DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM PART II

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

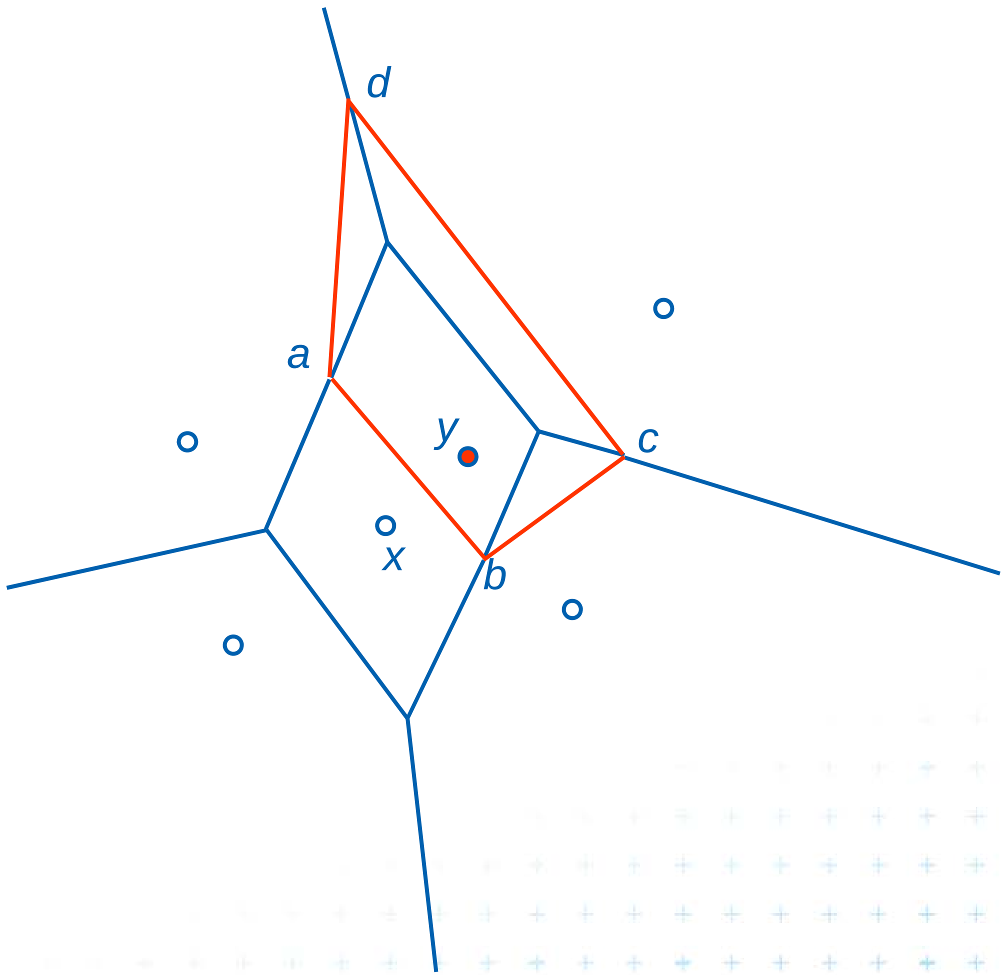
Based on [Berg], [Reiberg] and [Nandy]

Version from 13.11.2015

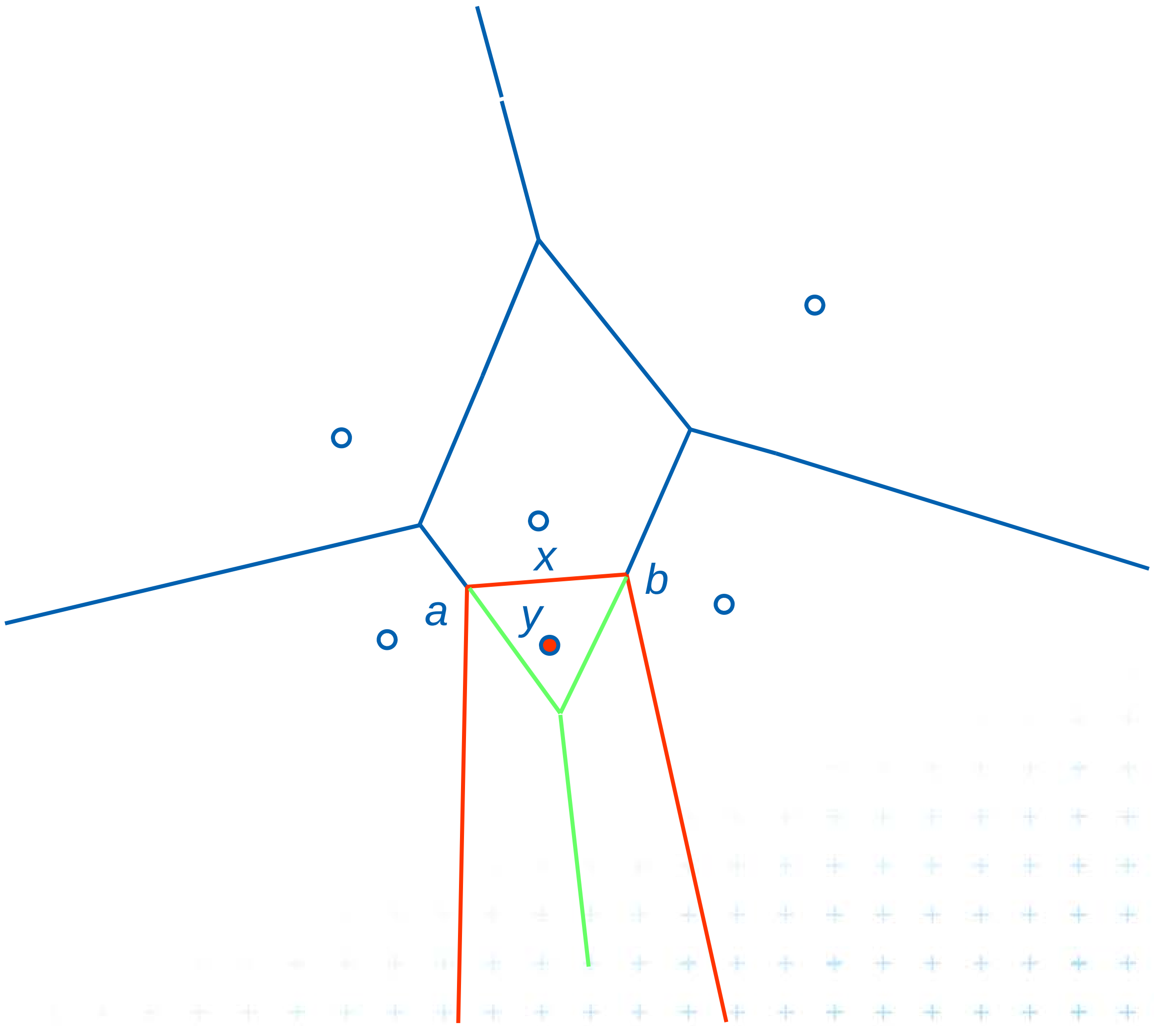
Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

Incremental construction – bounded cell



Incremental construction – unbounded cell



Incremental construction algorithm

InsertPoint(*S*, Vor(*S*), y) … **y = a new site**

Input: Point set *S*, its Voronoi diagram, and inserted point *y*Ù*S*

Output: VD after insertion of *y*

1. Find the cell V(*x*) in which *y* falls, set *c* = undef …O(log *n*)
2. Detect the intersections {*a*,*b*} of bisector *L*(*x*,*y*) with boundary of cell *V*(*x*)
   => * first edge *e* = *ab* on the border of cells of sites *x* and *y* …O(*n*)
3. *p* = *b*, site *z* = neighbor site across the border with point *b* …O(1)
4. **while**(exists*(p) and c* û *a*) // trace the bisectors from *b* in one direction
   a. Detect the intersection *c* of bisector *L*(*z*,*y*) with *V*(*z*)
   b. Report Voronoi edge *pc* } …O(n^2)
   c. *p* = *c*, *z*=neighbor site across border with *c*
5. **if**(c û a) **then** // trace the bisectors from *a* in other direction
   a. p = a …O(1)
   b. **while**(exists*(p)* and c û b)
      a. Detect the intersection *c* of bisector *L*(*z*,*y*) with *V*(*z*)
      b. Report Voronoi edge *pc* } …O(n^2)
      c. *p* = *c*, *z*=neighbor site across border with *c*

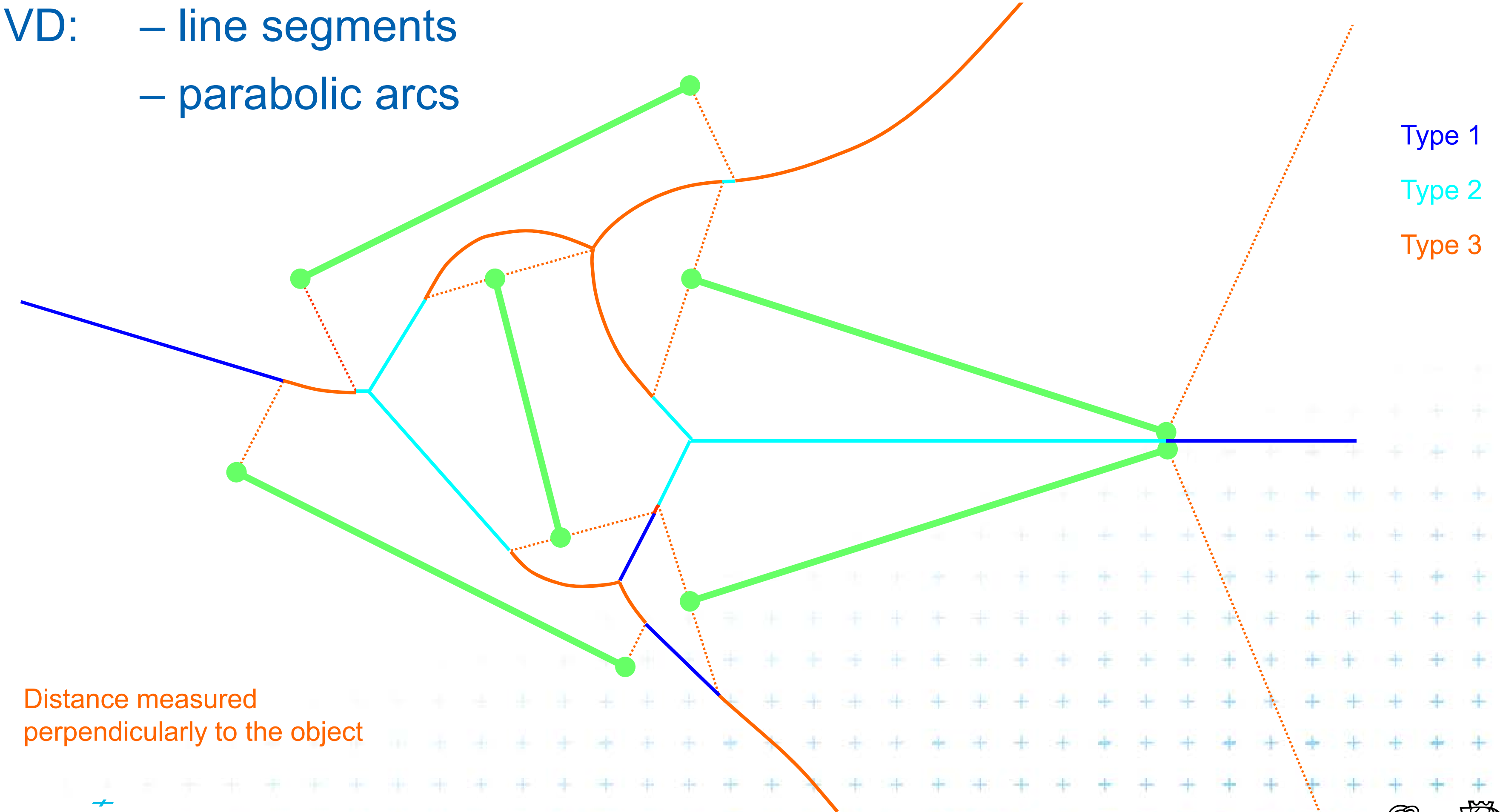
O(n^2) worst-case, O(*n*) expected time for some distributions

Voronoi diagram of line segments

Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)

