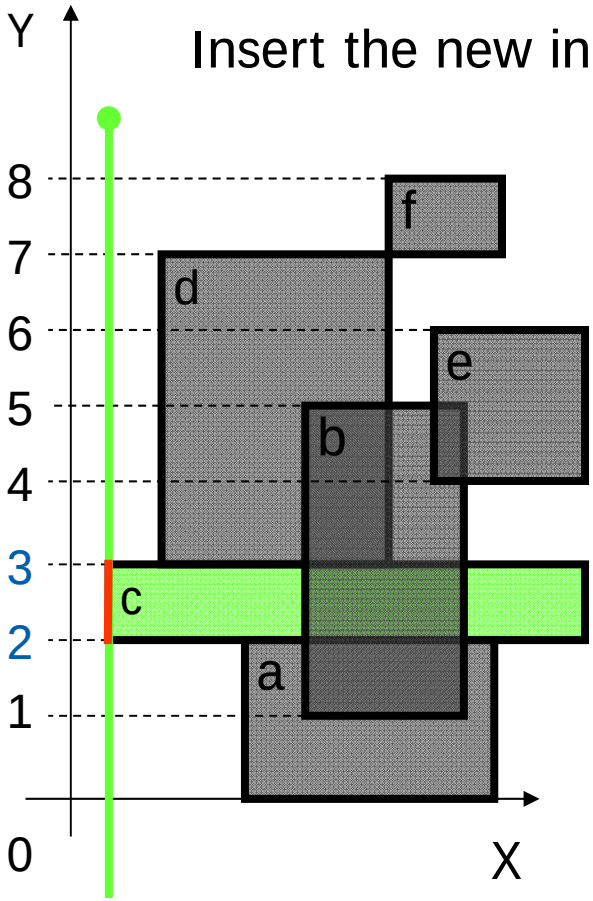
$b \le H(v) \le e$

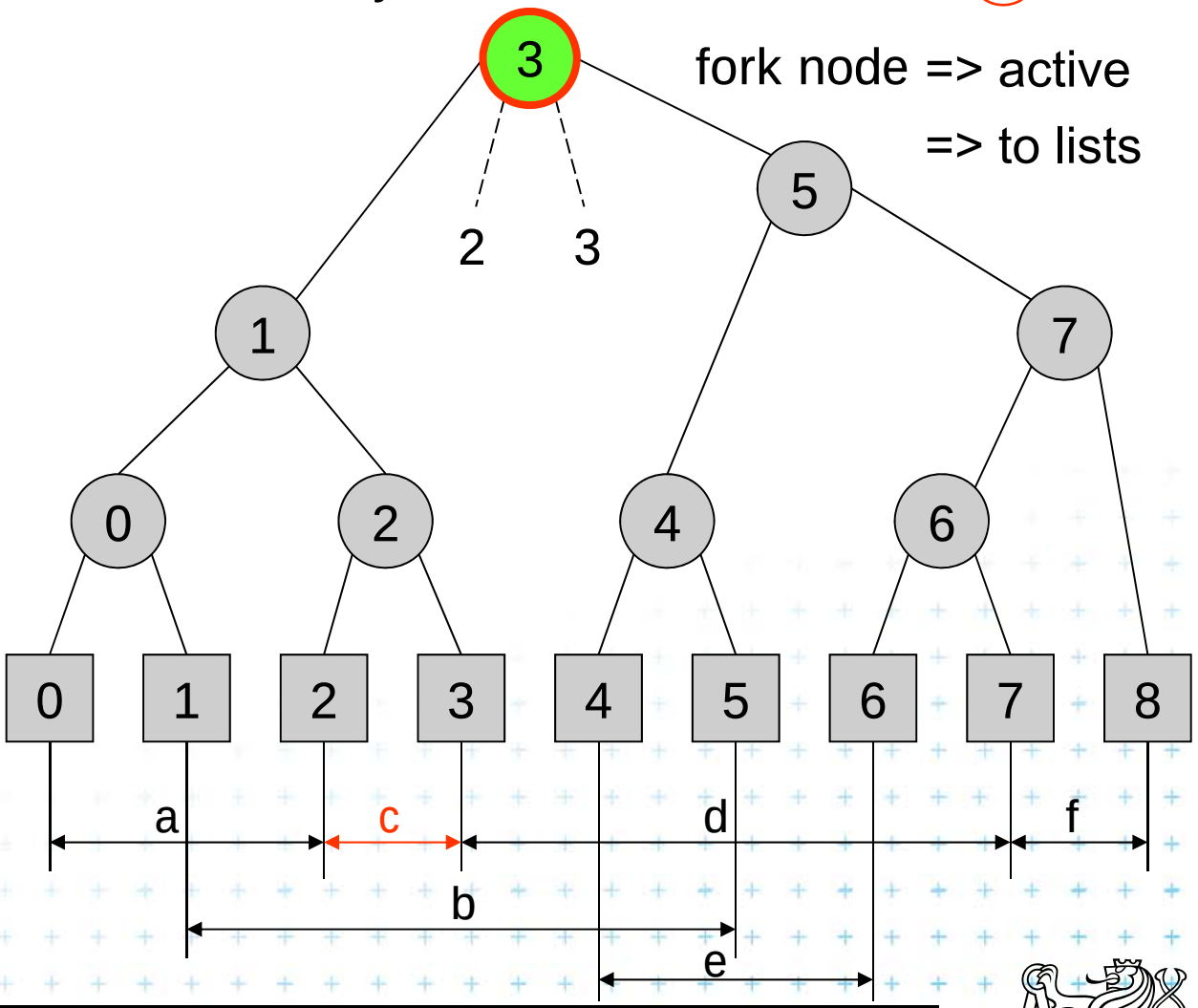
Insert the new interval to secondary lists

? 2 ≤ 3 ≤ 3 ?

fork node => active

=> to lists




Active rectangle

Current node

Active node

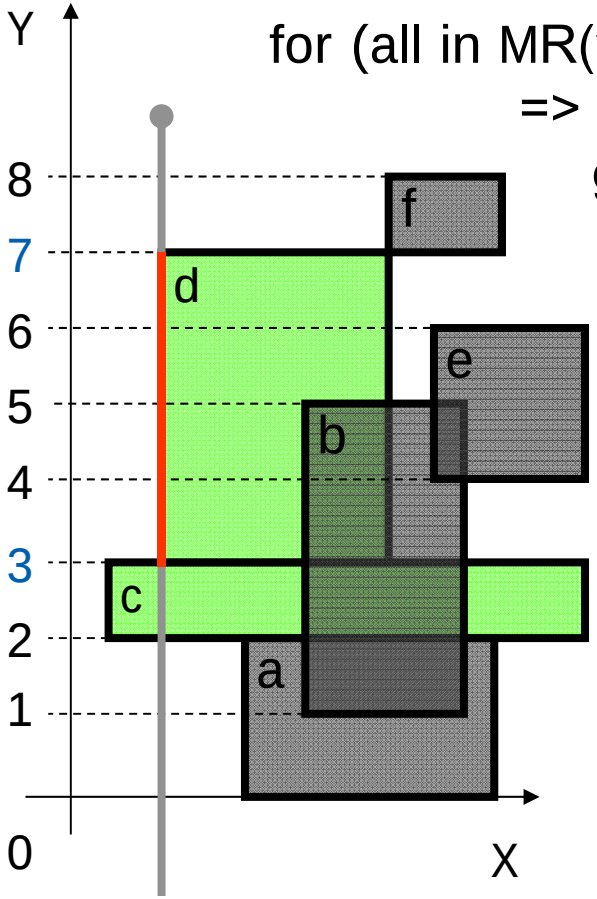
[Drtina]

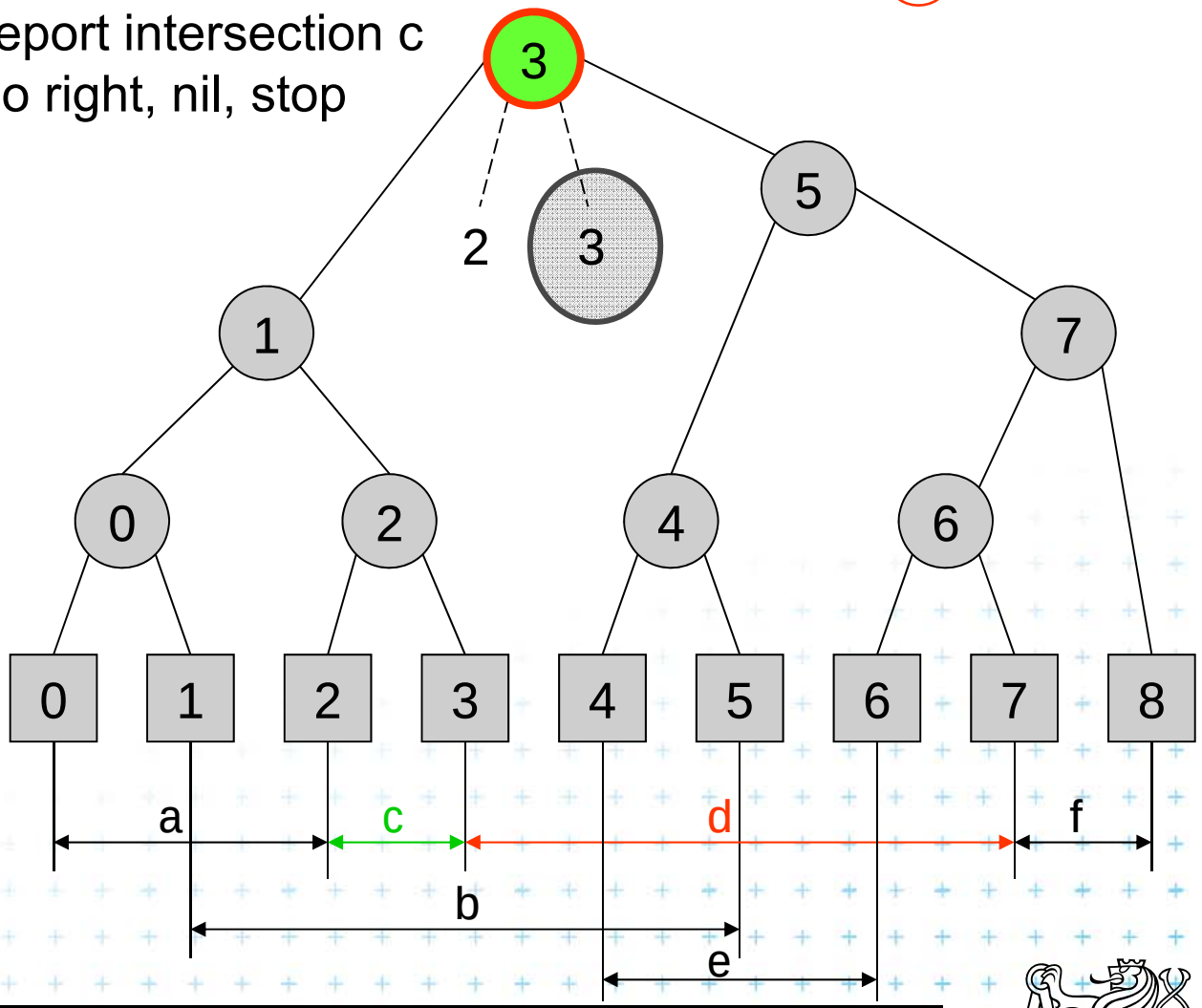
# Insert [3,7] a) Query Interval

$H(v) \le b < e$

for (all in MR(v)) test MR(v)[i] >= 3
=> report intersection c
go right, nil, stop

? 3 ≤ 3 < 7 ?




Active rectangle

Current node

Active node

[Drtina]

# Insert [3,7] b) Insert Interval

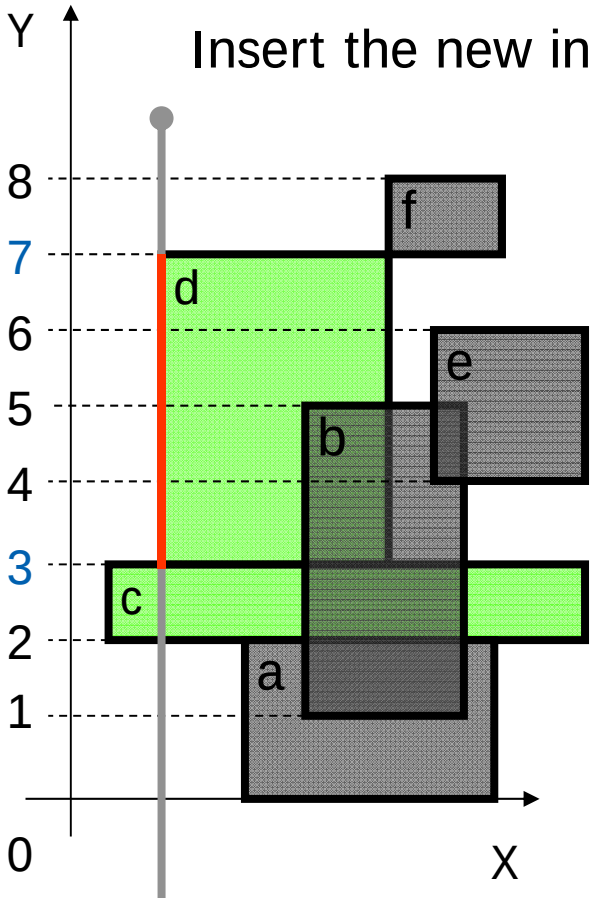
$b \le H(v) \le e$

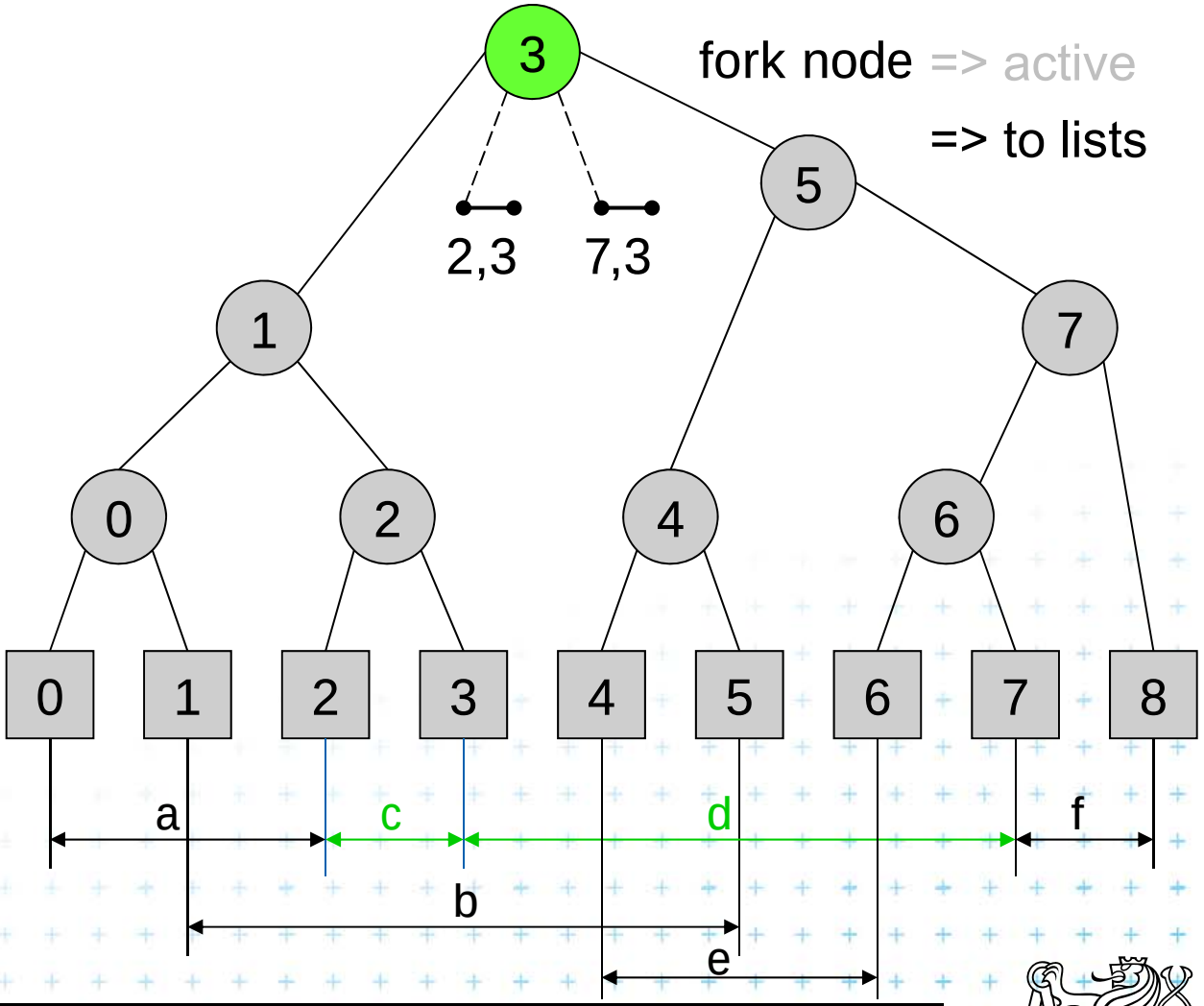
Insert the new interval to secondary lists