VD: – line segments

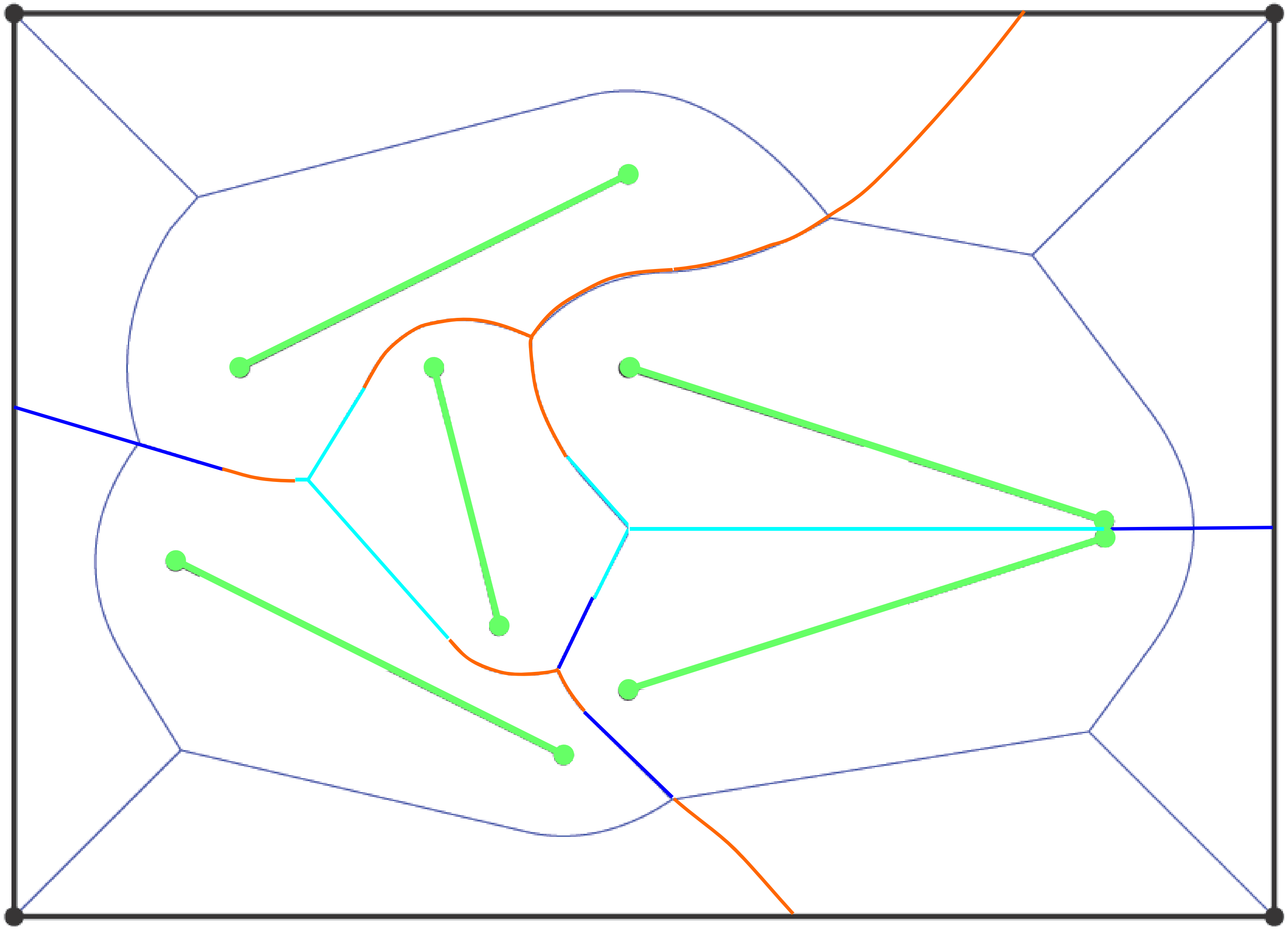
– parabolic arcs



Distance measured perpendicularly to the object

[Berg]

VD of line segments with bounding box



BBOX
=>
standard
DCEL

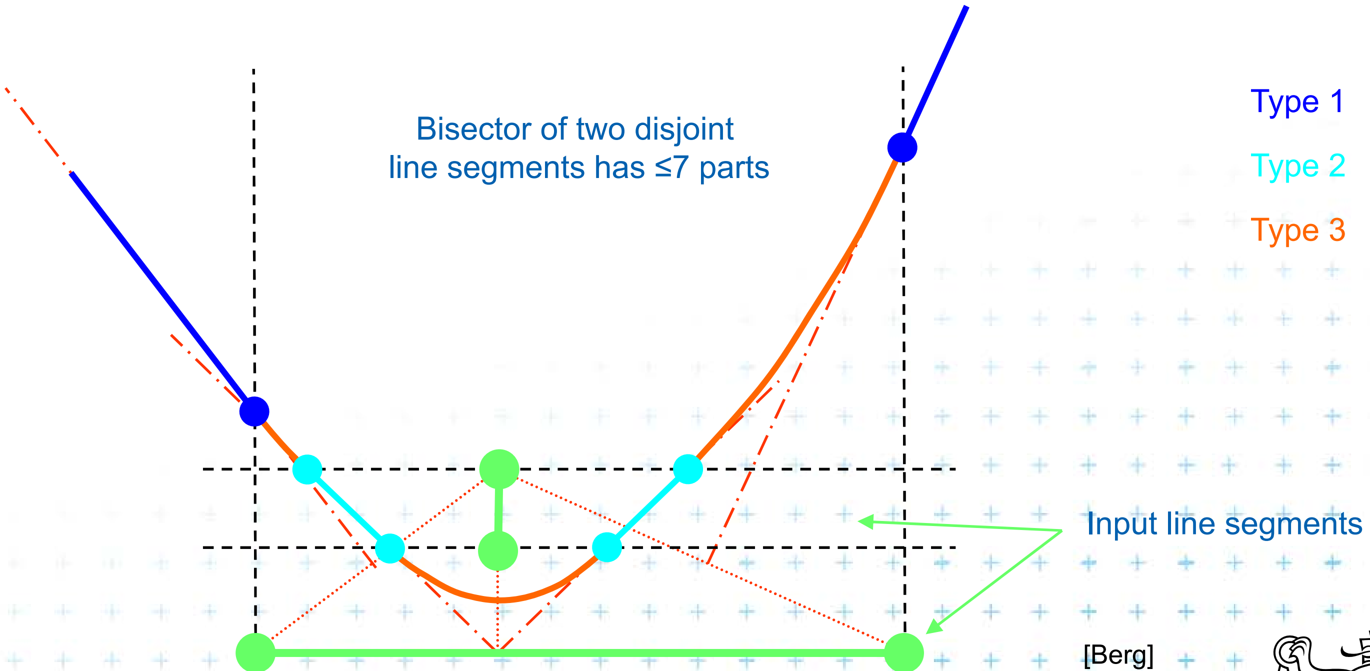
[Berg]

Bisector of 2 line-segments in detail

- Consists of line segments and parabolic arcs

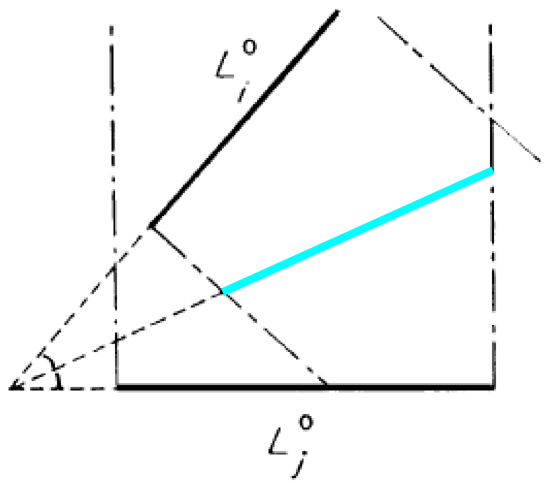
  Distance from point-to-object is measured to the closest point on the object (perpendicularly to the object silhouette)

  - Line segment – bisector of end-points(1) or of interiors(2)
  - Parabolic arc – of point and interior(3) of a line segment

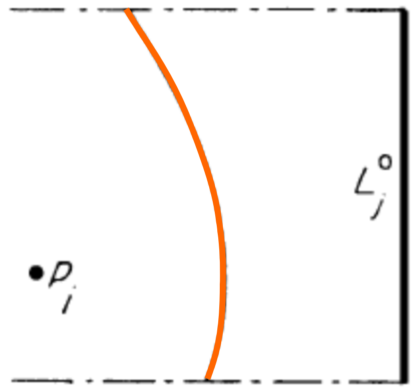


Bisector in greater details

Type 2



Type 3



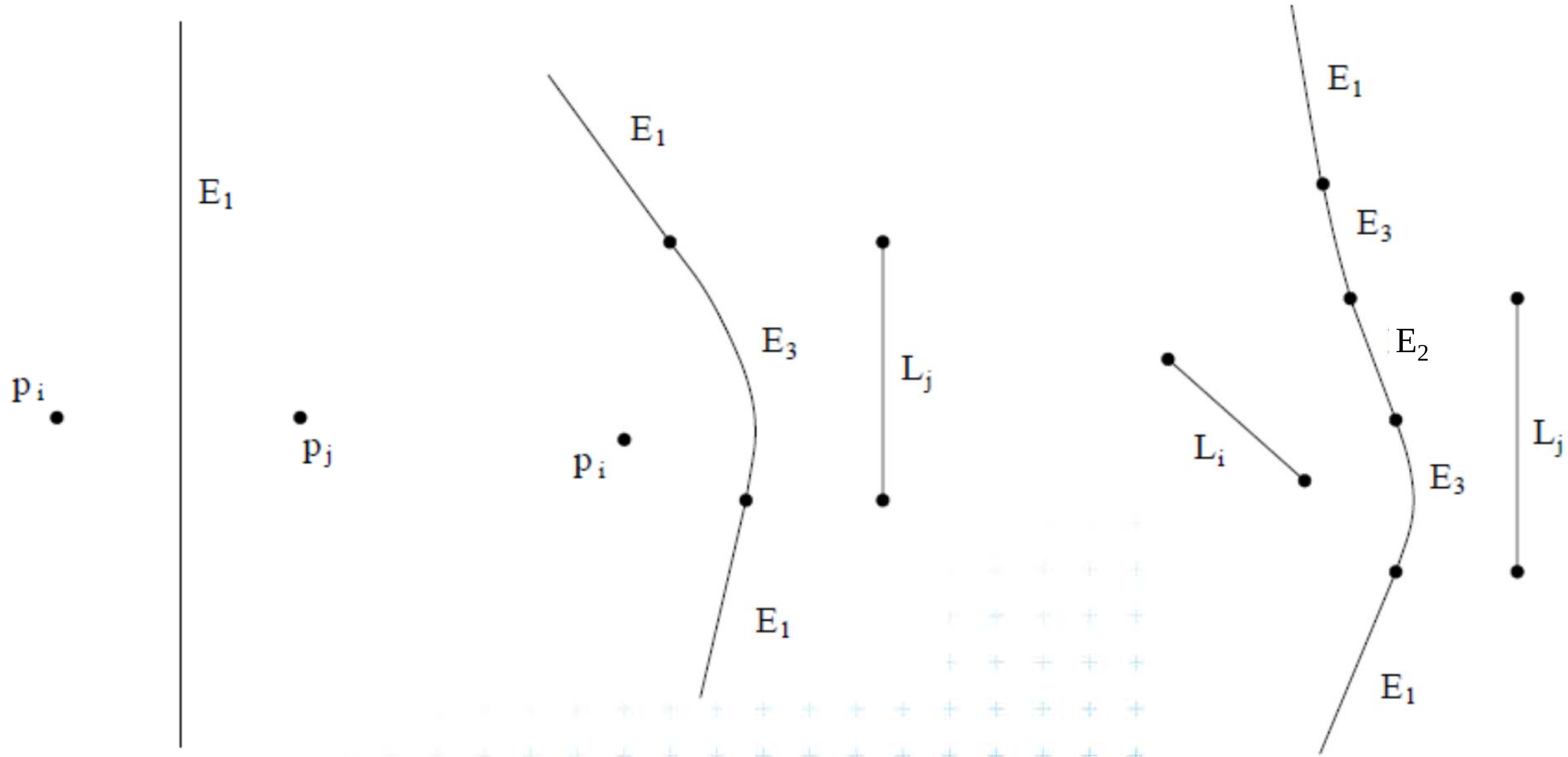
[Reiberg]

Bisector of two
line segment interiors
(in intersection of perpendicular slabs only)

Bisector of (end-)point and
line segment interior

VD of points and line segments examples

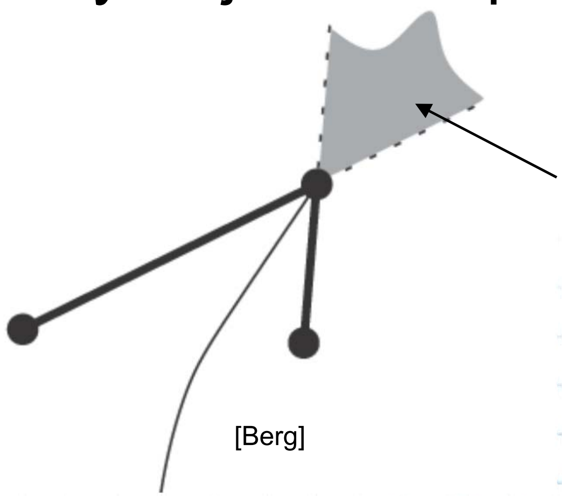
2 points   Point & segment   2 line segments



[Reiberg]

Voronoi diagram of line segments

- More complex bisectors of line segments
  - line segments and parabolic arcs
- Still combinatorial complexity of O(n)
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)