$3 \le 3 \le 7$

fork node => active

=> to lists




Active rectangle

Current node

Active node

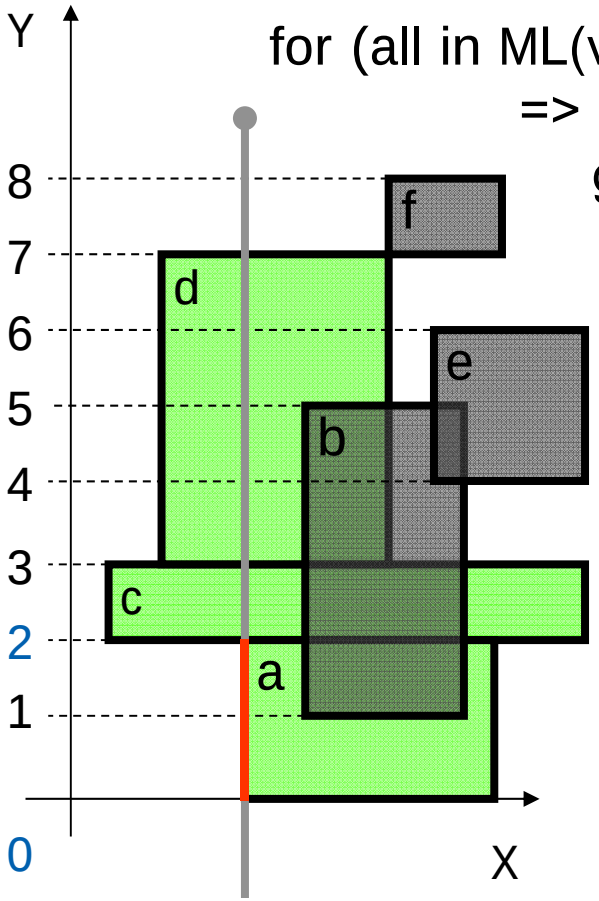
[Drtina]

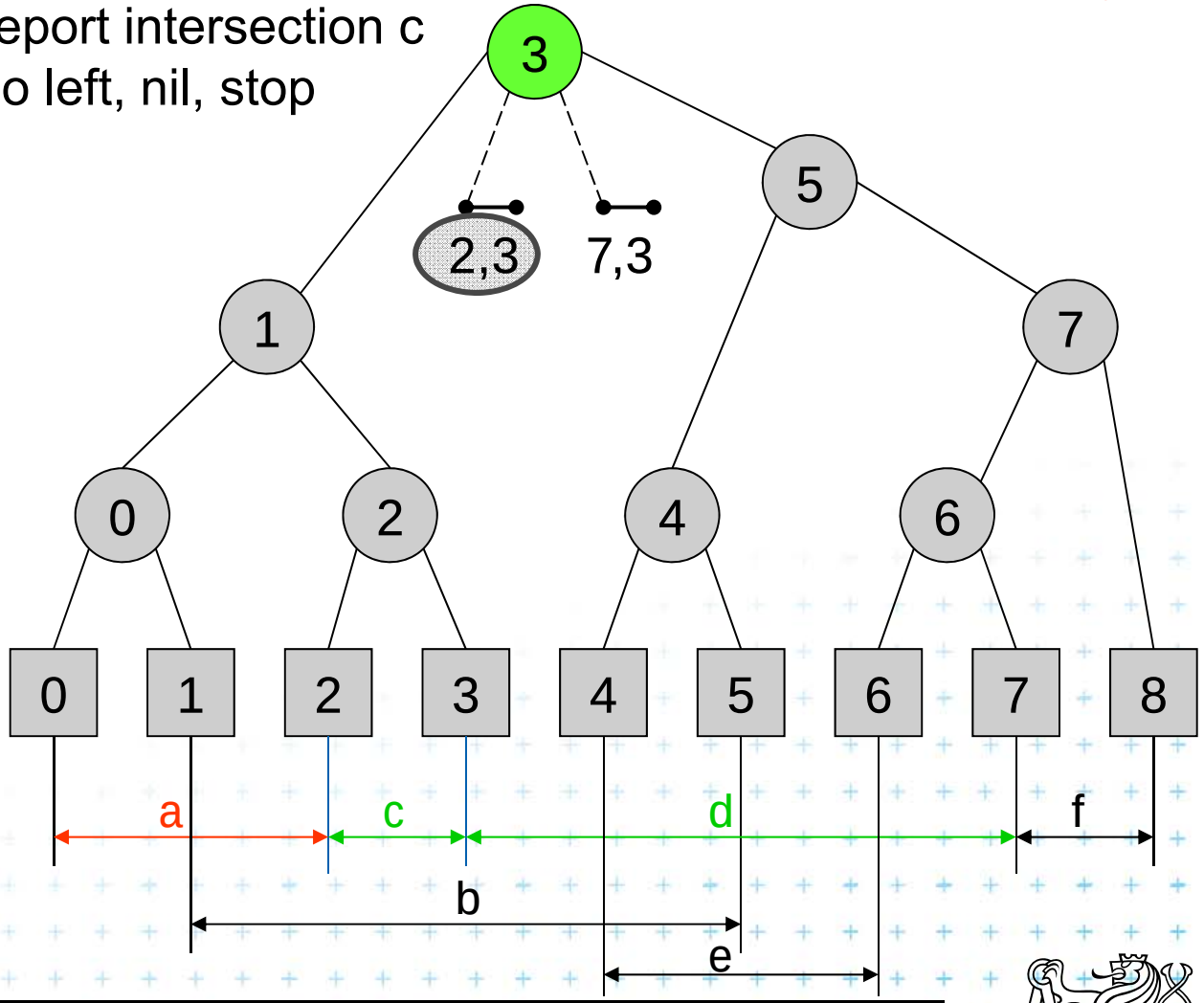
# Insert [0,2] a) Query Interval

$b < e \leq H(v)$

for (all in ML(v)) test ML(v).[i] $\leq 2$
=> report intersection c
go left, nil, stop

? $0 < 2 \leq 3$ ?




Active rectangle

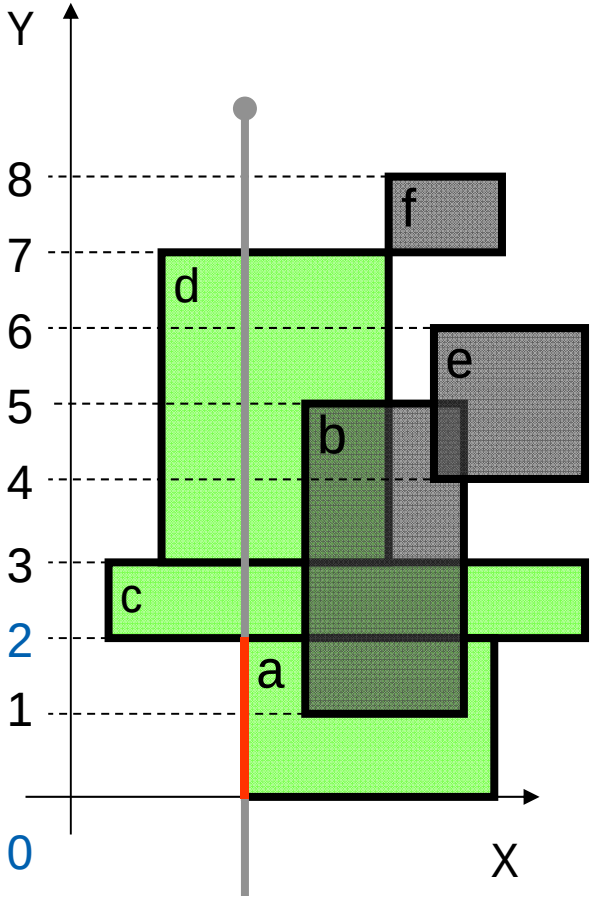
Current node

Active node

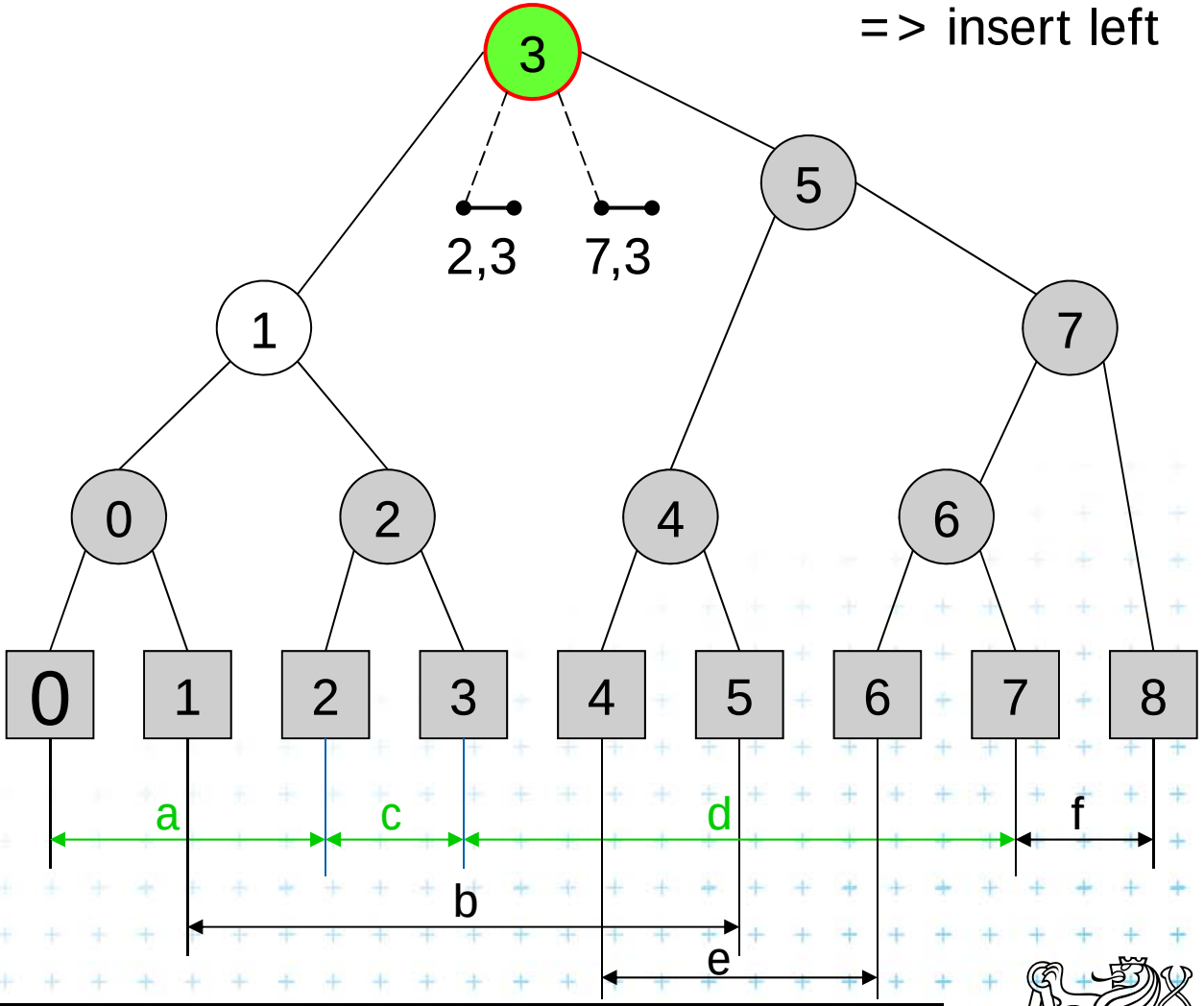
[Drtina]

# Insert [0,2] b) Insert Interval 1/2

b < e <H(v)




? 0 < 2 < 3 ?
=> insert left




Active rectangle

Current node

Active node

[Drtina]

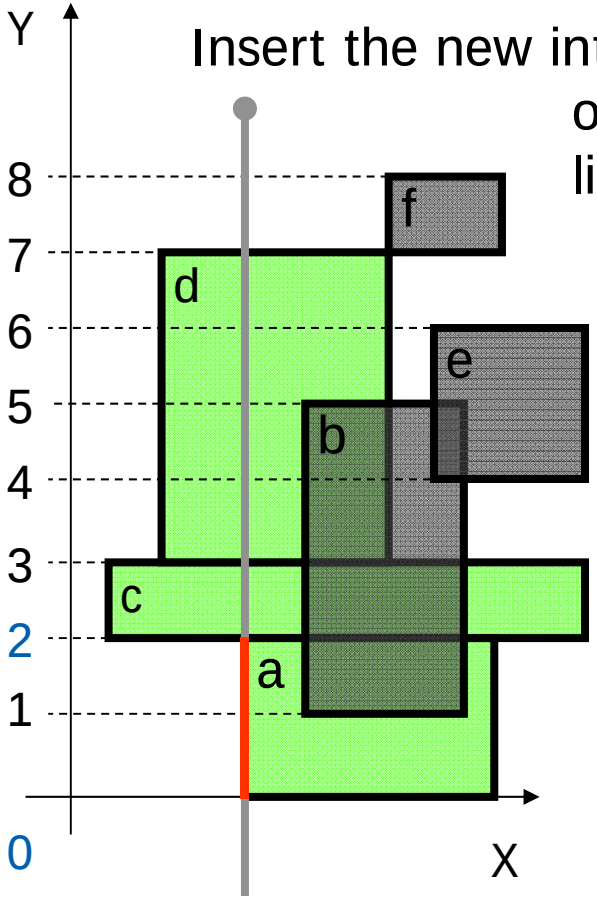
# Insert [0,2] b) Insert Interval 2/2

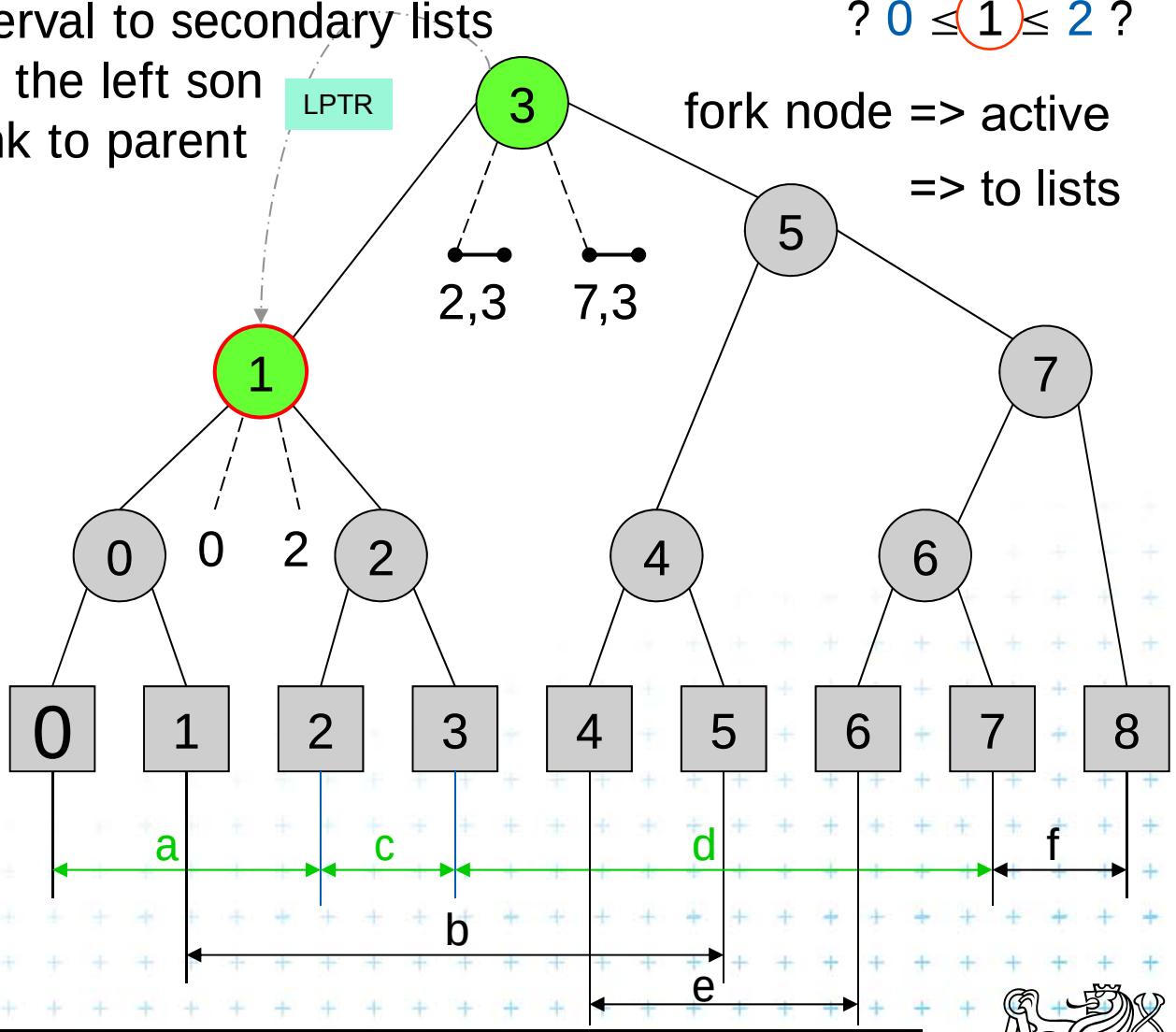
$b \le H(v) \le e$

Insert the new interval to secondary lists of the left son link to parent

? 0 ≤ 1 ≤ 2 ?

fork node => active => to lists




Active rectangle

Current node

Active node

[Drtina]

# Insert [1,5] a) Query Interval 1/2

b < H(v) < e

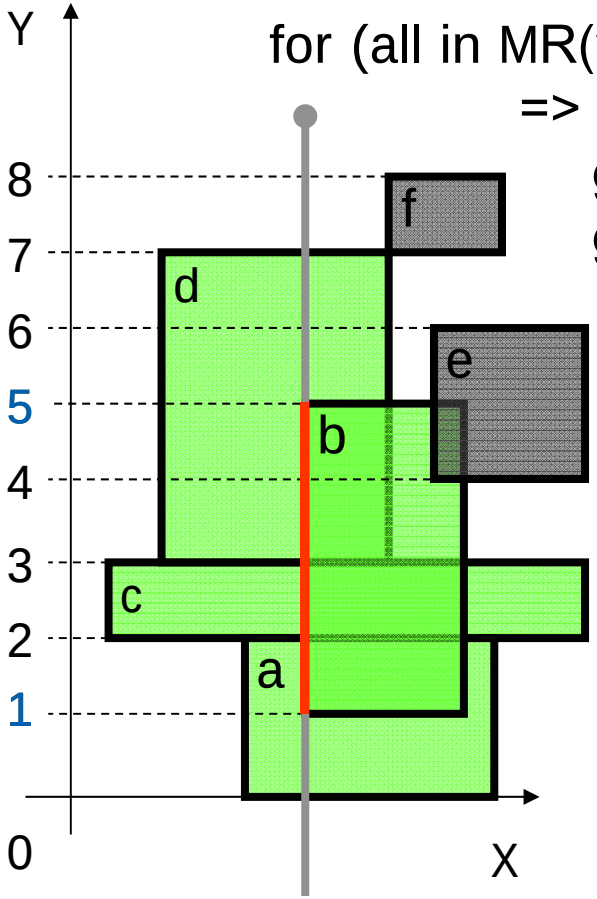
for (all in MR(v))

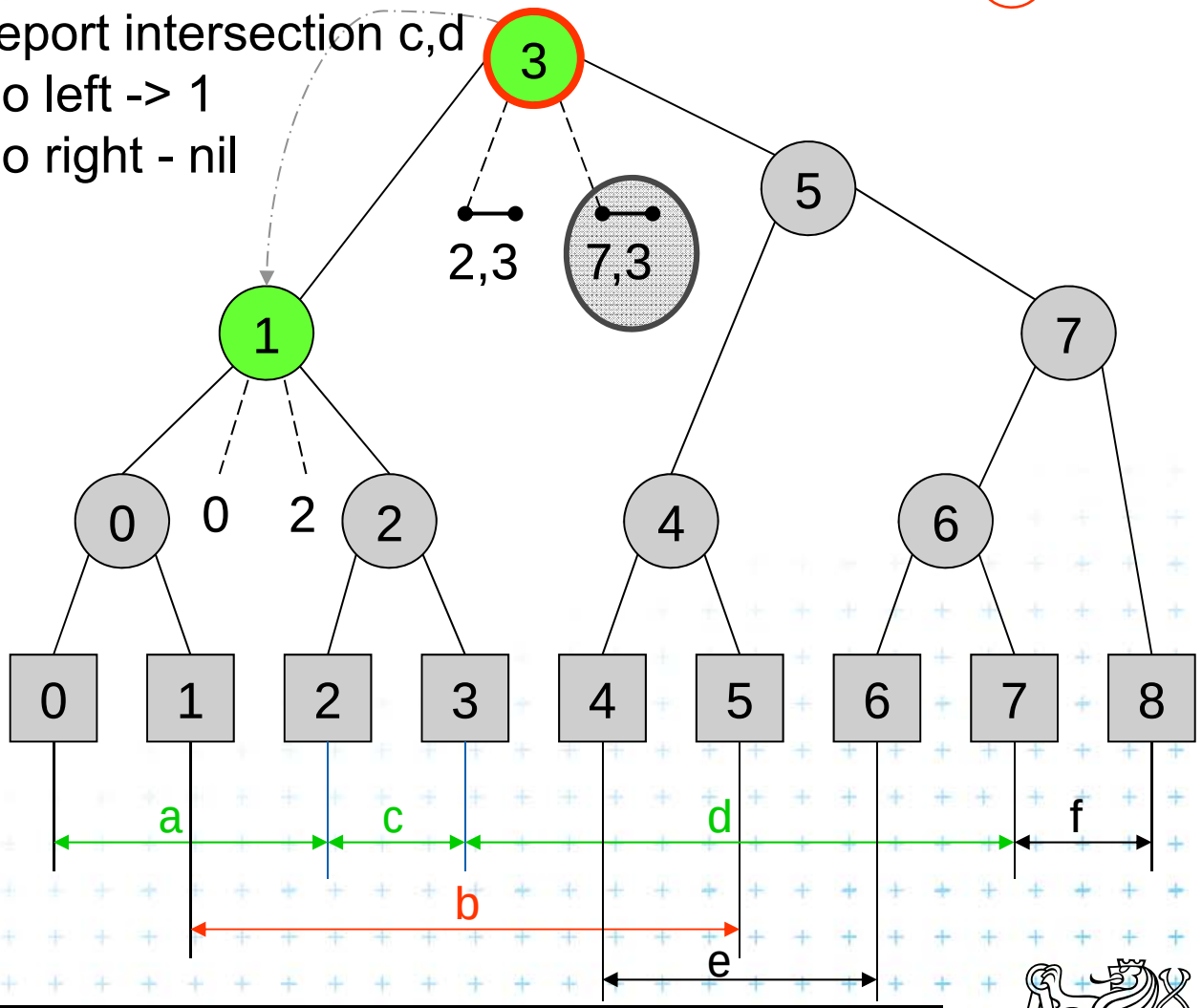
=> report intersection c,d

go left -> 1

go right - nil

? 1 < 3 < 5 ?




Active rectangle

Current node

Active node

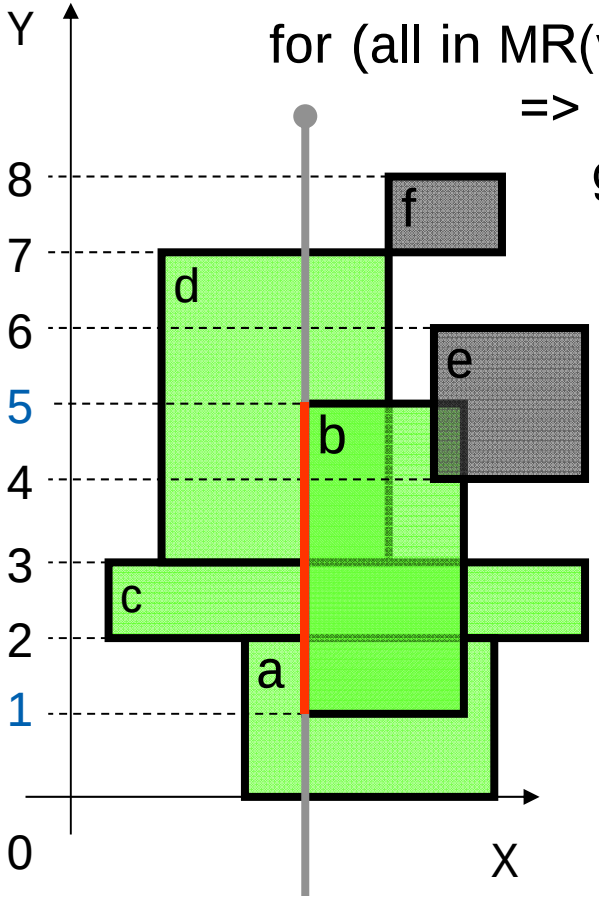
[Drtina]

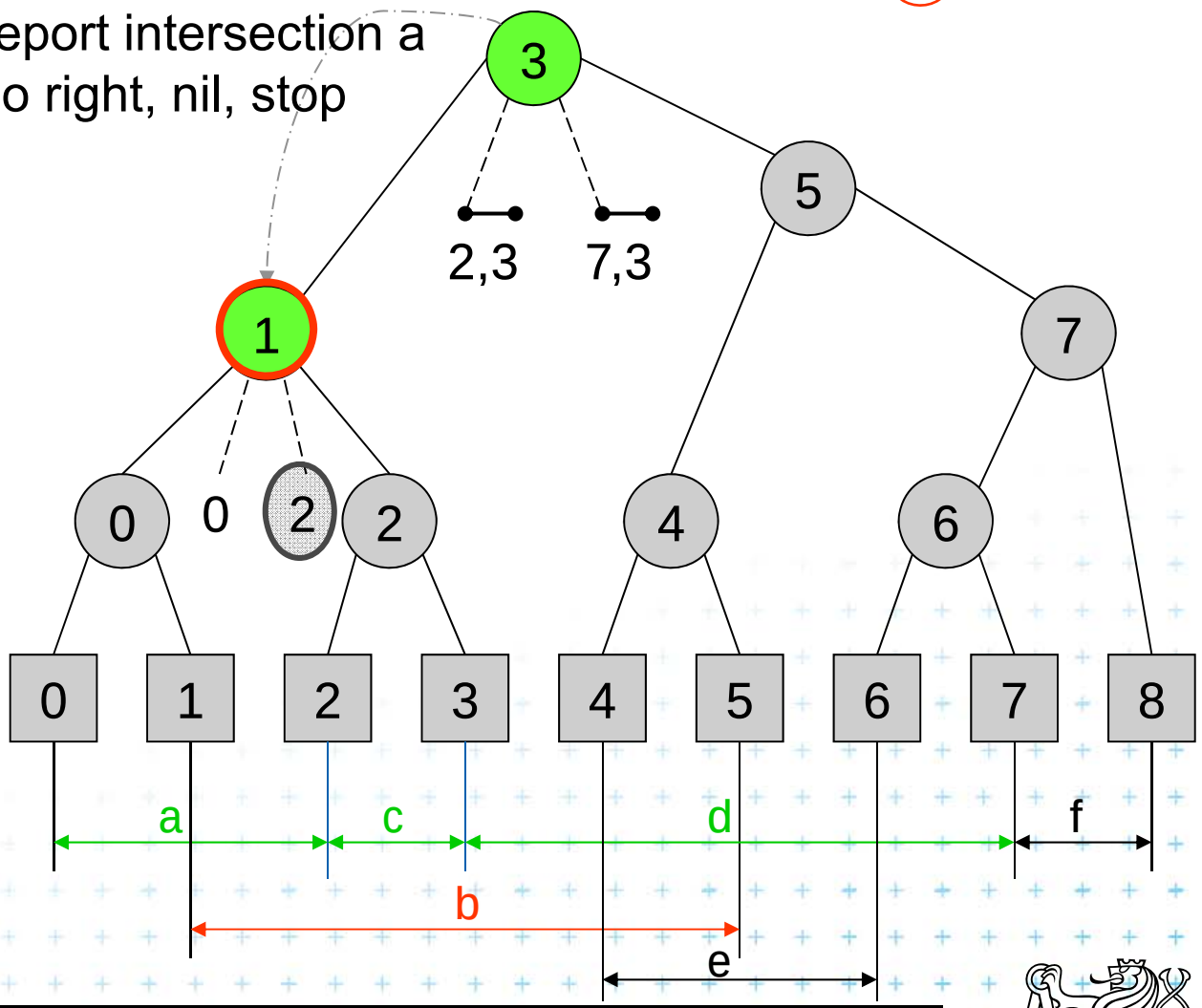
# Insert [1,5] a) Query Interval 2/2

H(v) ≤ b < e

for (all in MR(v)) test MR(v)[i] ≥ 1
=> report intersection a
go right, nil, stop

? 1 ≤ 1 < 5 ?