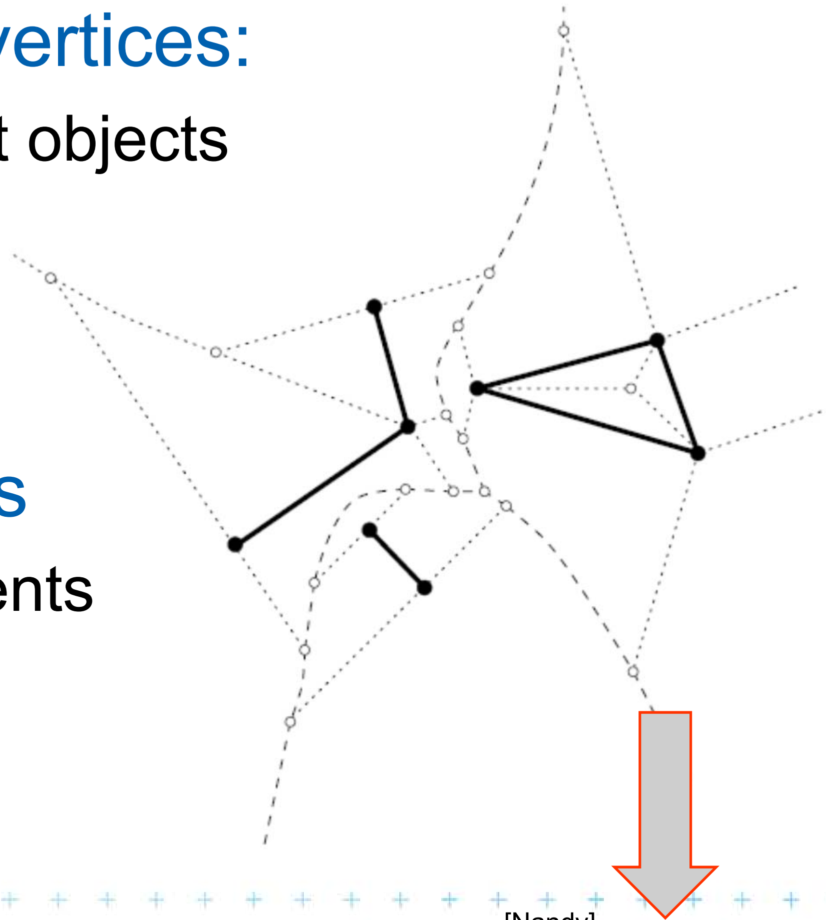
[Berg]

if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

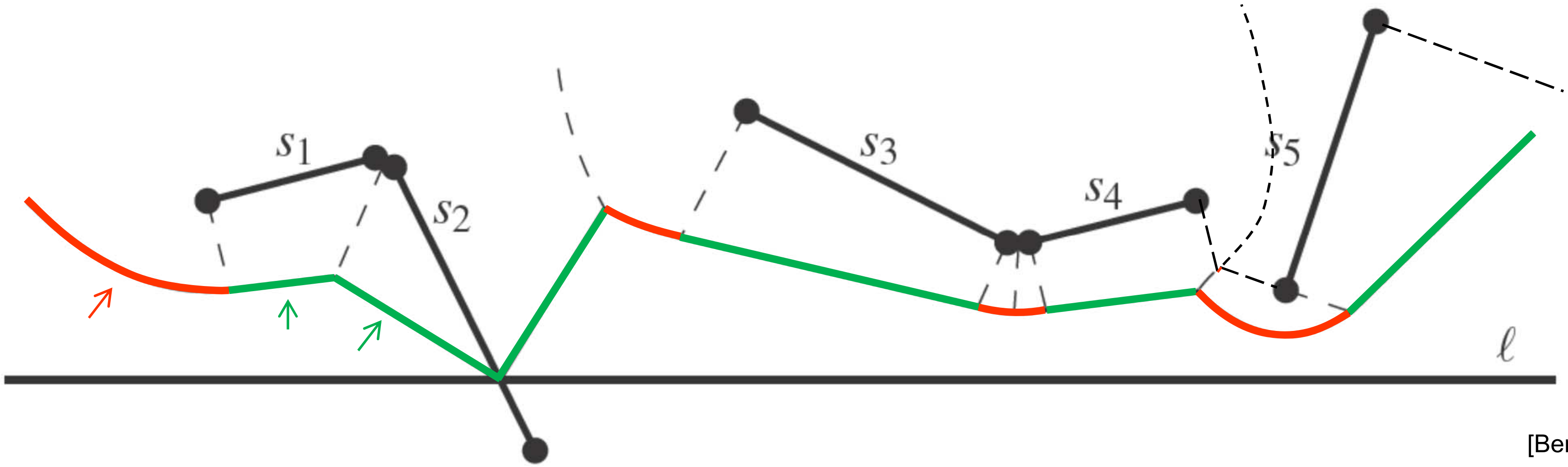
VD of line segments - touching segments

- Variant with touching segments in their end-points
- Two types of Voronoi vertices:
  - Type 3 – three different objects
  - Type 2 – two objects (segment and one of its end-points)
- Contains also 2D areas
  - Not only 1D line segments and parabolic arcs



[Nandy]

Shape of Beach line for line segments



[Berg]

= Points with distance to the closest site above sweep line l equal to the distance to l

- Beach line contains
  - *parabolic arcs* when closest to a site end-point
  - *straight line segments* when closest to a site interior (or just the part of the site interior above l if the site s intersects l)

(This is the shape of the beach line)

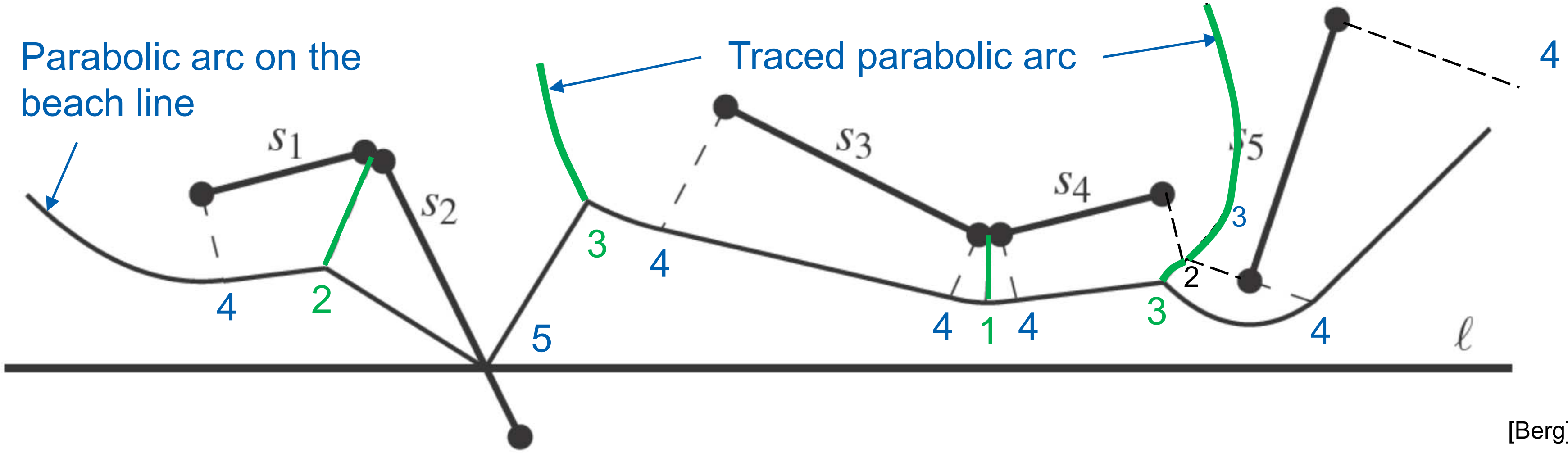
Beach line breakpoints types

Breakpoint *p* is equidistant from *l* and equidistant and closest to:

1. two site end-points => *p* traces a VD line segment
2. two site interiors => *p* traces a VD line segment
3. end-point and interior => *p* traces a VD parabolic arc
4. one site end-point => *p* traces a line segment (border of the slab perpendicular to the site)
5. site interior intersects the scan line *l* => *p* = intersection, traces the input line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg.only)

Breakpoints types and what they trace



[Berg]

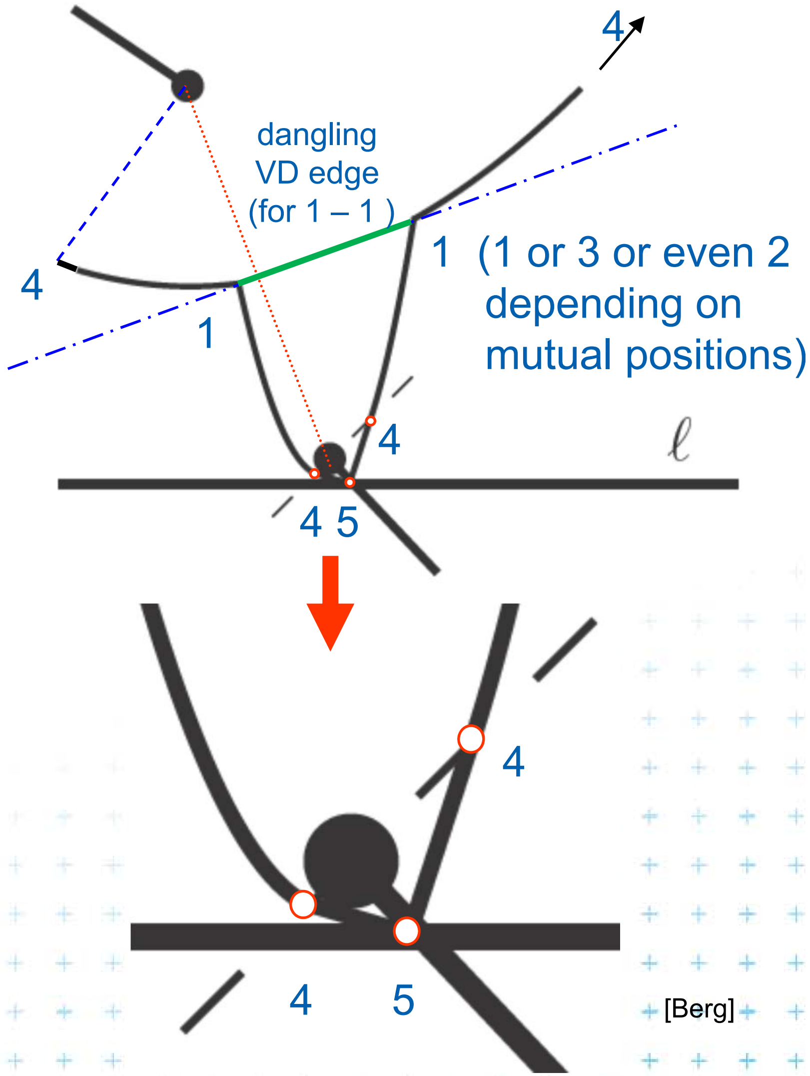
- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Site event – sweep line reaches an endpoint

I. At upper endpoint of

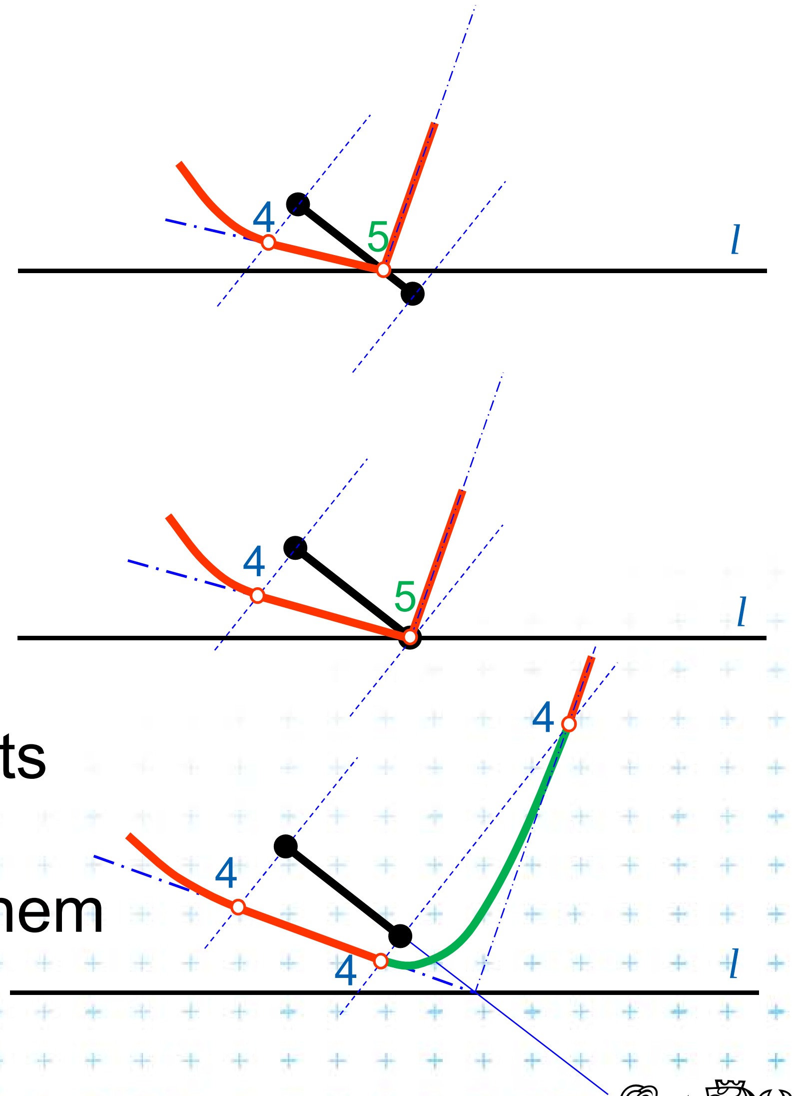
- Arc above is split into two
- 4 new arcs are created (2 segments + 2 parabolas)
- Breakpoints for 2 segments are of type 4-5-4
- Breakpoints for parabolas depend on the surrounding sites
  - Type 1 for two end-points
  - Type 3 for endpoint and interior
  - etc…



Site event – sweep line reaches an endpoint

II. At lower endpoint of

- Intersection with interior (breakpoint of type 5)