Active rectangle

Current node

Active node

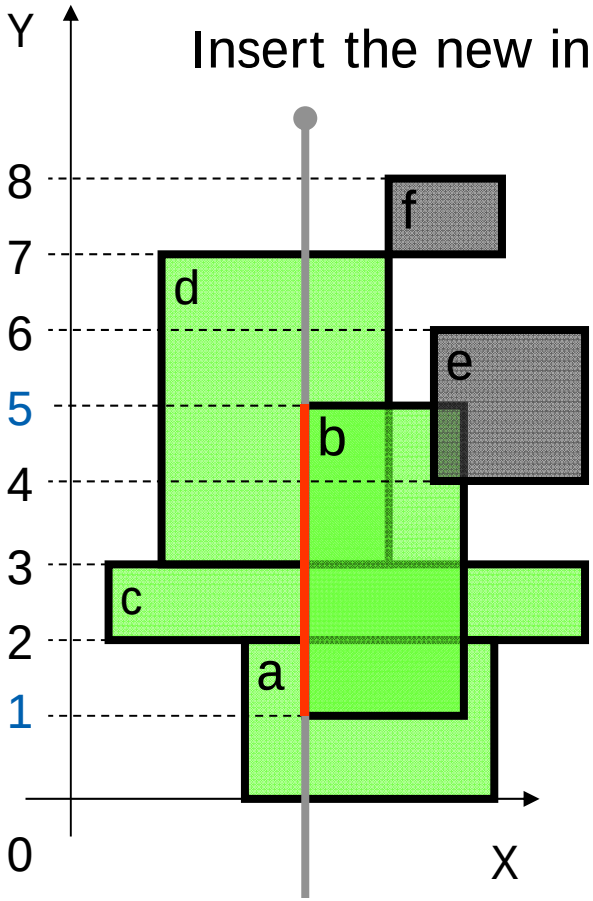
[Drtina]

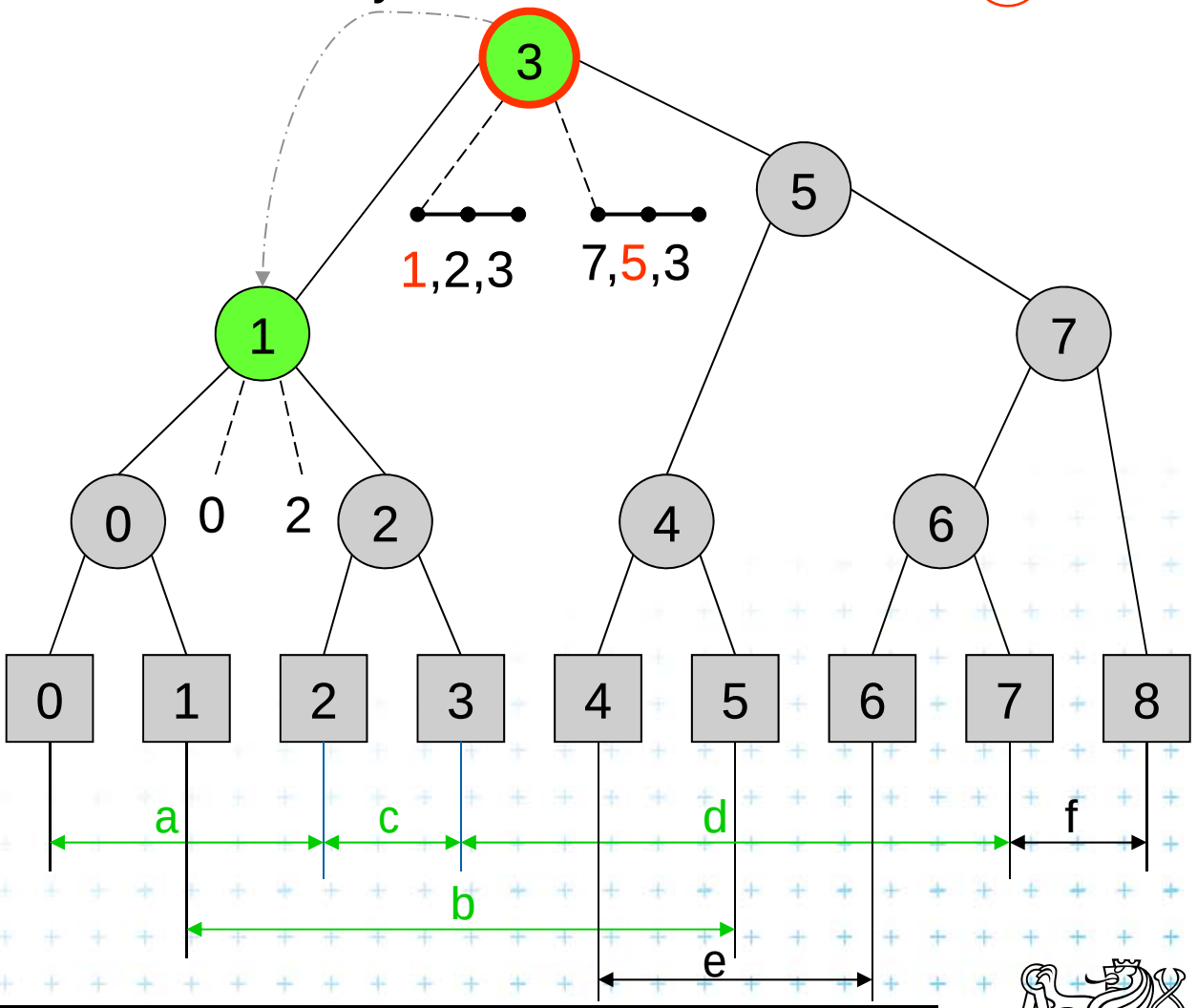
# Insert [1,5] b) Insert Interval

$b \le H(v) \le e$

Insert the new interval to secondary lists

? 1 ≤ 3 ≤ 5 ?




Active rectangle

Current node

Active node

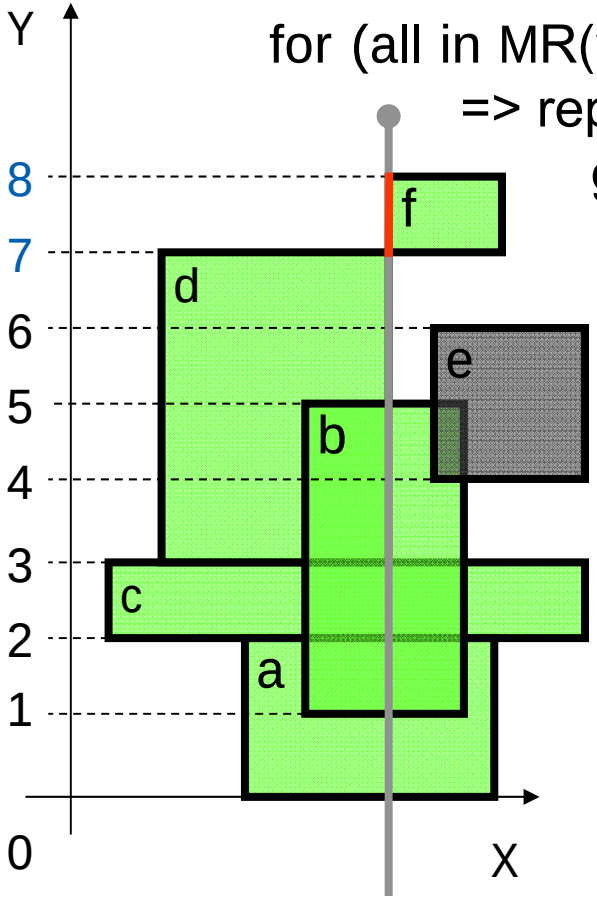
[Drtina]

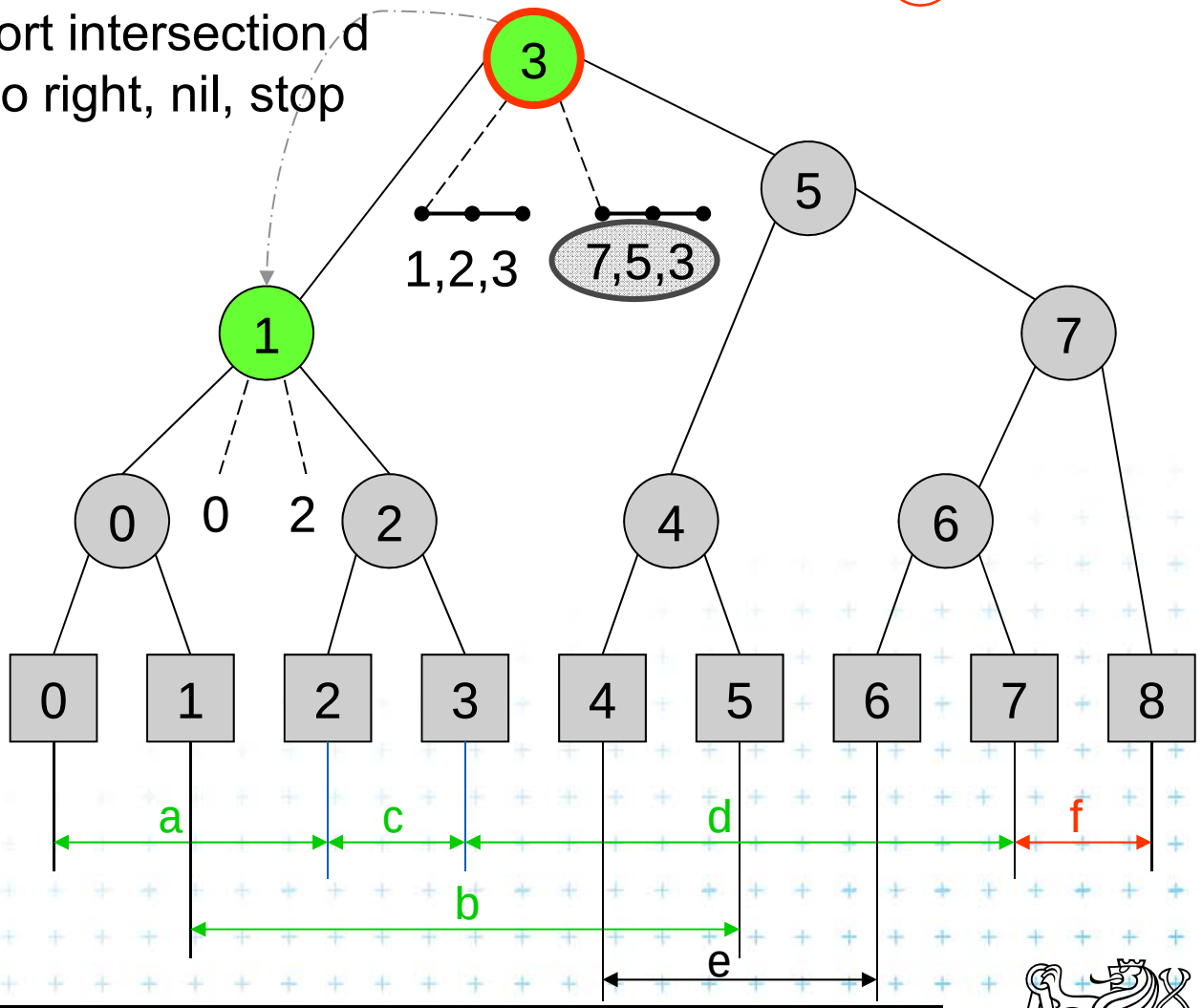
# Insert [7,8] a) Query Interval

H(v) ≤ b < e

for (all in MR(v)) test MR(v).[i] ≥ 7
=> report intersection d
go right, nil, stop

? 3 ≤ 7 < 8 ?




Active rectangle

Current node

Active node

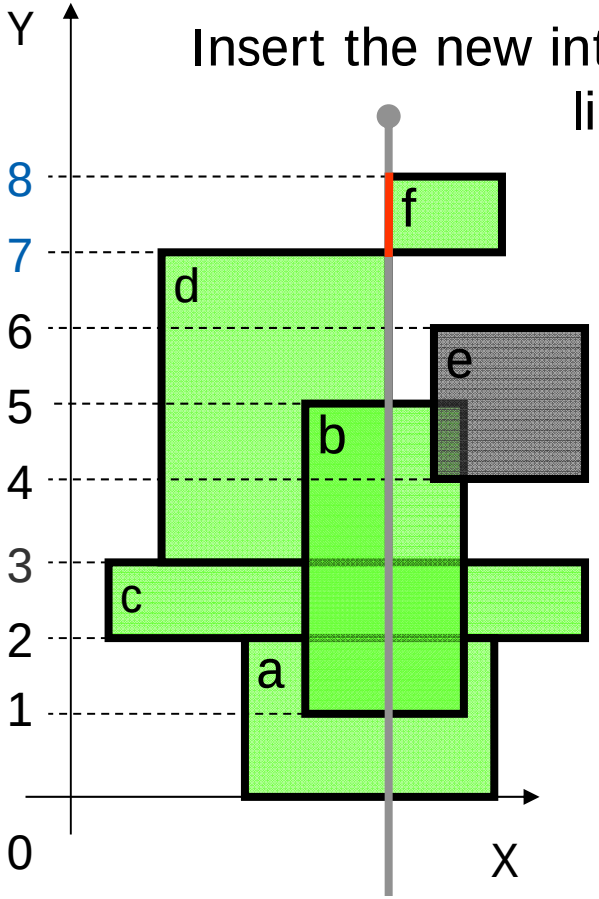
[Drtina]