- is replaced by two breakpoints (of type 4) with parabolic arc between them

Circle event – lower point of circle of 3 sites

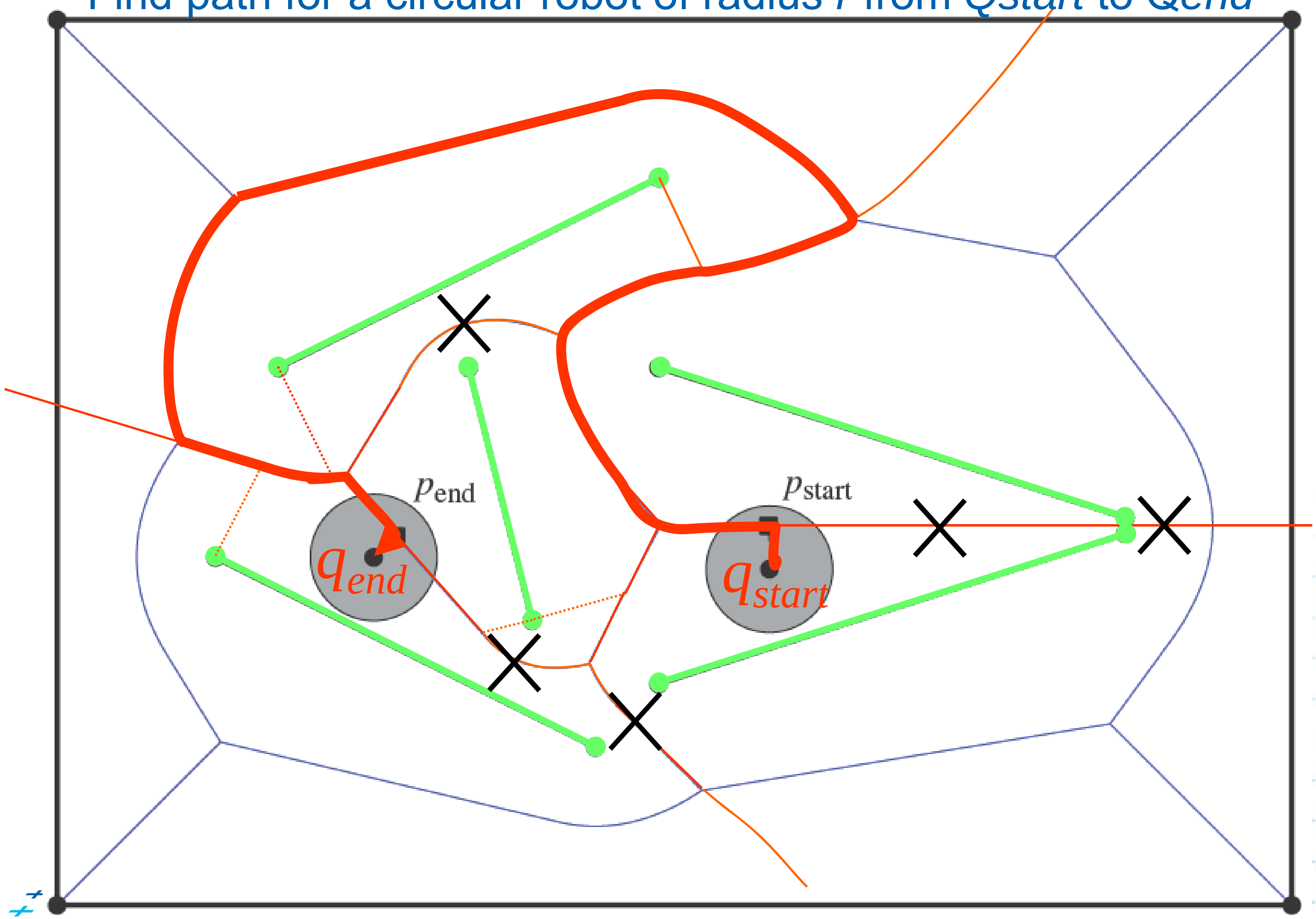
- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types meet
    - – 3 sites involved – Voronoi vertex created
  - Type 4 with something else
    - – two sites involved – breakpoint changes its type
    - – Voronoi vertex not created
      (Voronoi edge may change its shape)
  - Type 5 with something else
    - – never happens for disjoint segments
      (meet with type 4 happens before)

Summary of the VD terms

- Site = input point, line segment, …
- Cell = area around the site, in VD_1 the nearest to site
- Edge, arc = part of Voronoi diagram (border between cells)
- Vertex = intersection of VD edges

Motion planning example - retraction **Rušení hran**

Find path for a circular robot of radius *r* from *Qstart* to *Qend*



[Berg]

Motion planning example - retraction Rušení hran

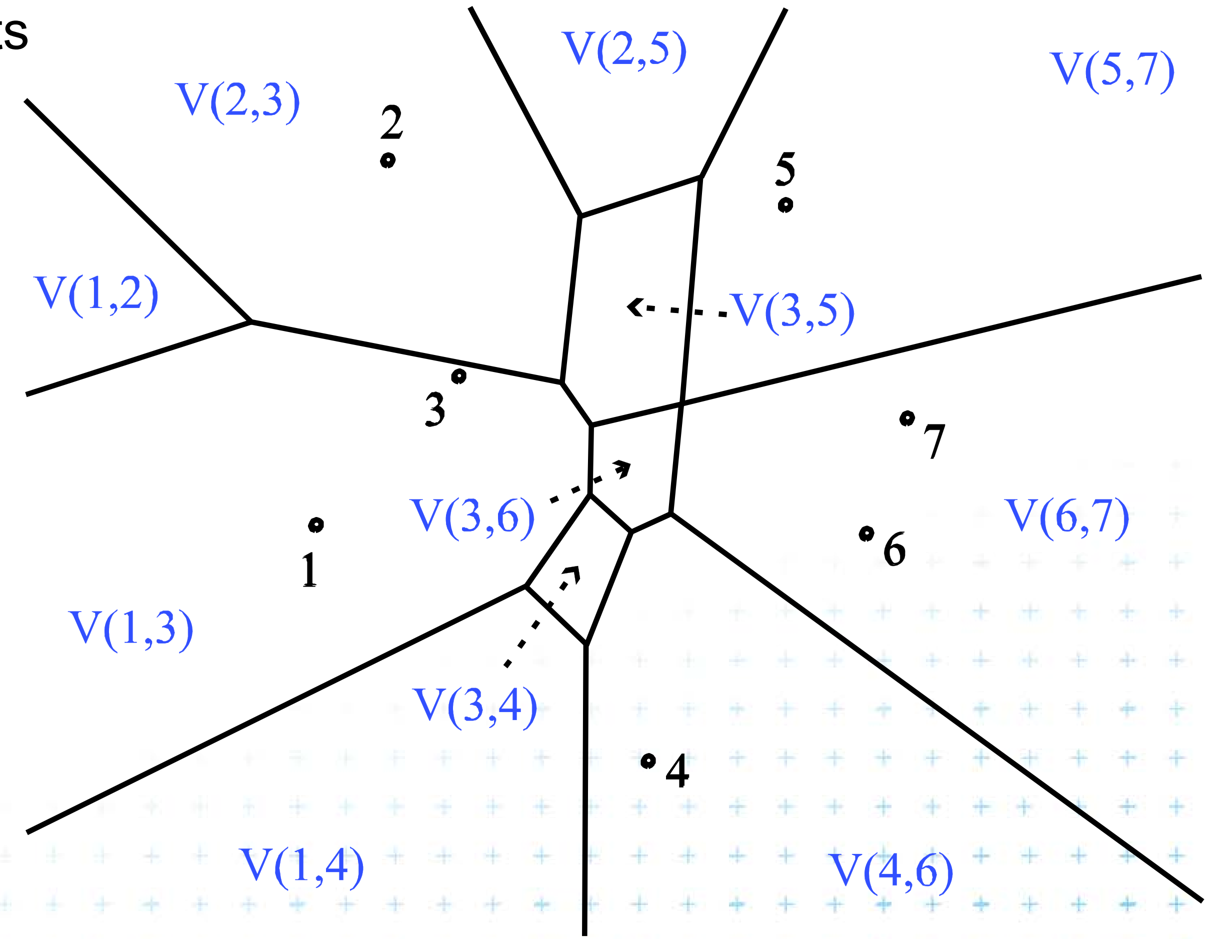
Find path for a circular robot of radius r from Q_{start} to Q_{end}

- Create Voronoi diagram of line segments, take it as a graph
- Project Q_{start} to P_{start} *on VD and* Q_{end} to P_{end}
- Remove segments with distance to sites smaller than radius r of a robot
- Depth first search if path from P_{start} to P_{end} *exists*
- Report path $Q_{start}\,P_{start}$*…path…* P_{end} to Q_{end}
- $O(n \log n)$ time using $O(n)$ storage

Order-2 Voronoi diagram

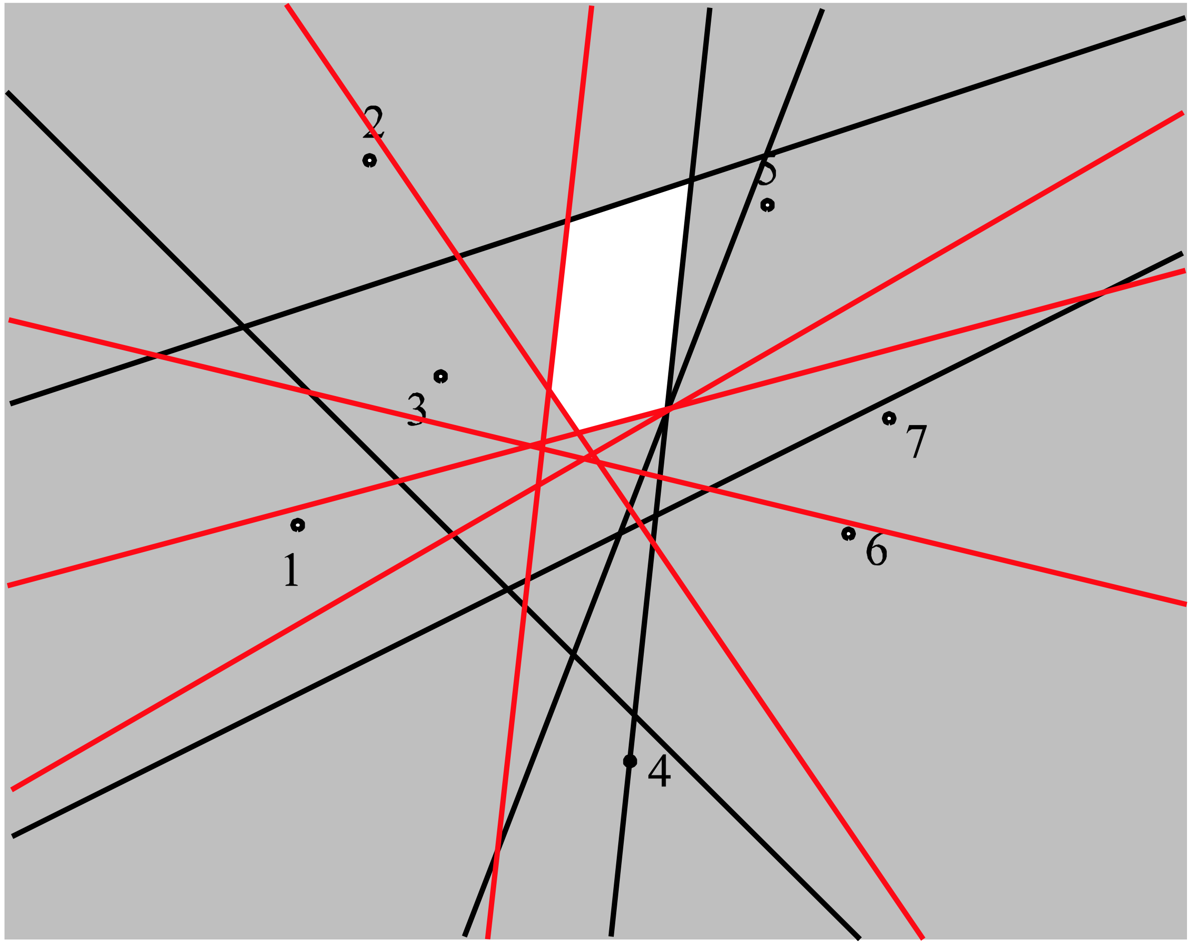
$V(p_i,p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site

Property
The order-2 Voronoi regions are convex



[Nandy]

Construction of V(3,5) = V(5,3)



[Nandy]