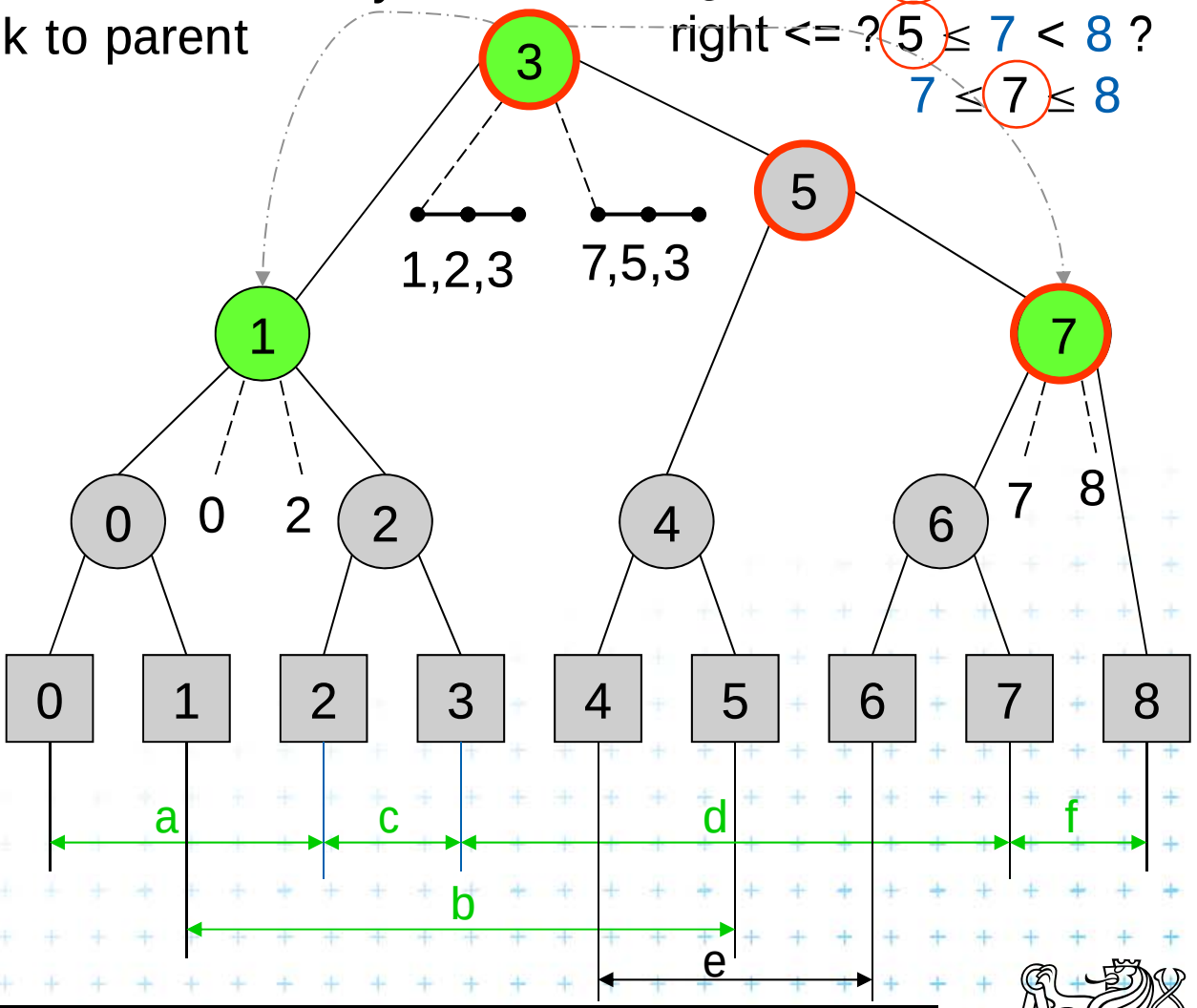
# Insert [7,8] b) Insert Interval

$b \le H(v) \le e$

Insert the new interval to secondary lists
link to parent

right <= ? 3 ≤ 7 < 8 ?
right <= ? 5 ≤ 7 < 8 ?
7 ≤ 7 ≤ 8




Active rectangle
Current node
Active node

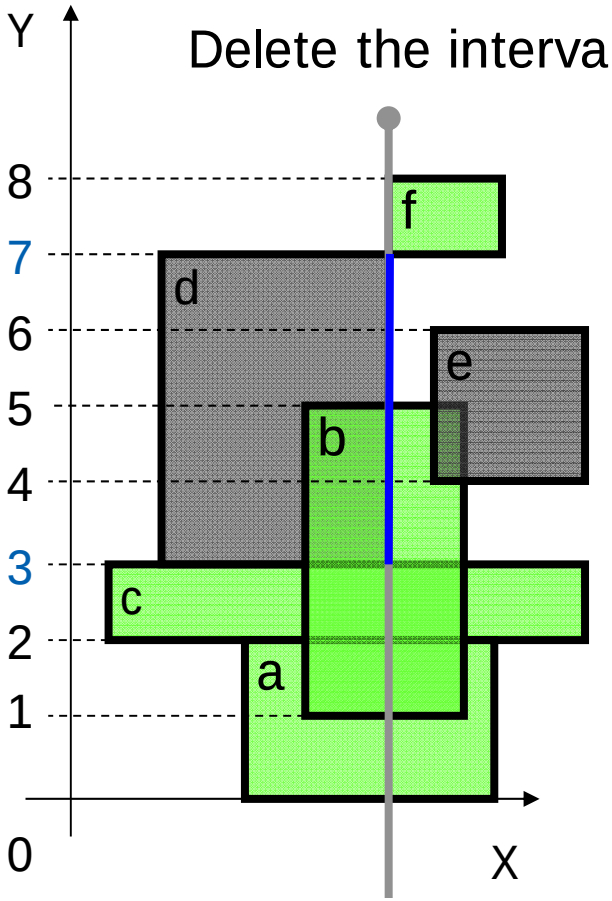
[Drtina]

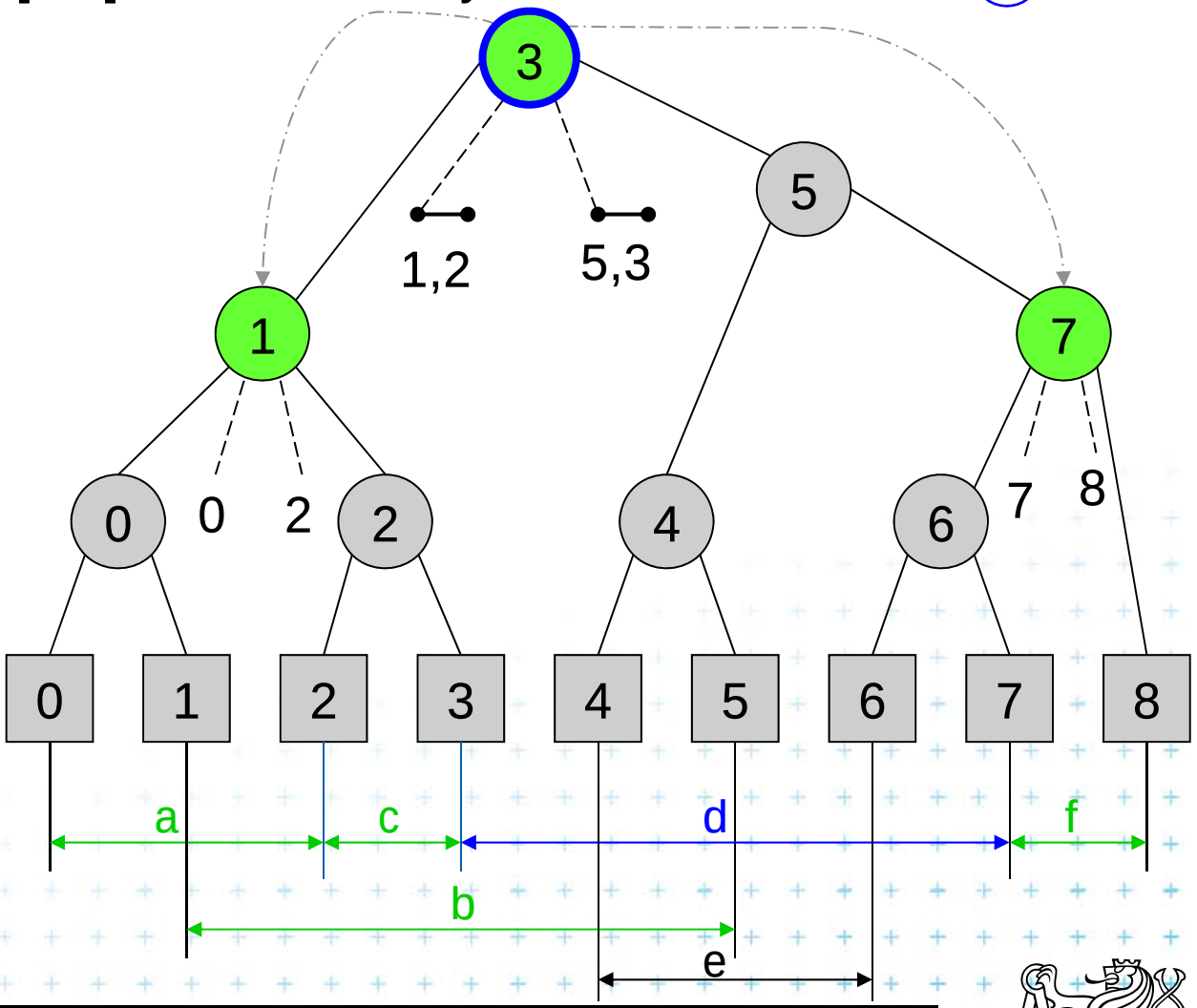
# Delete [3,7] Delete Interval

$b \le H(v) \le e$

Delete the interval [3,7] from secondary lists

? 3 ≤ 7 ≤ 8 ?




Active rectangle

Current node

Active node

[Drtina]

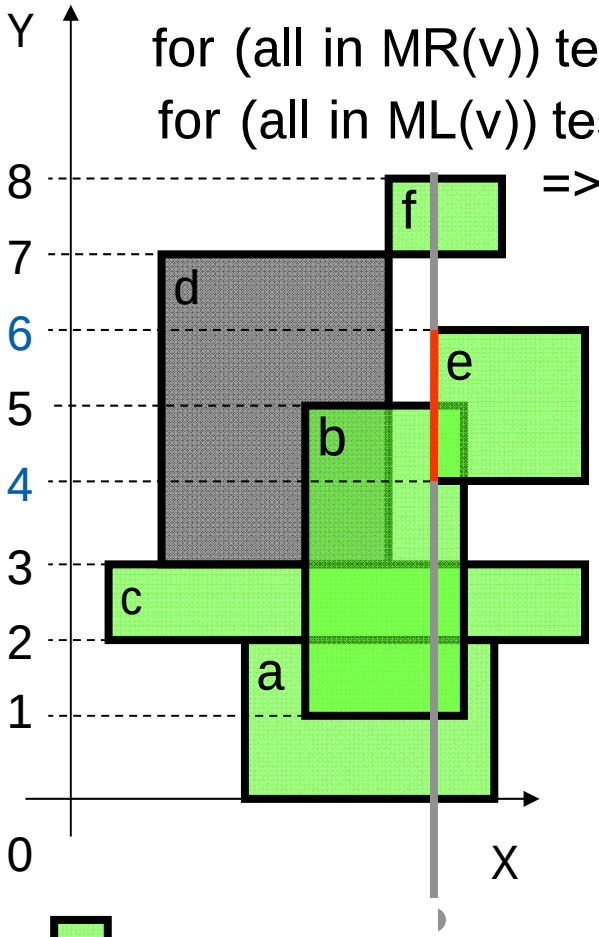
# Insert [4,6] a) Query Interval

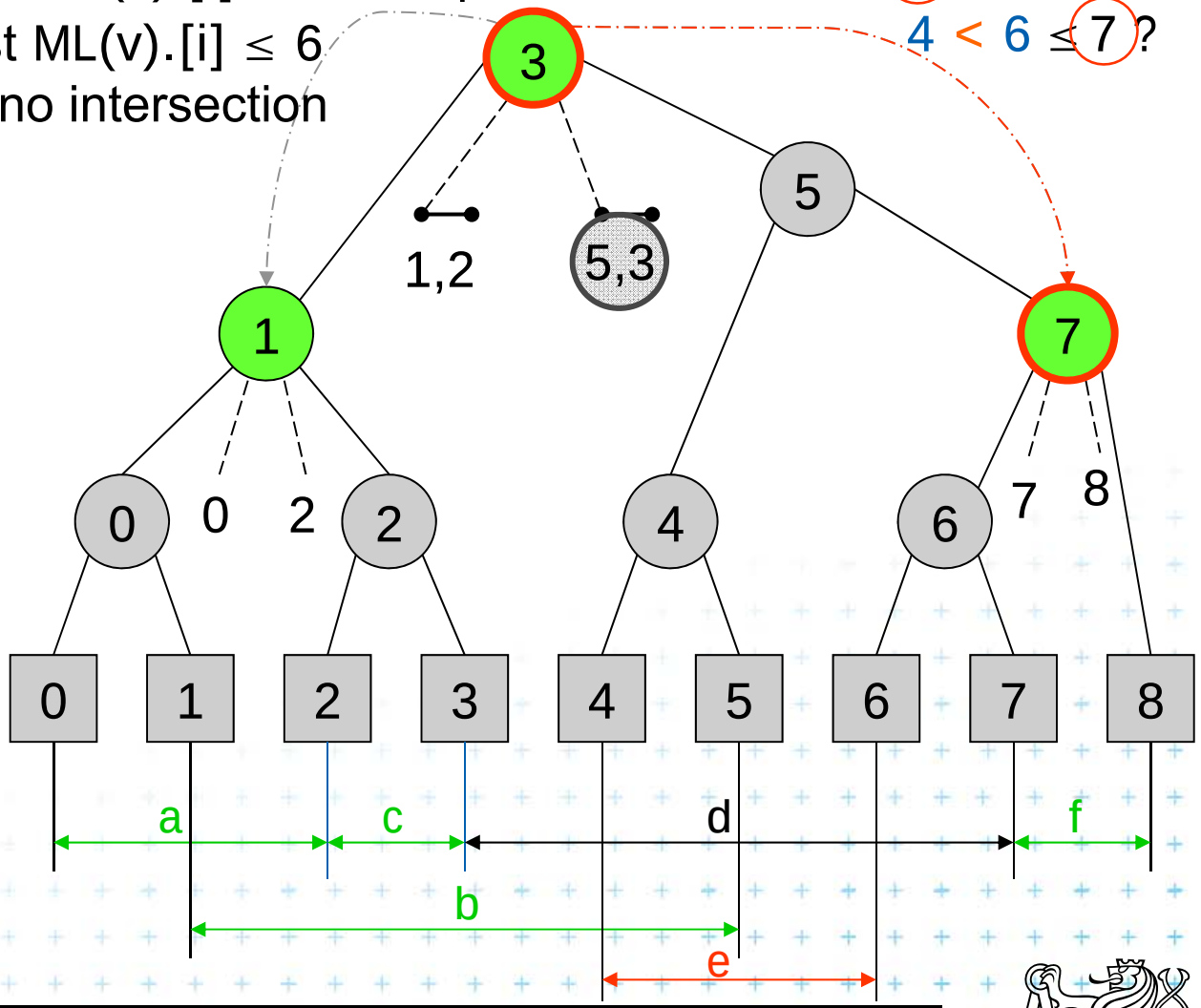
$H(v) \le b < e$

for (all in MR(v)) test MR(v).[i] $\ge 4$ => report intersection b $3 \le 4 < 6$ ?

for (all in ML(v)) test ML(v).[i] $\le 6$ $4 < 6 \le 7$ ?

=> no intersection




Active rectangle

Current node

Active node

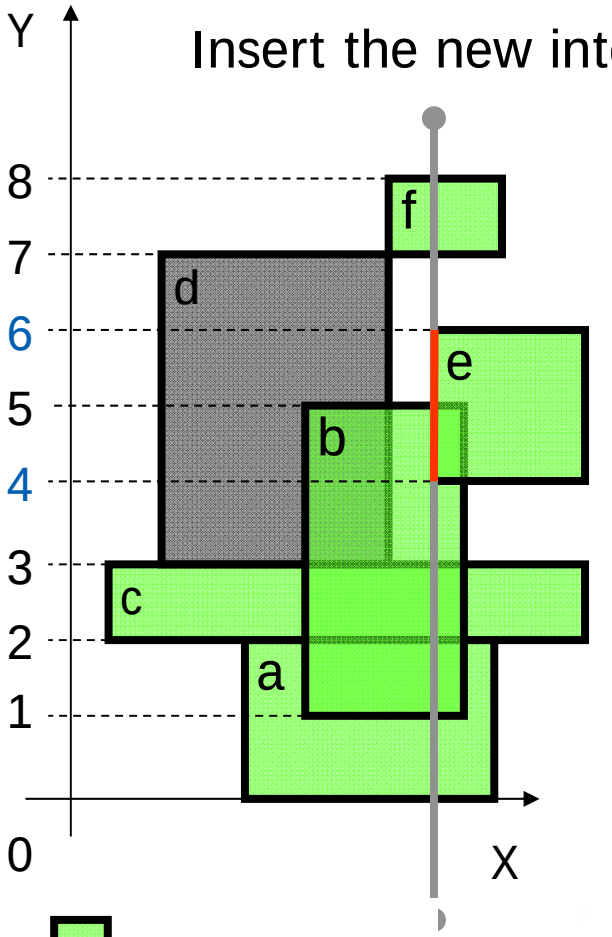
[Drtina]

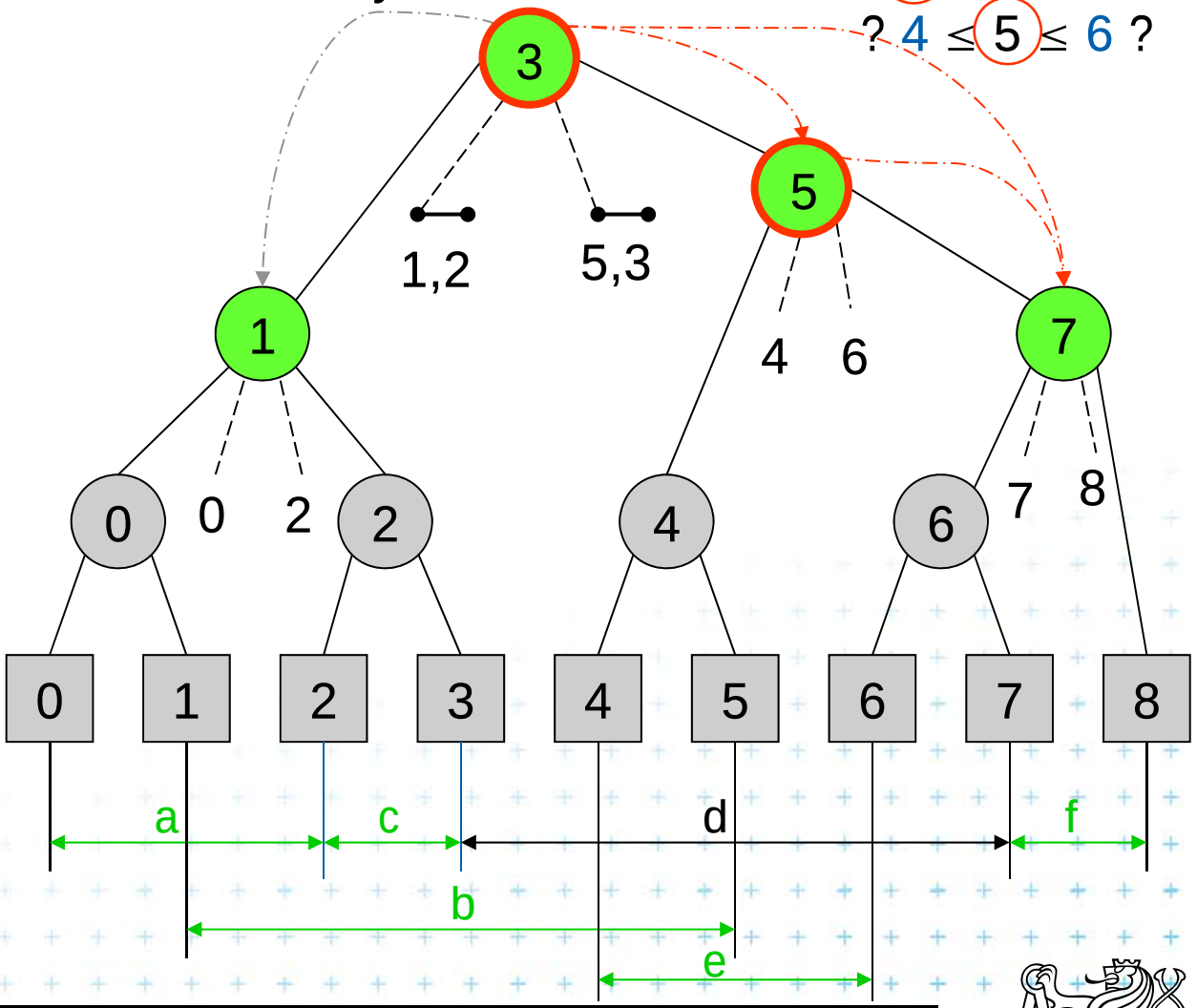
# Insert [4,6] b) Insert Interval

$H(v) \le b < e$

Insert the new interval to secondary lists

? 3 ≤ 4 < 6 ?
? 4 ≤ 5 ≤ 6 ?




Active rectangle

Current node

Active node

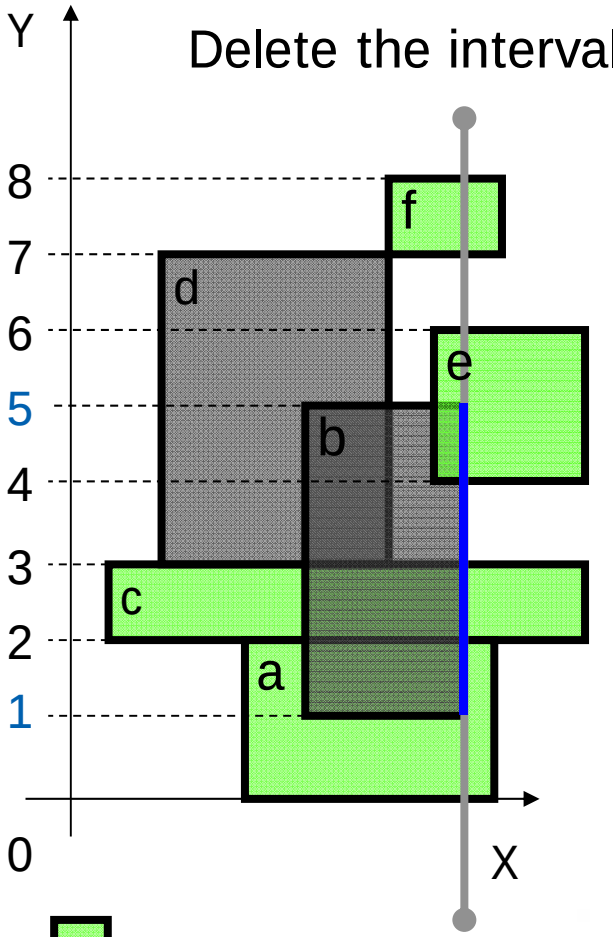
[Drtina]

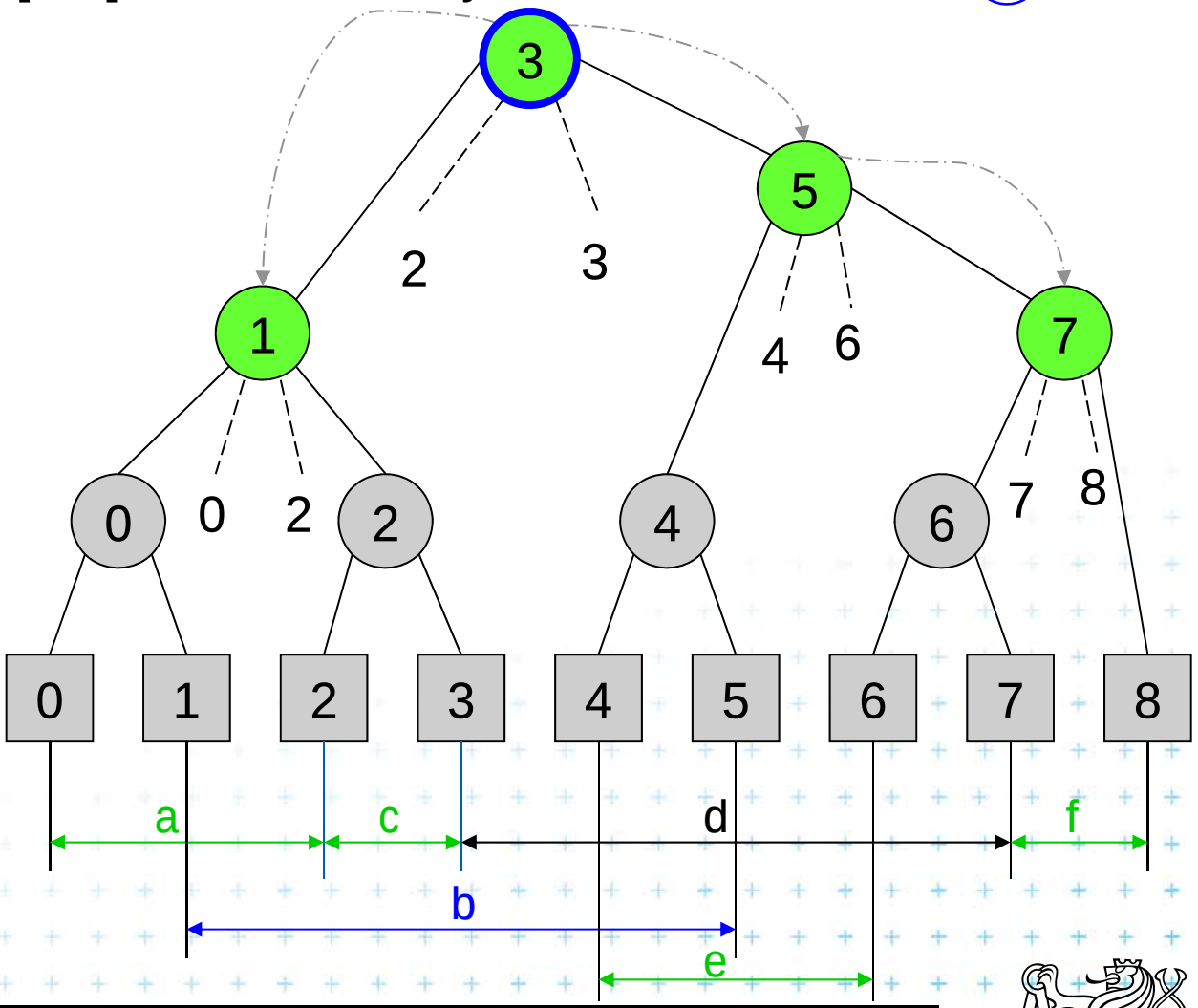
# Delete [1,5] Delete Interval

$b \le H(v) \le e$

Delete the interval [1,5] from secondary lists

? 1 ≤ 3 ≤ 5 ?




Active rectangle

Current node

Active node

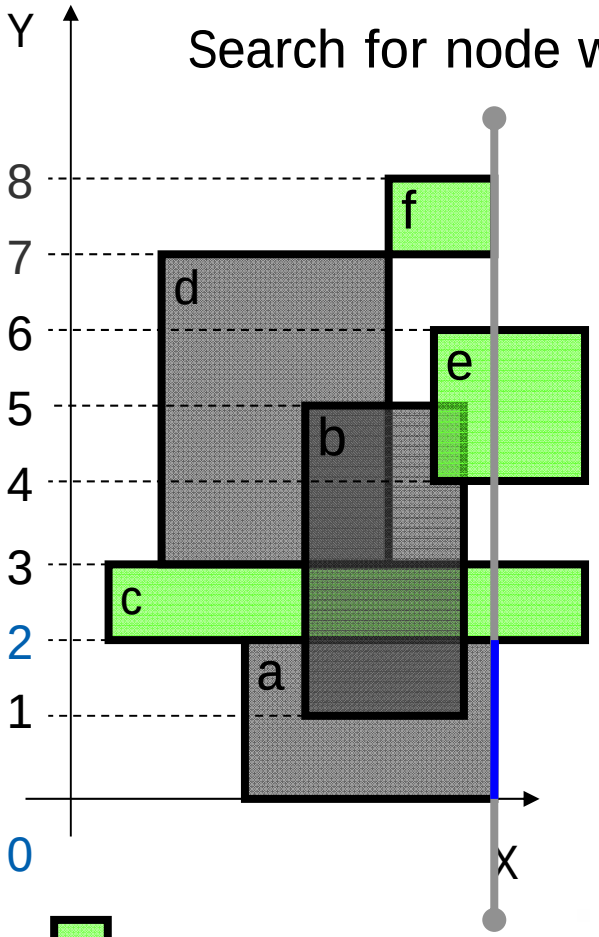
[Drtina]

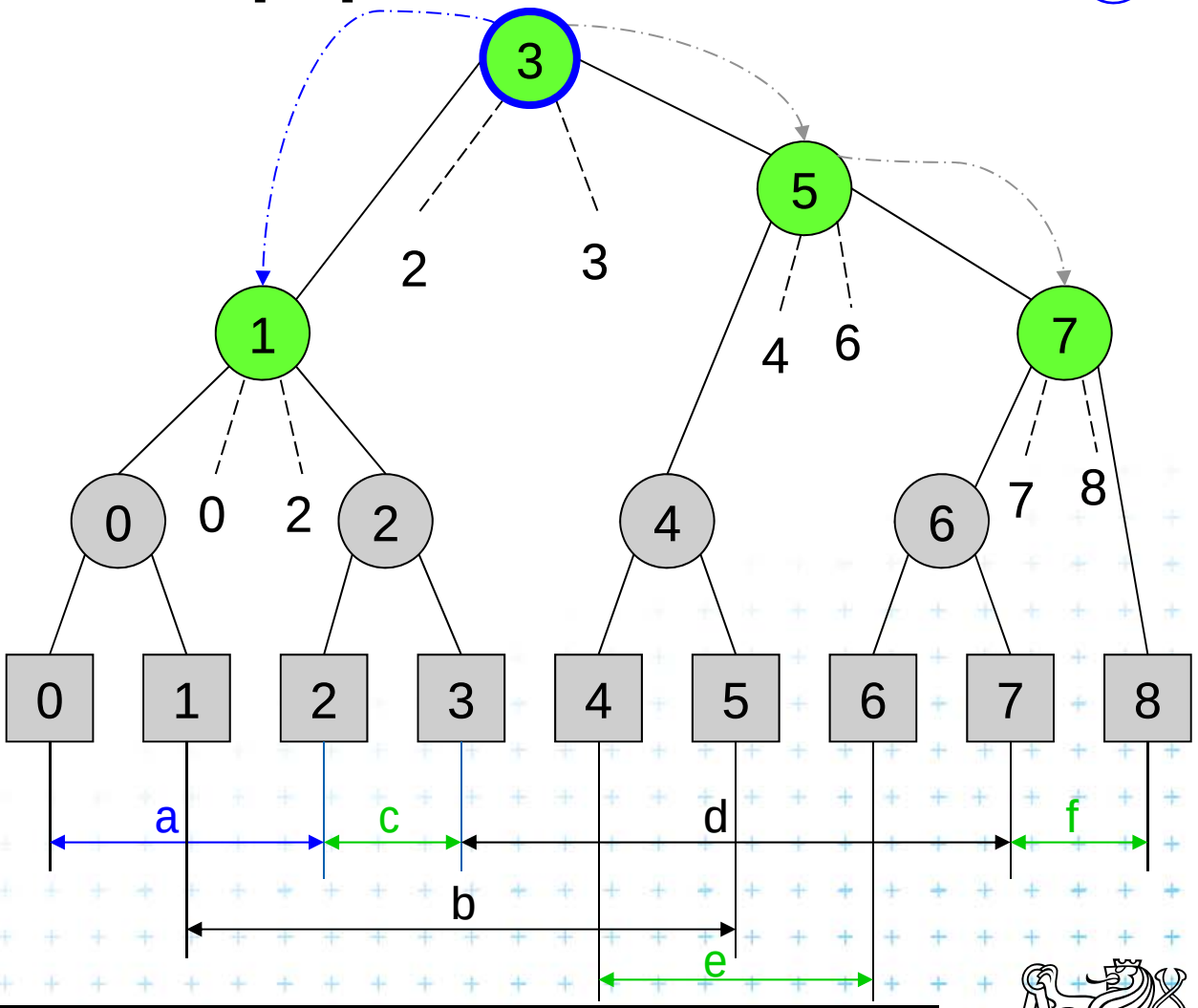
# Delete [0,2] Delete Interval 1/2

$b < e \leq H(v)$

Search for node with interval [0,2]

? 0 < 2 ≤ 3?