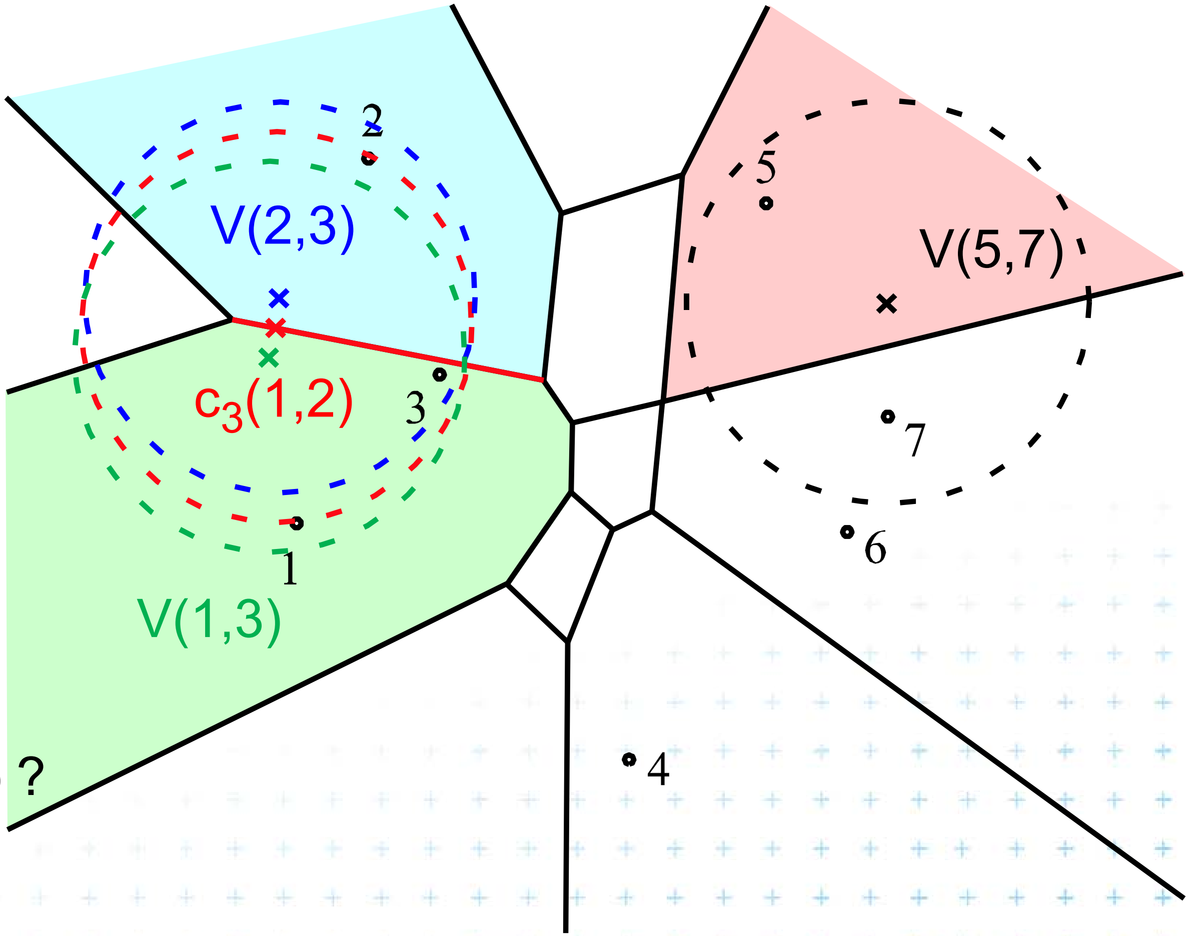
Intersection of all halfplanes except H(3,5) and H(5,3)

$$\bigcap_{x \neq 5} h(3,x) \cap \bigcap_{x \neq 3} h(5,x)$$

Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites *s* and *t* and containing one site *p*

=> $c_p(s,t)$



Question

Which are the regions on both sides of $c_p(s,t)$?

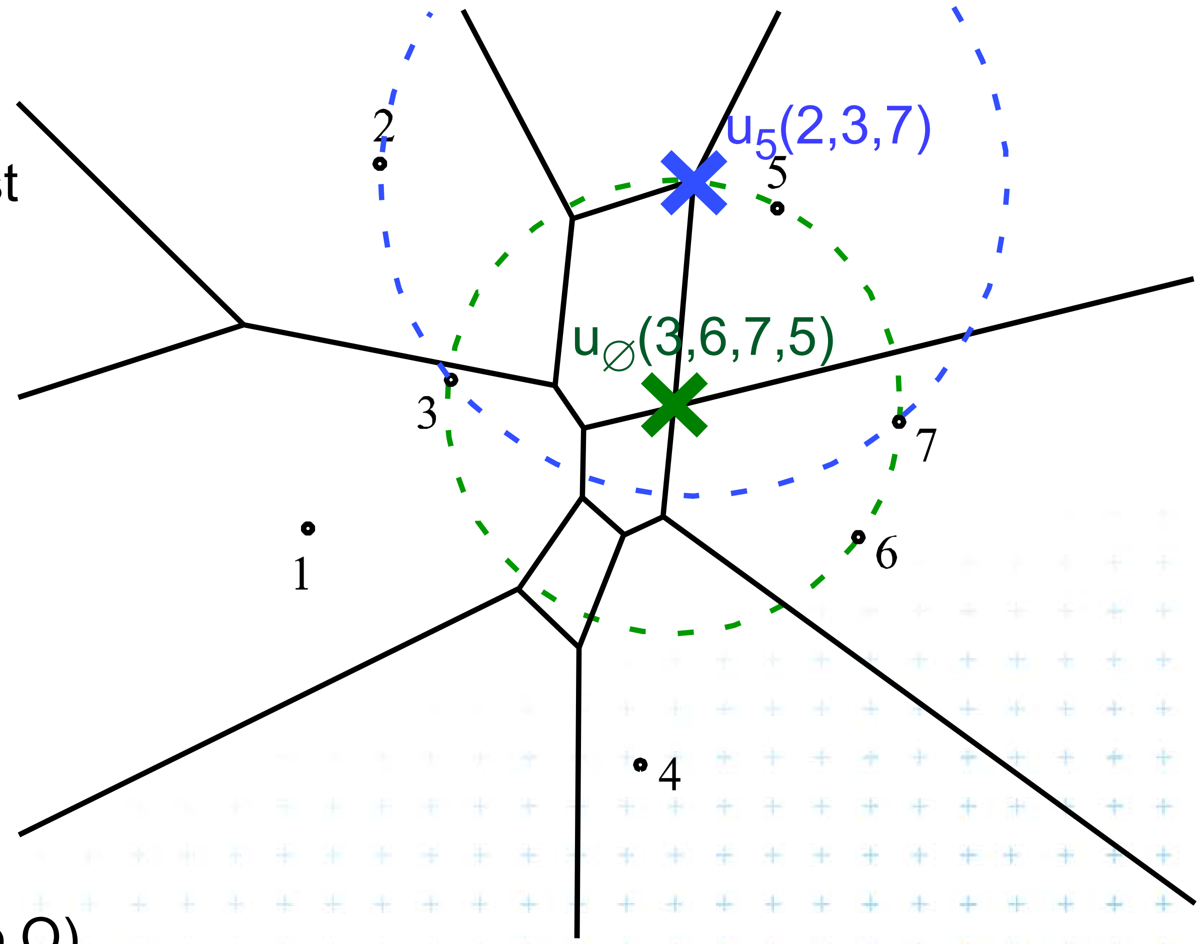
=> V(p,s) and V(p,t)

[Nandy]

Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

$\Rightarrow u_p(Q)$ or $u_\varnothing(Q + p)$
$u_5(2,3,7)$, $u_\varnothing(3,6,7)$



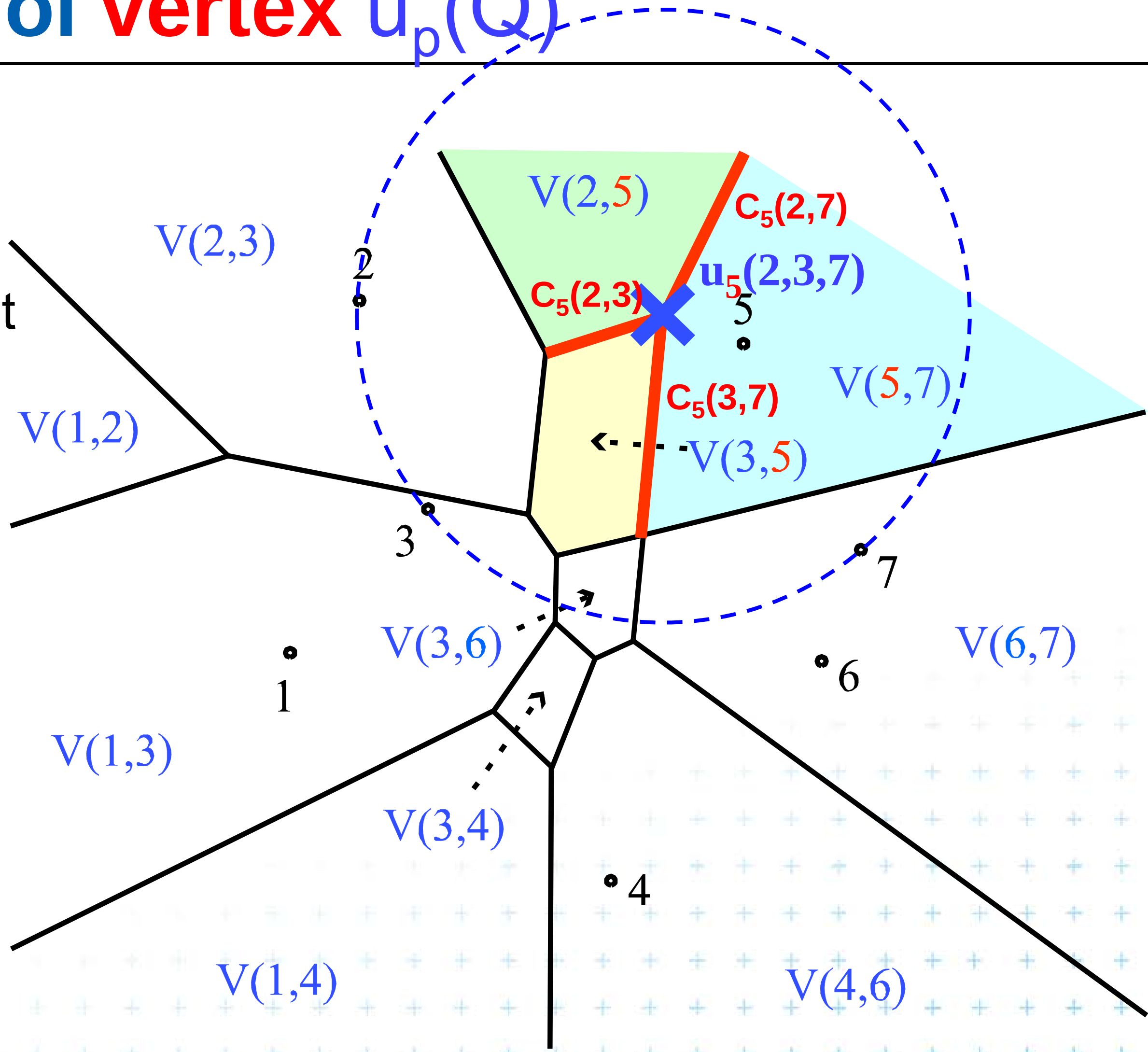
(circle circumscribed to Q)

[Nandy]

Order-2 Voronoi vertex $u_p(Q)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_p(Q)$
$u_5(2,3,7)$

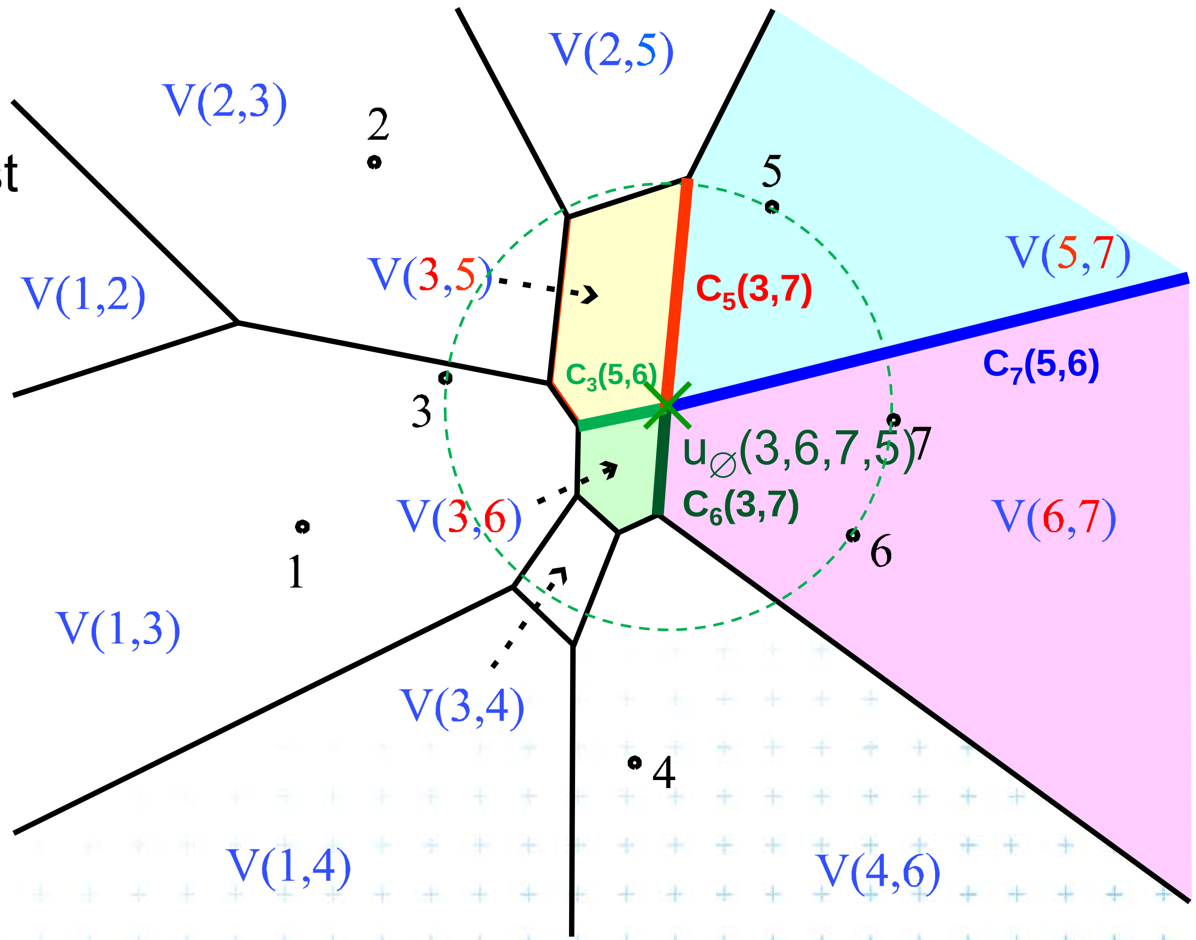


[Nandy]

Order-2 Voronoi vertex $u_\varnothing(Q + p)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_\varnothing(Q + p)$
$u_\varnothing(3,6,7,5)$



[Nandy]