Active rectangle

Current node

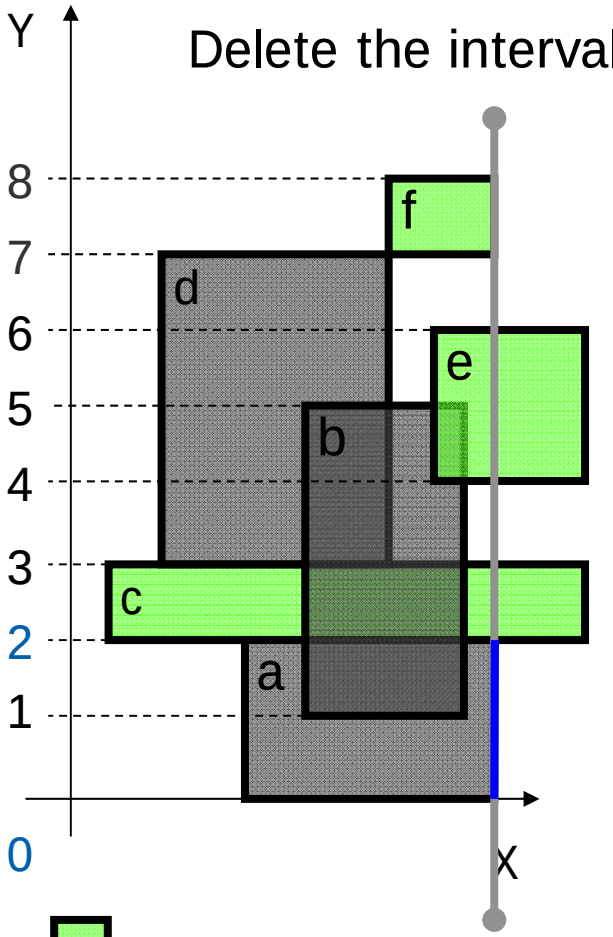
Active node

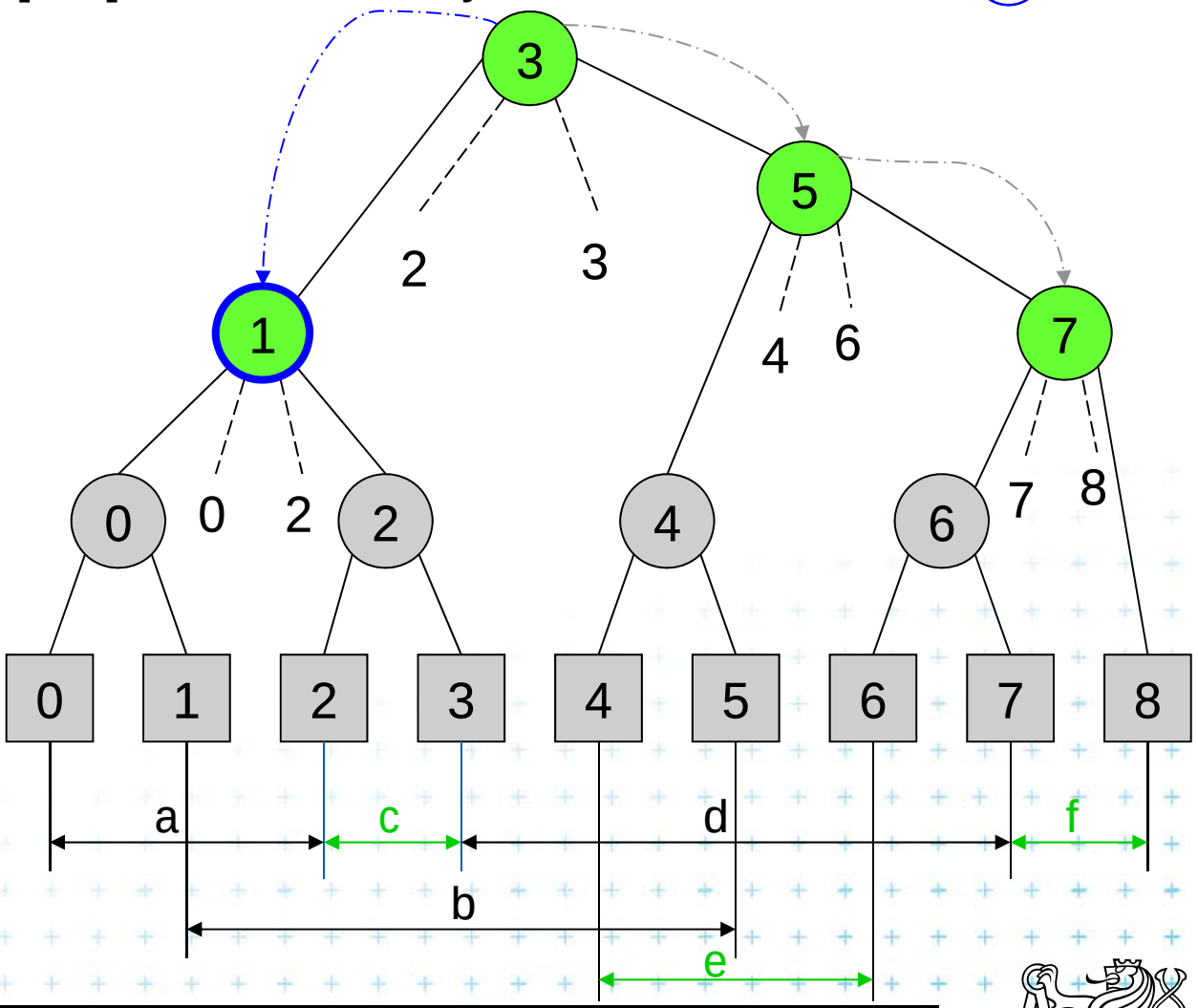
[Drtina]

# Delete [0,2] Delete Interval 2/2

$b \le H(v) \le e$

Delete the interval [0,2] from secondary lists of node 1 ? 0 ≤ 1 ≤ 2 ?




Active rectangle

Current node

Active node

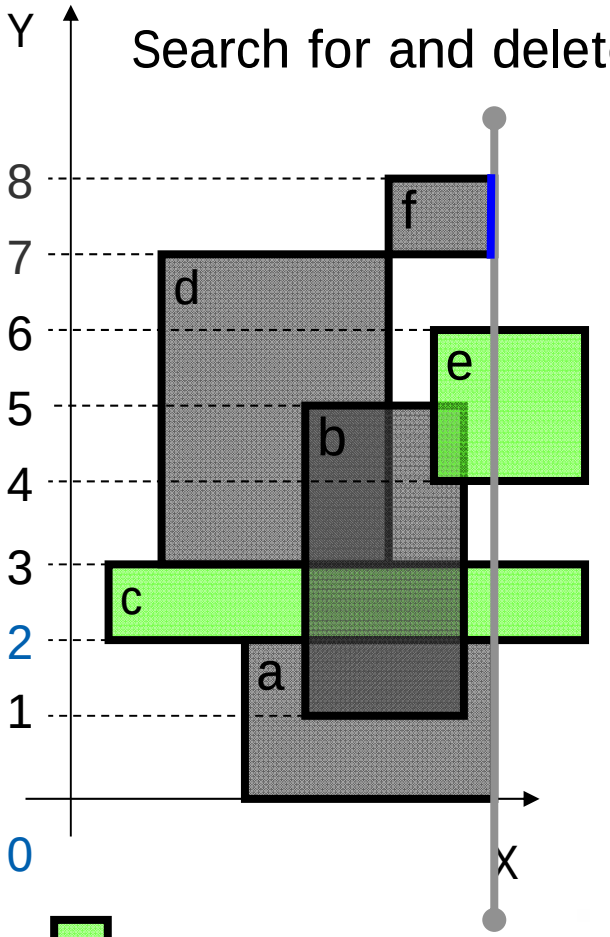
[Drtina]

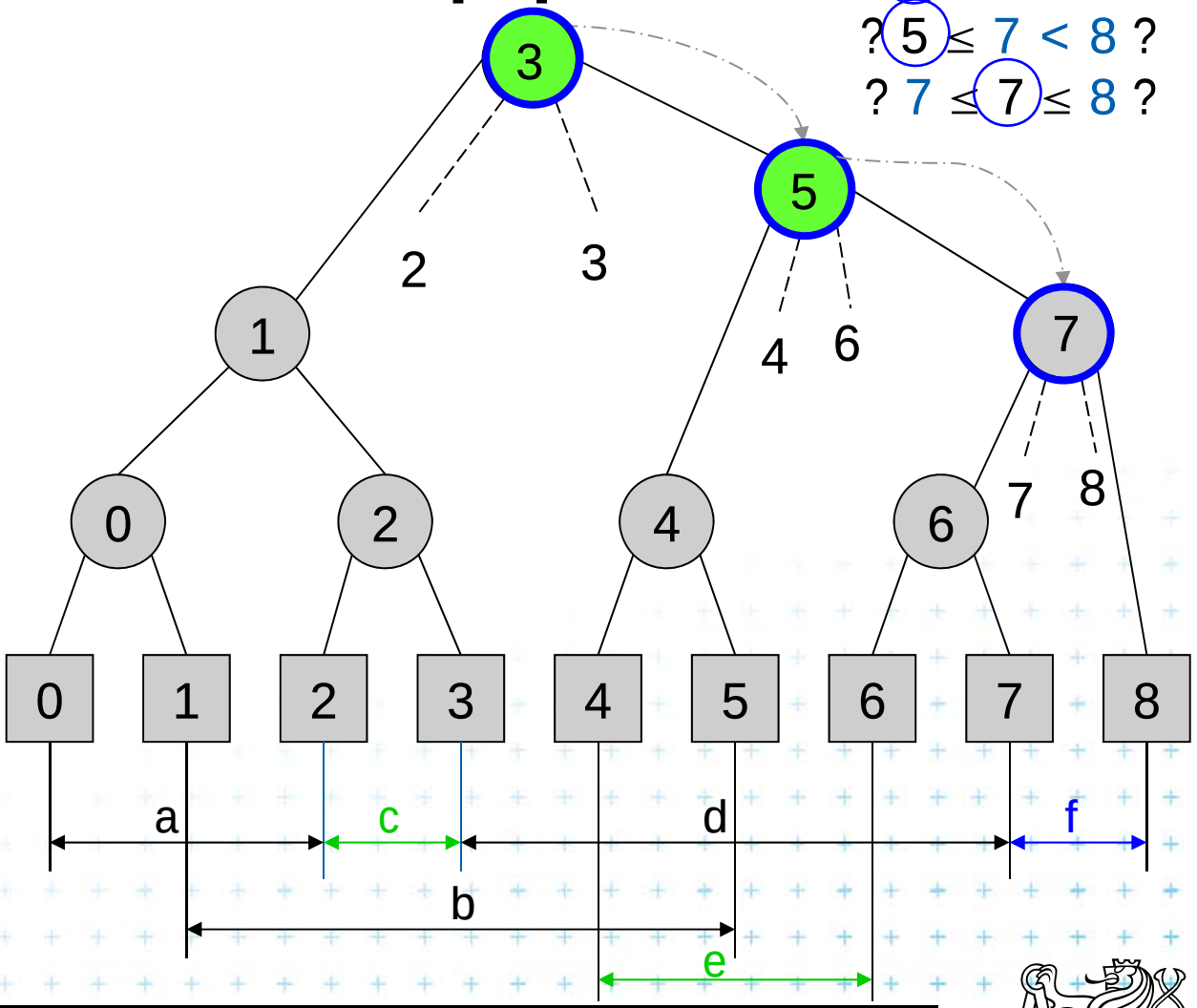
# Delete [7,8] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [7,8]

? 3 ≤ 7 < 8 ?
? 5 ≤ 7 < 8 ?
? 7 ≤ 7 ≤ 8 ?




Active rectangle

Current node

Active node

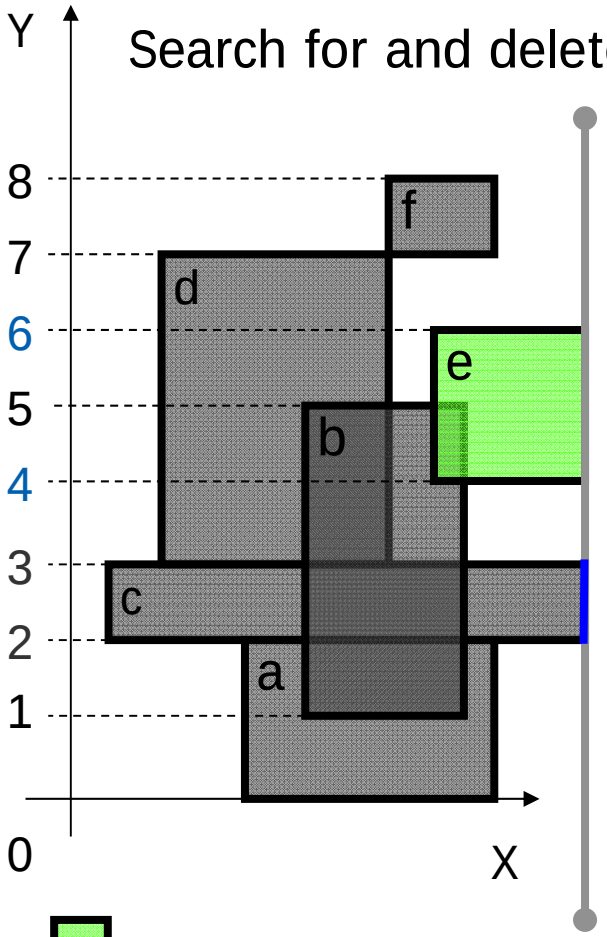
[Drtina]