Order-k Voronoi Diagram

Theorem věta

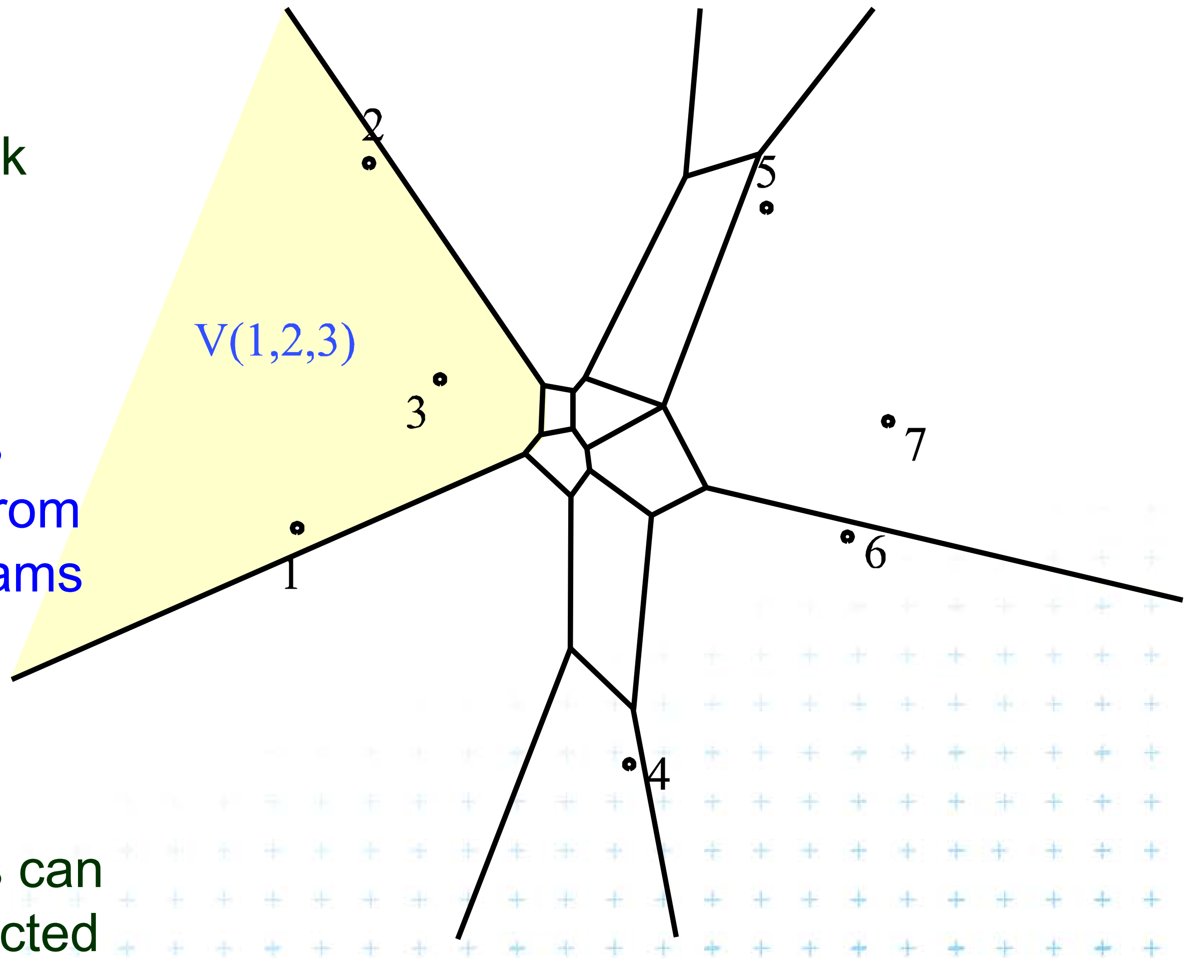
The size of the order-k diagrams is O(k(n-k))

Theorem věta

The order-k diagrams can be constructed from the order-(k-1) diagrams in O(k(n-k)) time

Corollary důsledek

The order-k diagrams can be iteratively constructed in $O(n \log n + k^2(n-k))$ time

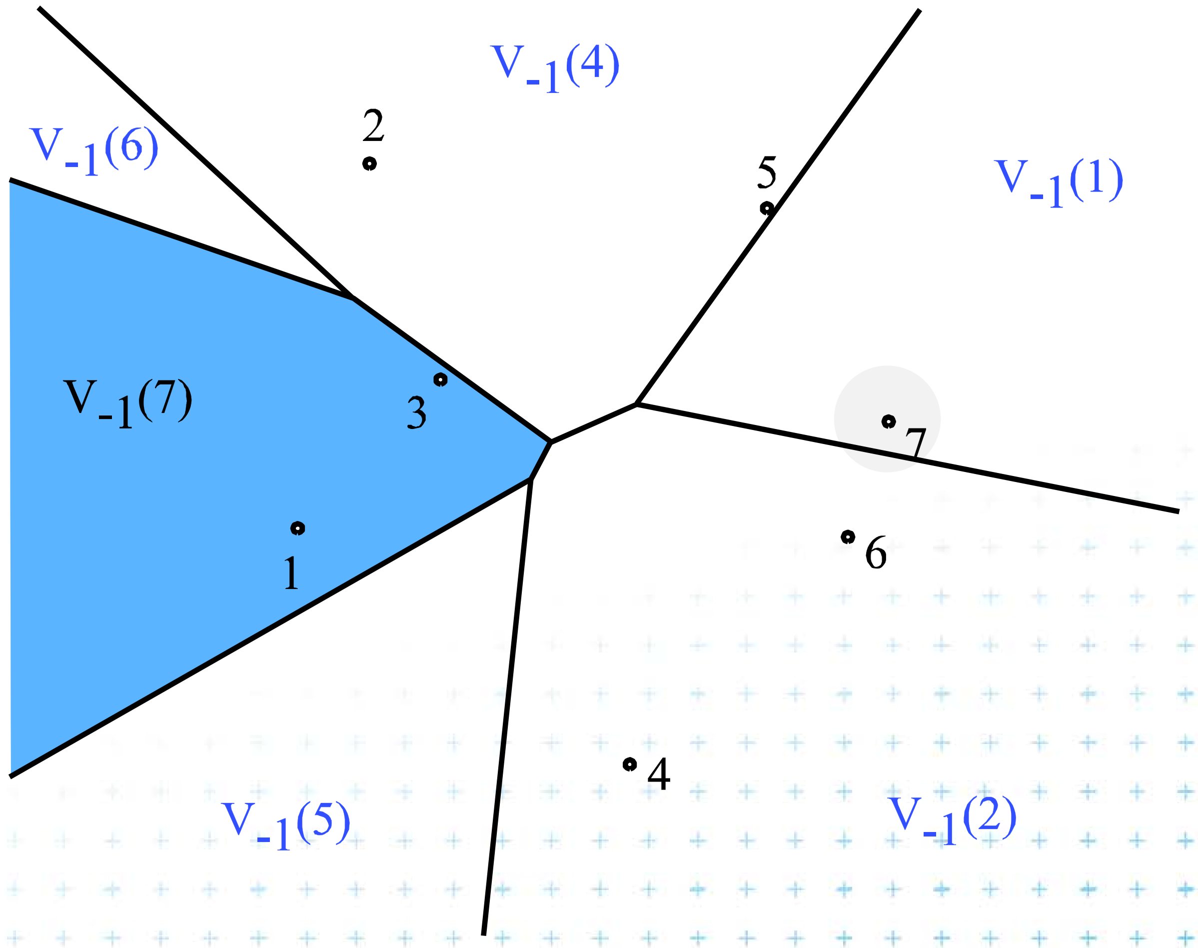


[Nandy]

Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from $p_i=7$ than from any other site

$Vor_{-1}(P) = Vor_{n-1}(P)$
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



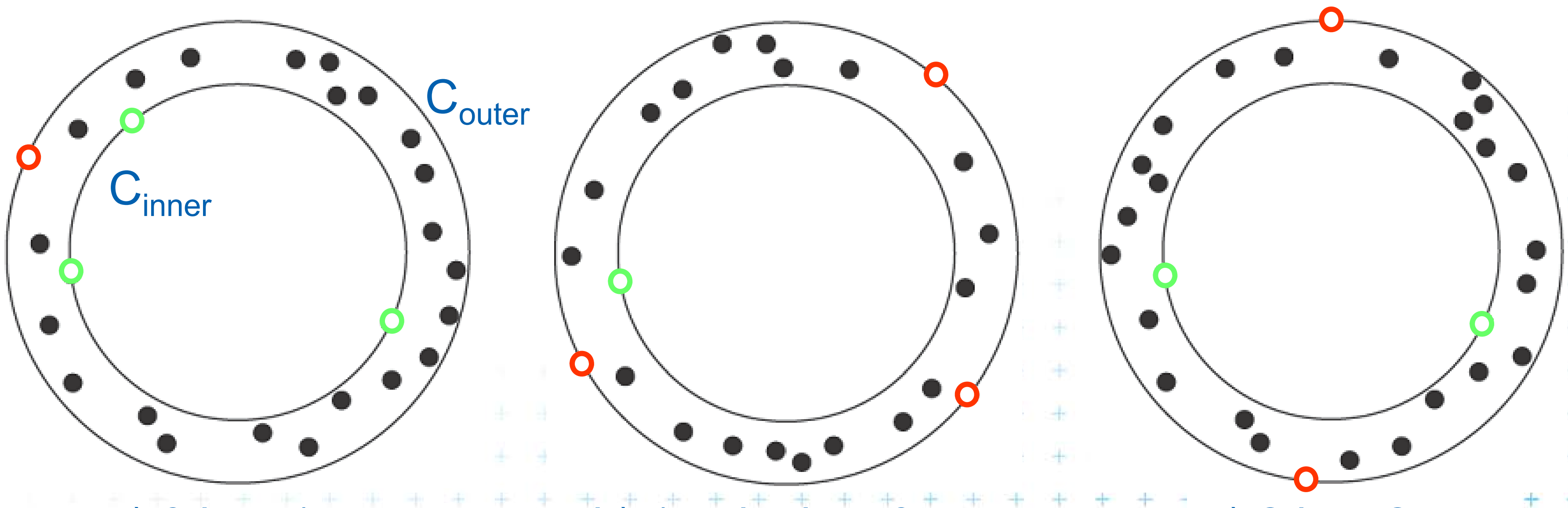
[Nandy]

Farthest-point Voronoi diagrams example

Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles C_{inner} and C_{outer})

Three cases to test – one will win:



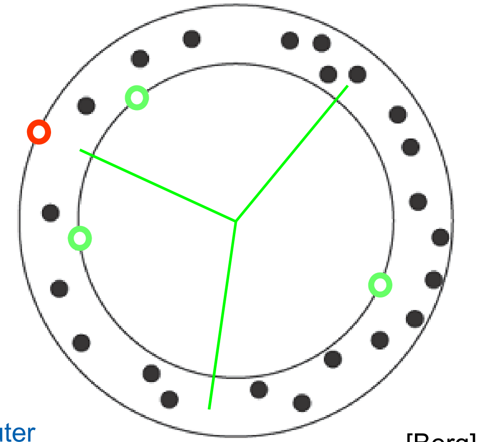
a) 3 in – 1 out b) 1 point in – 3 out c) 2 in – 2 out

[Berg]

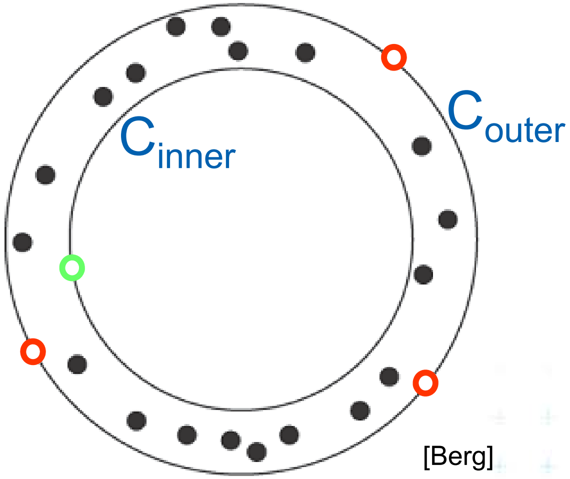
Smallest width annulus – cases with 3 pts

a) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on C_{outer} in O(n) for each vertex
  - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C outer
  - ⇒ $O(n^2)$



3 in – 1 out



1 point in – 3 out

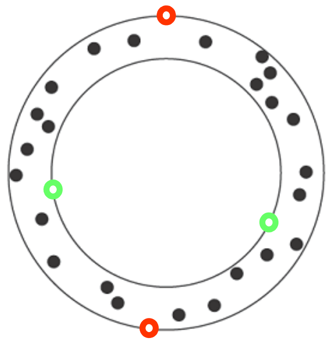
b) C_{outer} contains at least 3 points

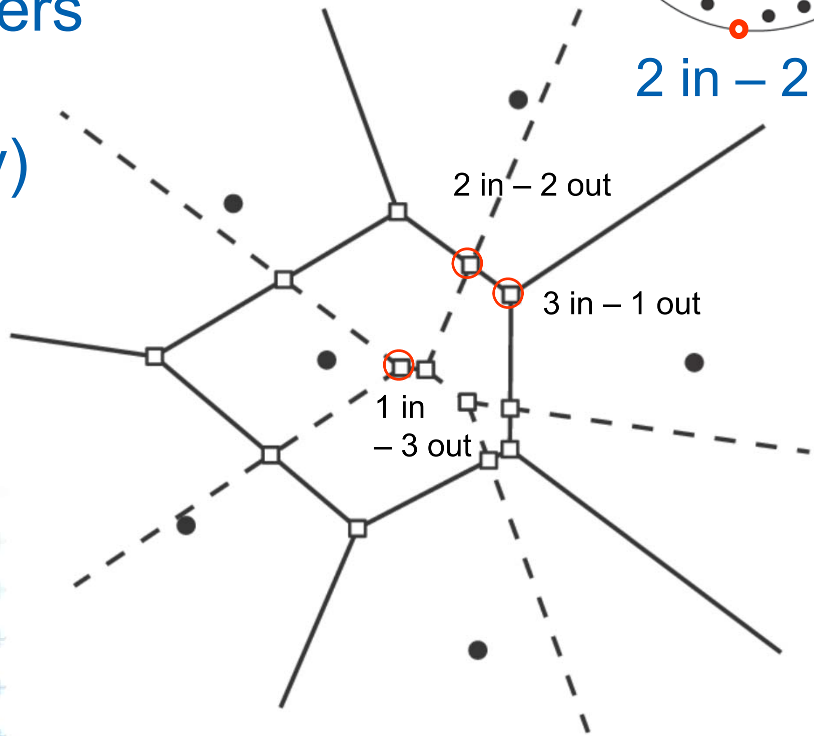
- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on C_{inner} in O(n)
  - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C inner
  - ⇒ $O(n^2)$

Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (__) and farthest-point Voronoi (- - -) diagrams => $O(n^2)$ candidates for centers
  (we need only vertices,
  not the complete overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)
- $O(n^2)$





[Berg]

Smallest width annulus

Smallest-Width-Annulus
Input: Set P of n points in the plane
Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor(P)
   and farthest-point Voronoi diagram $Vor_{-1}(P)$ of P
2. For each vertex of Vor(P) (r) determine the *farthest point* (R) from P
   => $O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $Vor_{-1}(P)$ (R) determine the *closest point* (r) from P
   => $O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges Vor(P) and $Vor_{-1}(P)$ test if they intersect
   => another set of four points defining candidate annulus – c)
5. For all candidates of all three types
   chose the smallest-width annulus

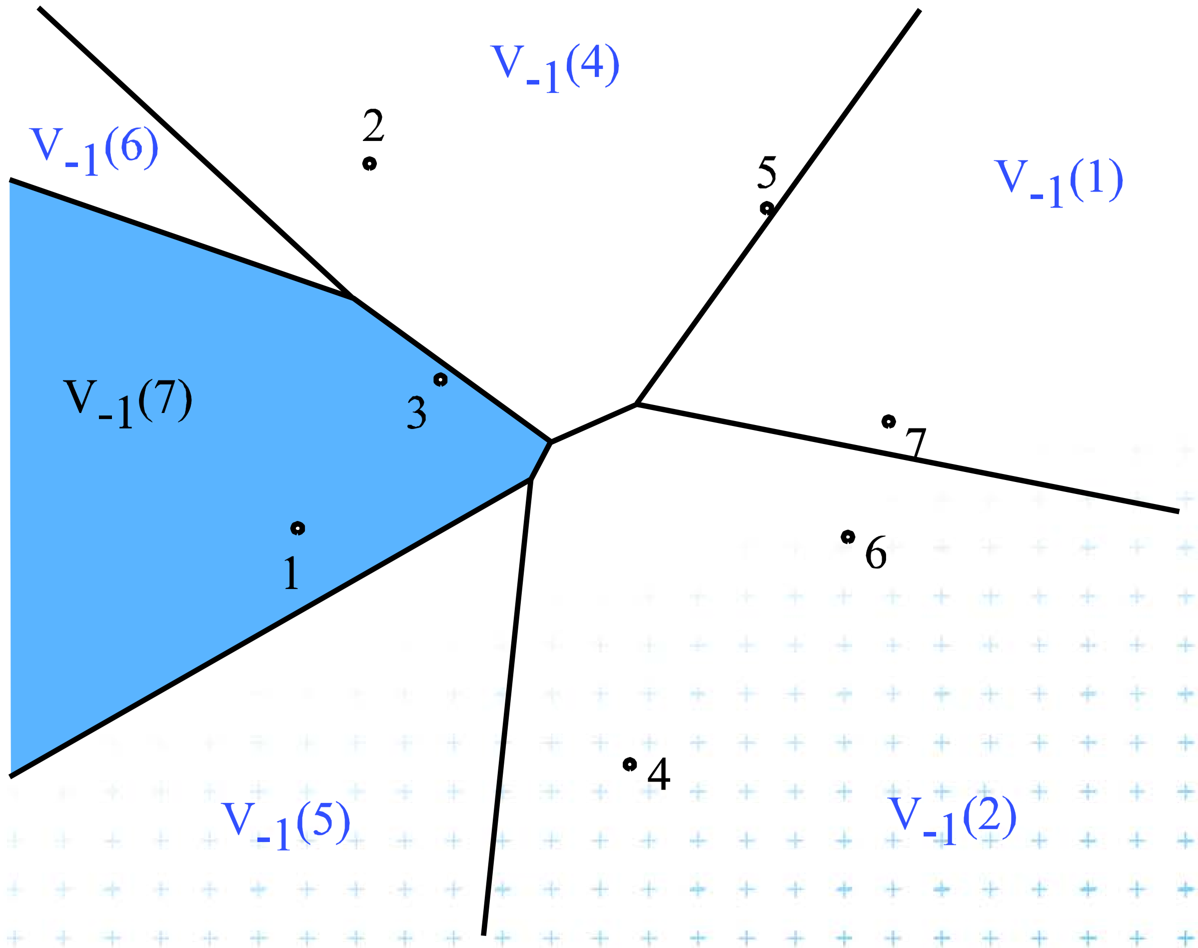
1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

$O(n^2)$ time using $O(n)$ storage

Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
= set of points in the plane farther from p_i than from any other site

$Vor_{-1}(P)$ diagram
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



[Nandy]

Farthest-point Voronoi region (cell)

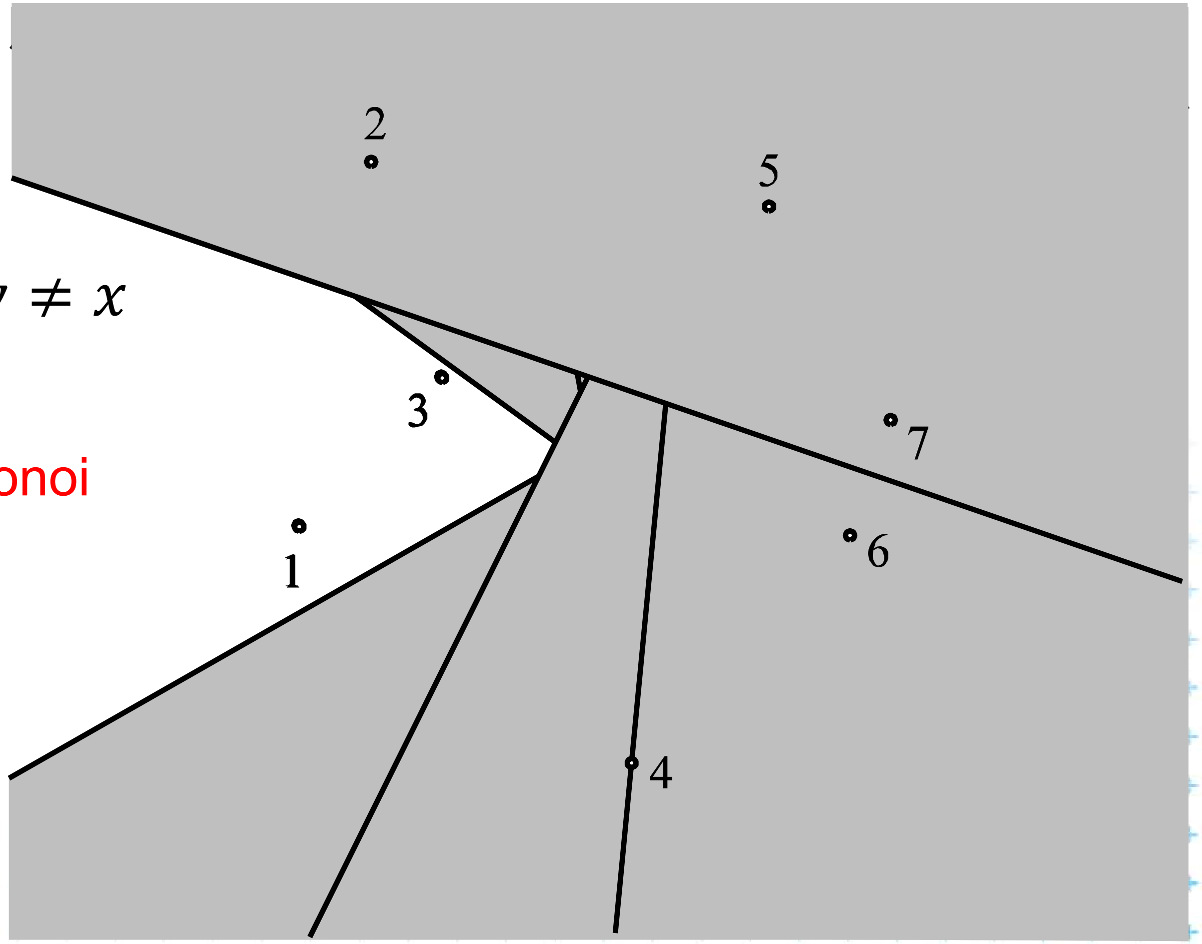
Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

$$V_{-1} = \bigcap_{x=1}^{n} h(y, x),\ y \neq x$$

Property

The farthest point Voronoi regions are convex and unbounded

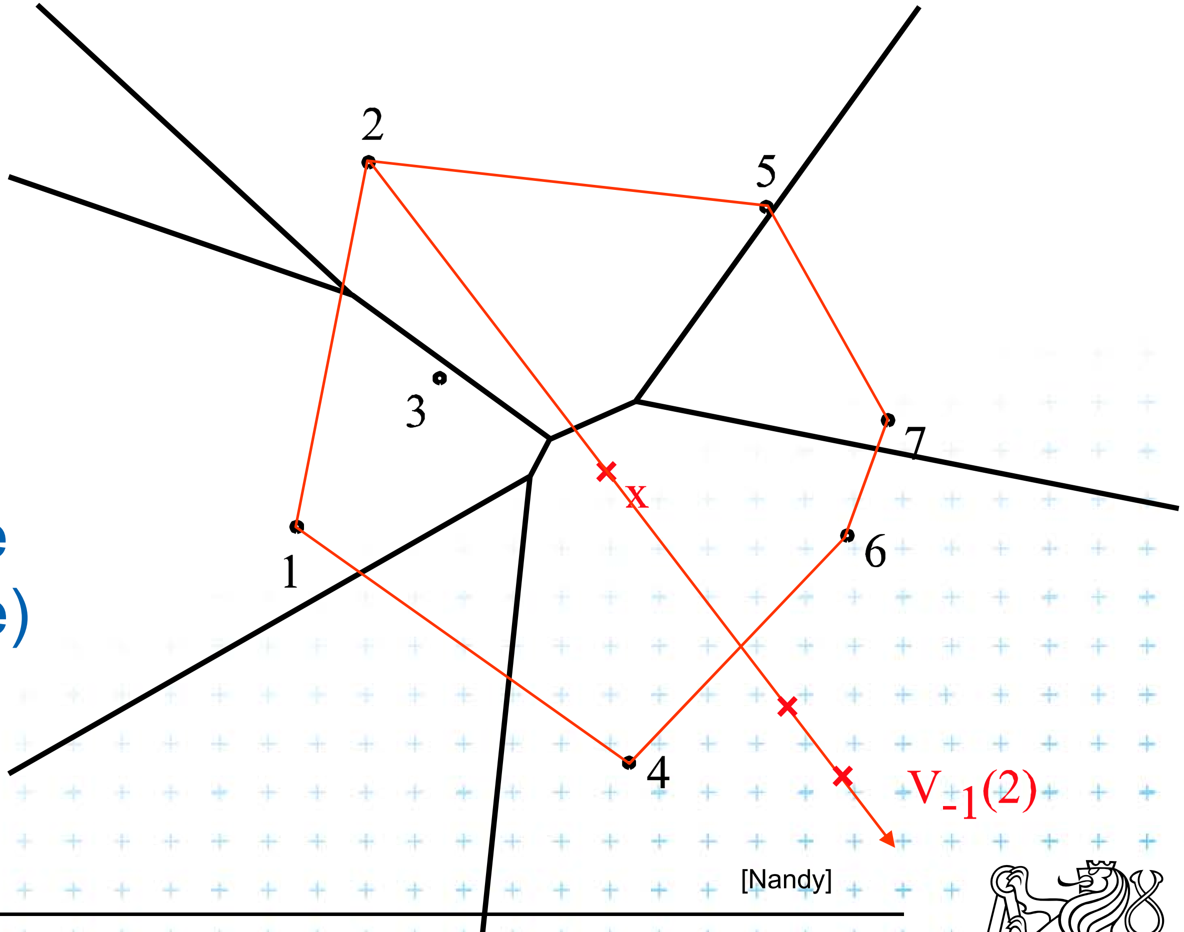


[Nandy]