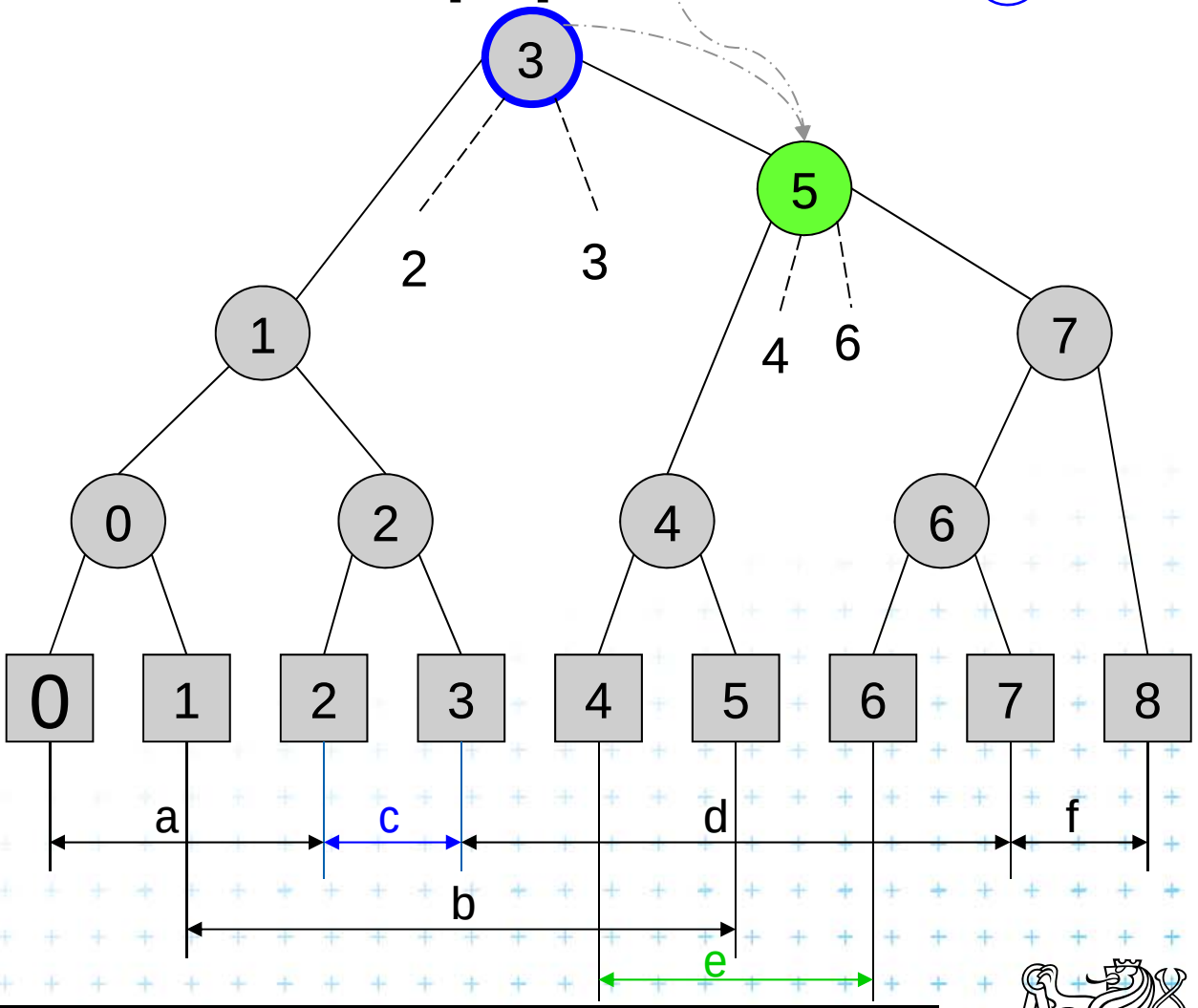
# Delete [2,3] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [2,3]

? 2 ≤ 3 ≤ 3 ?




Active rectangle

Current node

Active node

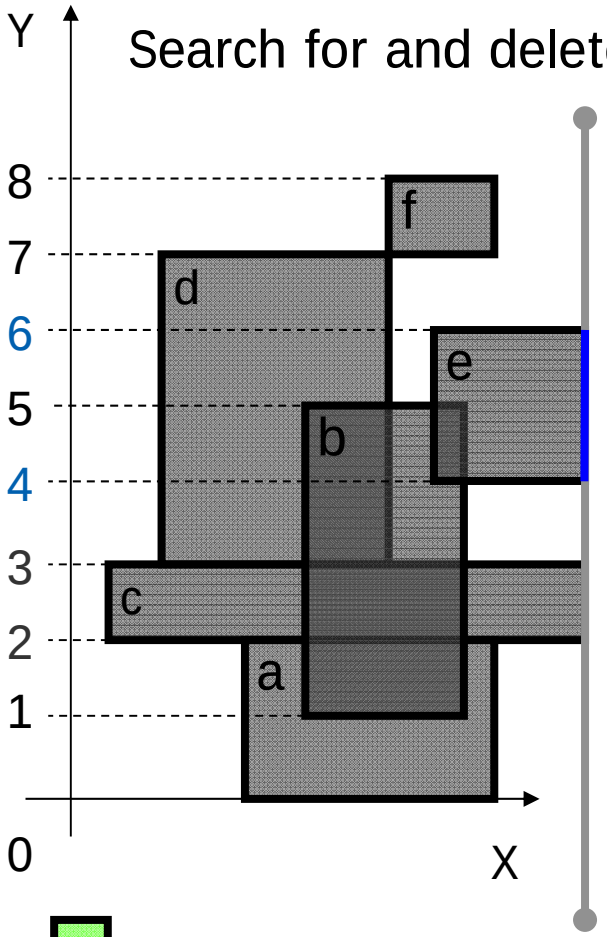
[Drtina]

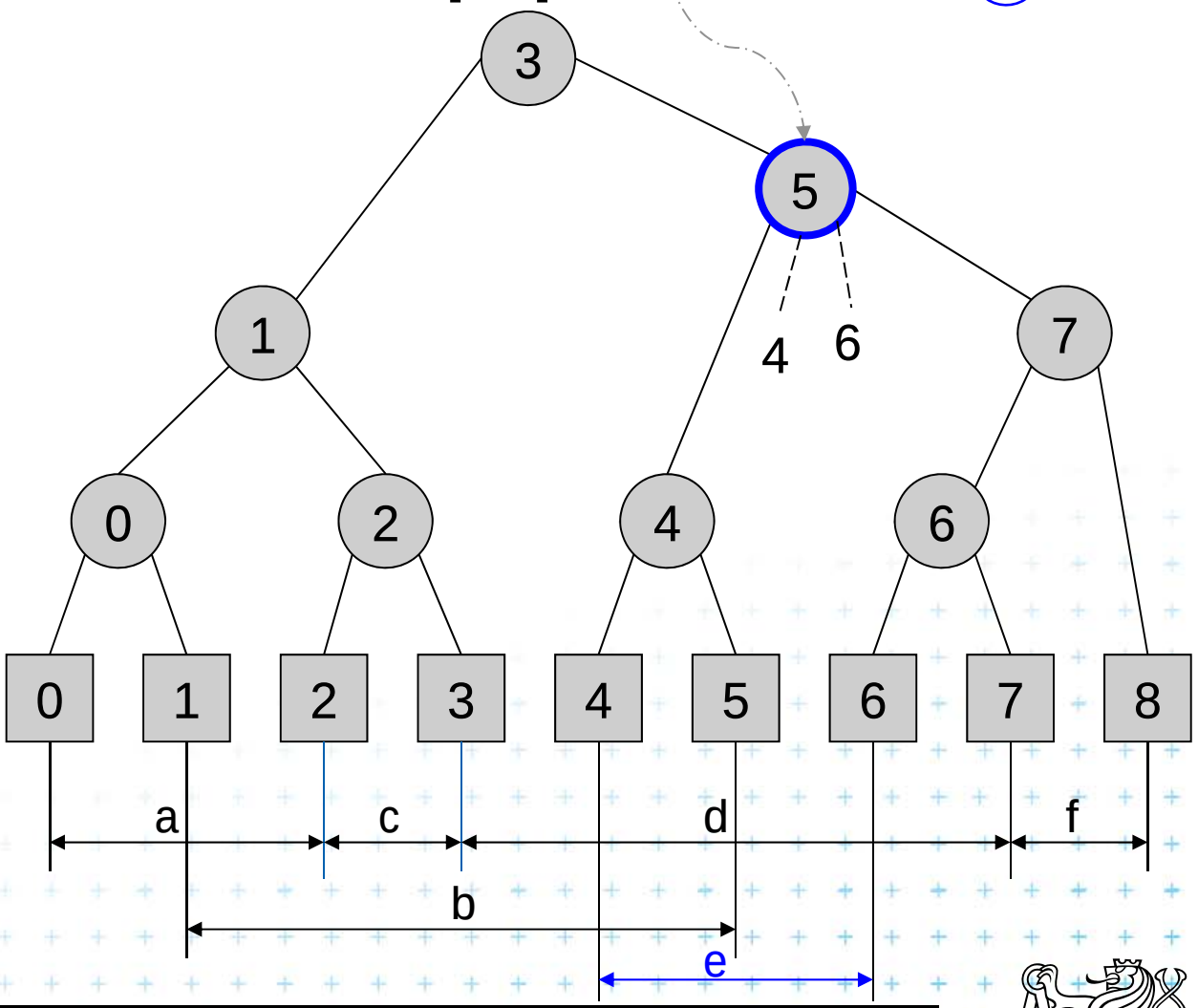
# Delete [4,6] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [4,6]

? 4 ≤ 5 ≤ 6 ?




Active rectangle

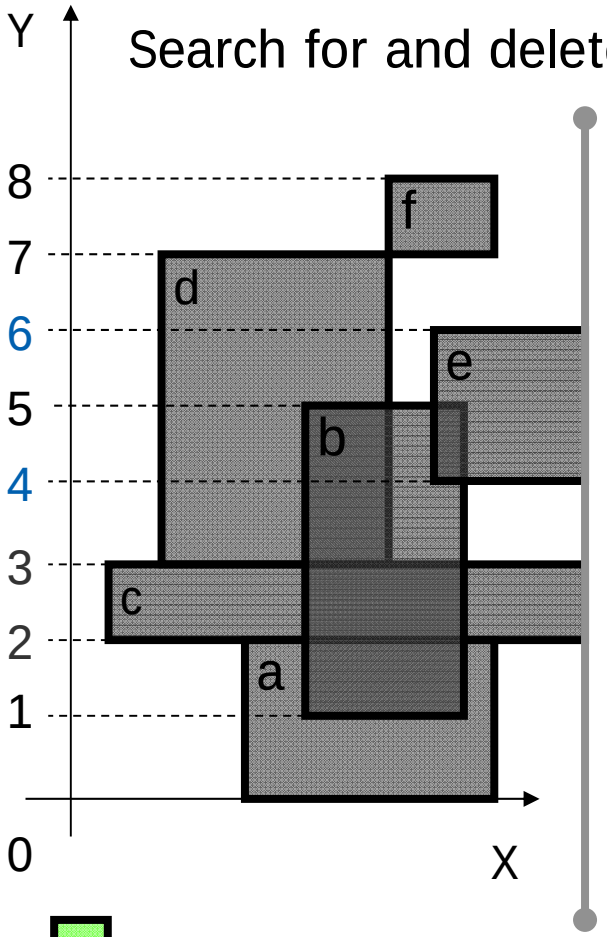
Current node

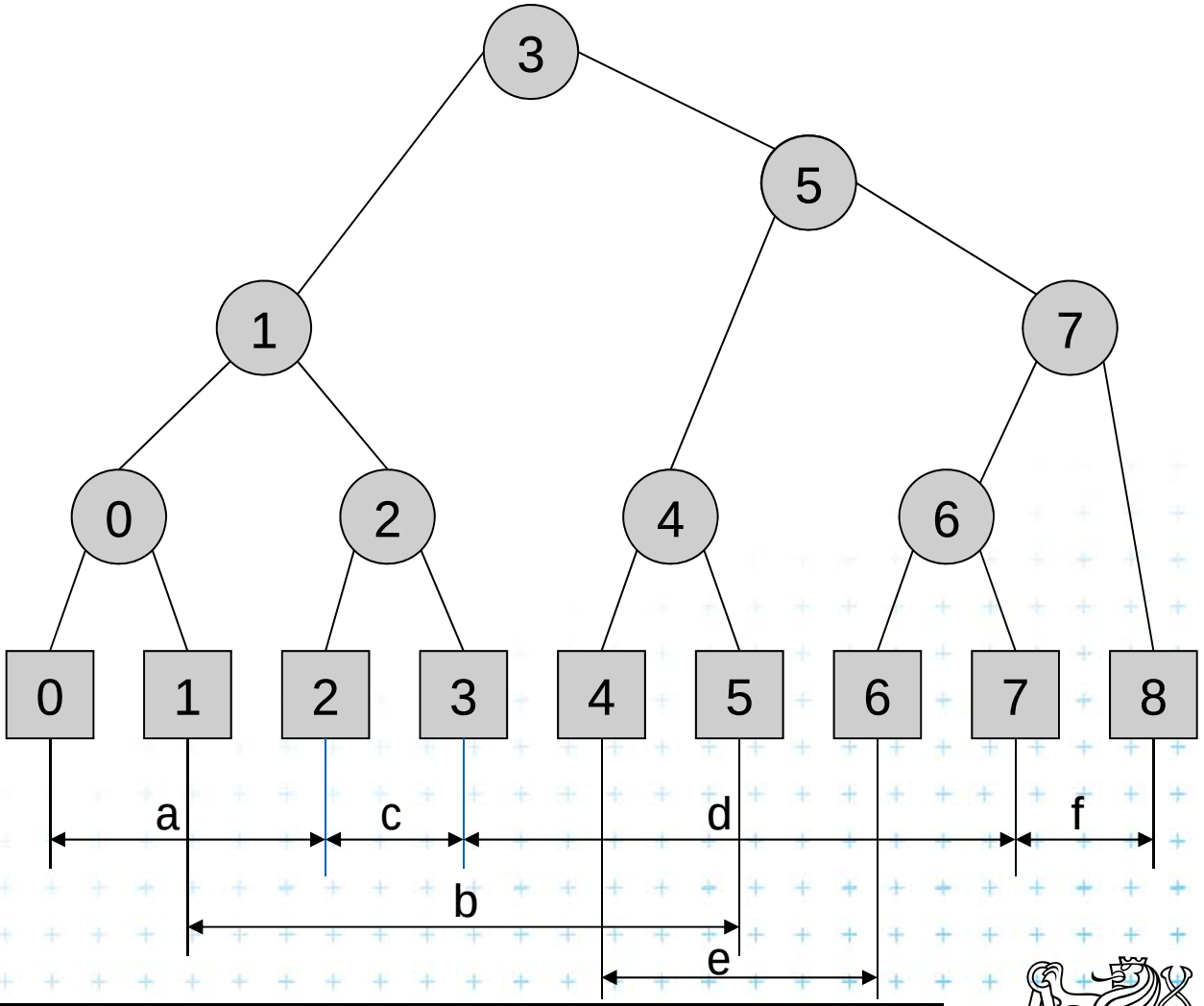
Active node

[Drtina]

# Empty tree

Search for and delete node with interval [4,6]




Active rectangle

Current node

Active node

[Drtina]

# Complexities of rectangle intersections

- $n$ rectangles, $s$ intersected pairs found
- $O(n \log n)$ preprocessing time to separately sort
  - x-coordinates of the rectangles for the plane sweep
  - the y-coordinates for initializing the interval tree.
- The plane sweep itself takes $O(n \log n + s)$ time, so the overall time is $O(n \log n + s)$
- $O(n)$ space
- This time is optimal for a decision-tree algorithm (i.e., one that only makes comparisons between rectangle coordinates).

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