Farthest-point Voronoi region

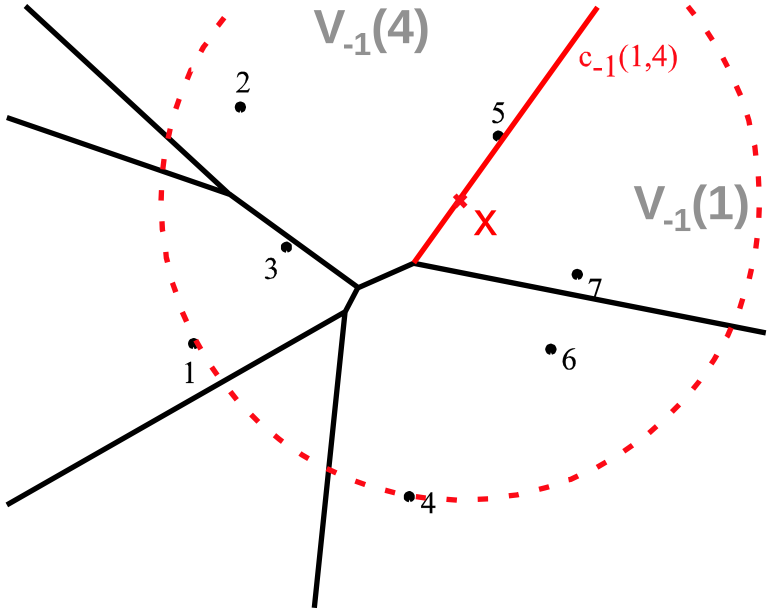
Properties:

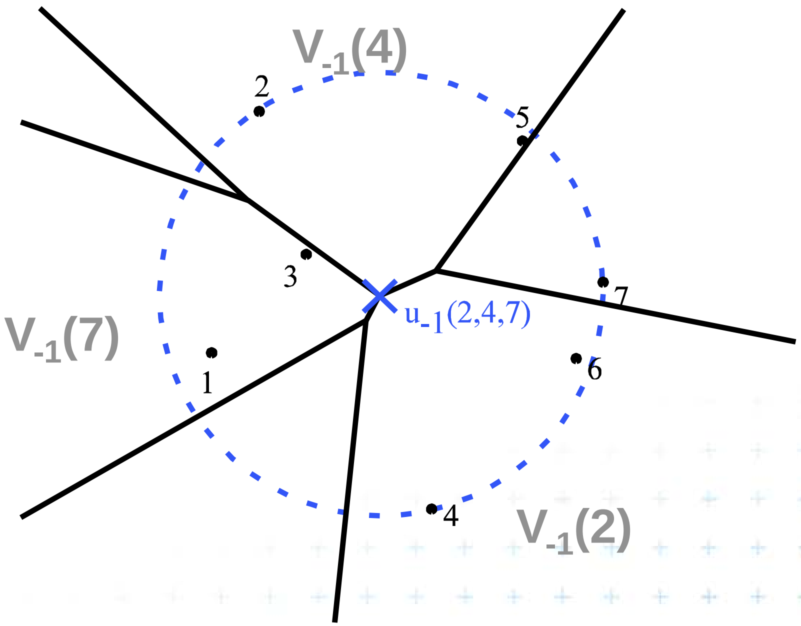
- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)



[Nandy]

Farthest point Voronoi edges and vertices





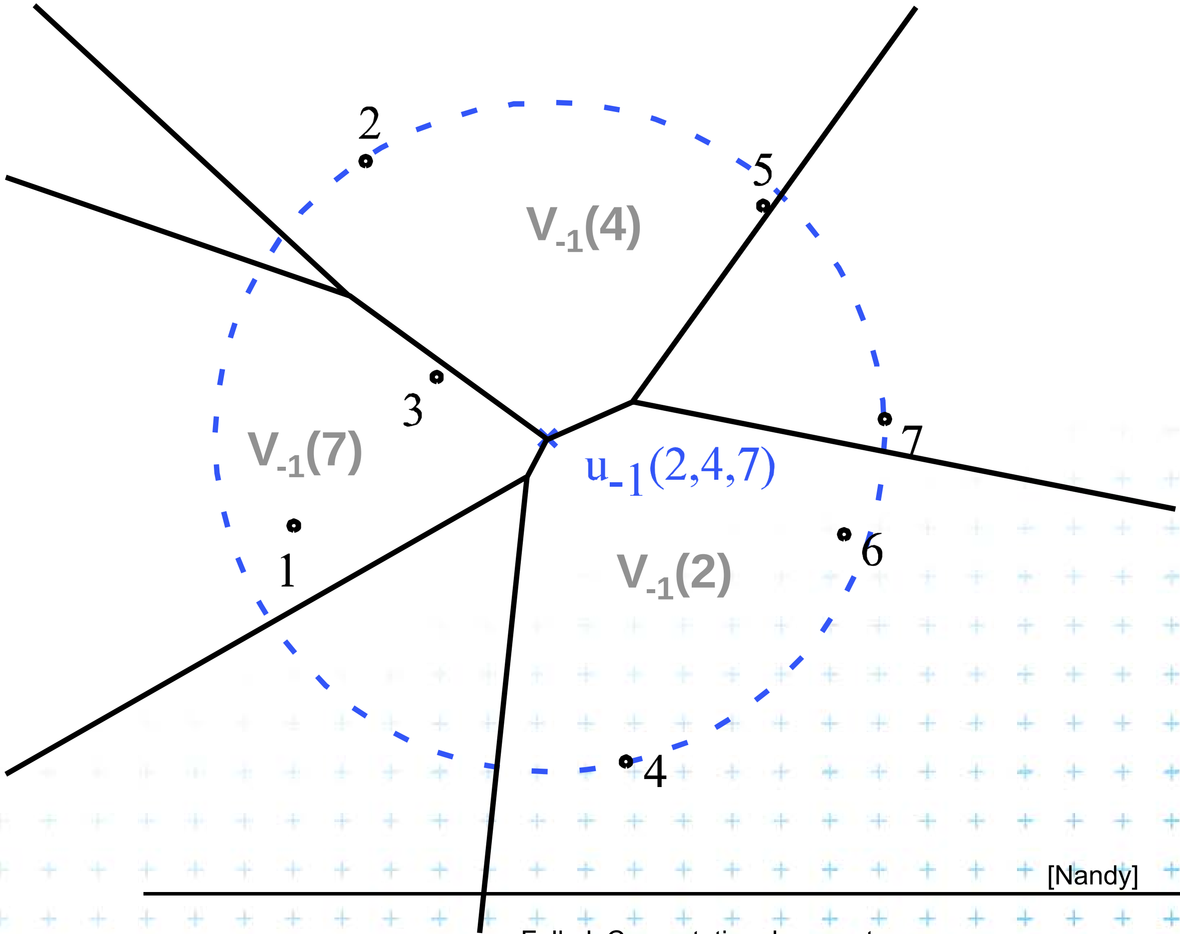
edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites

[Nandy]

Application of $Vor_{-1}(P)$: Smallest enclosing circle

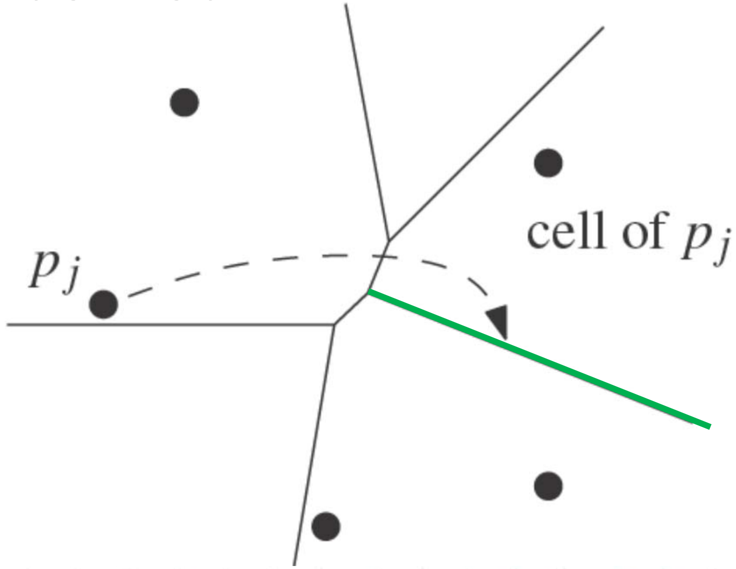
- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges



[Nandy]

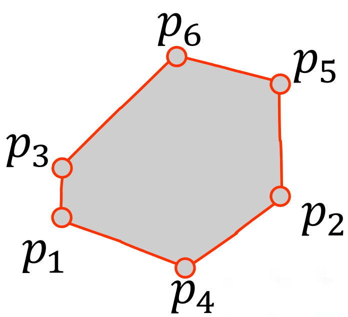
Modified DCEL for farthest-point Voronoi d

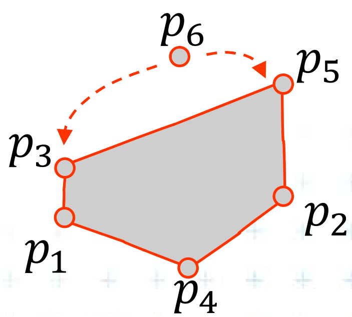
- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



Idea of the algorithm

1. Create the convex hull and number the CH points randomly
2. Remove the points starting in the last of this random order and store *cw*(p_i) and *ccw*(p_i) points at the time of removal.
3. Include the points back and compute V_{-1}





p_i	$ccw(p_i)$	$cw(p_i)$
p_6	p_3	p_5
p_5	p_3	p_2
…		

Farthest-point Voronoi d. construction

Farthest-pointVoronoi O(nlog n) time in O(n) storage

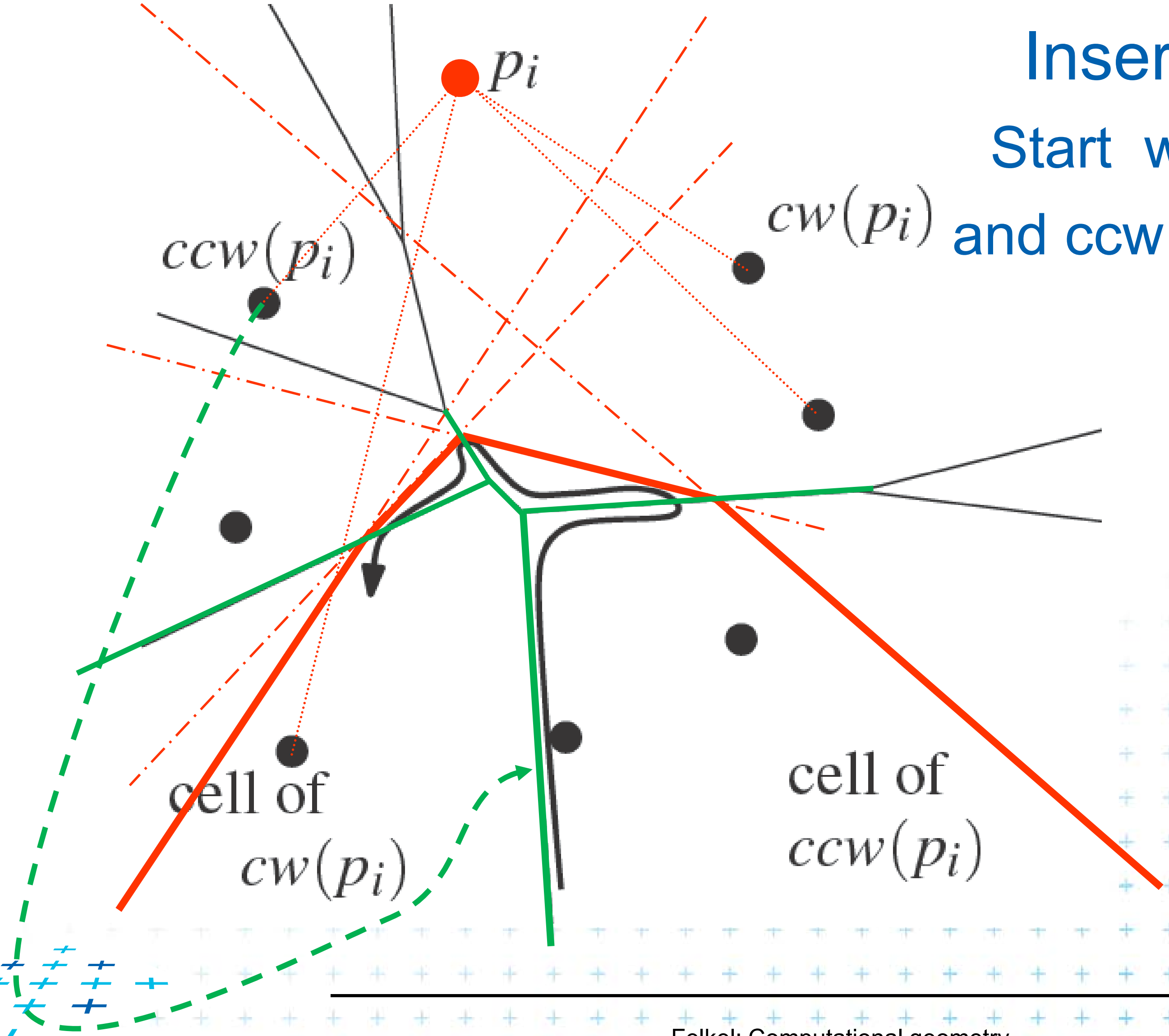
Input: Set of points P in plane

Output: Farthest-point VD $Vor_{-1}(P)$

1. Compute convex hull of P
2. Put points in CH(P) of P in random order $p_1,\dots,p_h$
3. Remove $p_h, \dots, p_4$ from the cyclic order (around the CH). When removing p_i, store the neighbors: $cw(p_i)$ and $ccw(p_i)$ at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $Vor_{-1}(\{ p_1, p_2, p_3 \})$ as init
5. **for** i = 4 **to** h **do**
6. Add site p_i to $Vor_{-1}(\{ p_1, p_2,\dots, p_{i-1} \})$ between site $cw(p_i)$ and $ccw(p_i)$
7. - start at most CCW edge of the cell $ccw(p_i)$
8. - continue CW to find intersection with bisector($ccw(p_i), p_i$)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i

DCGI

Farthest-point Voronoi d. construction



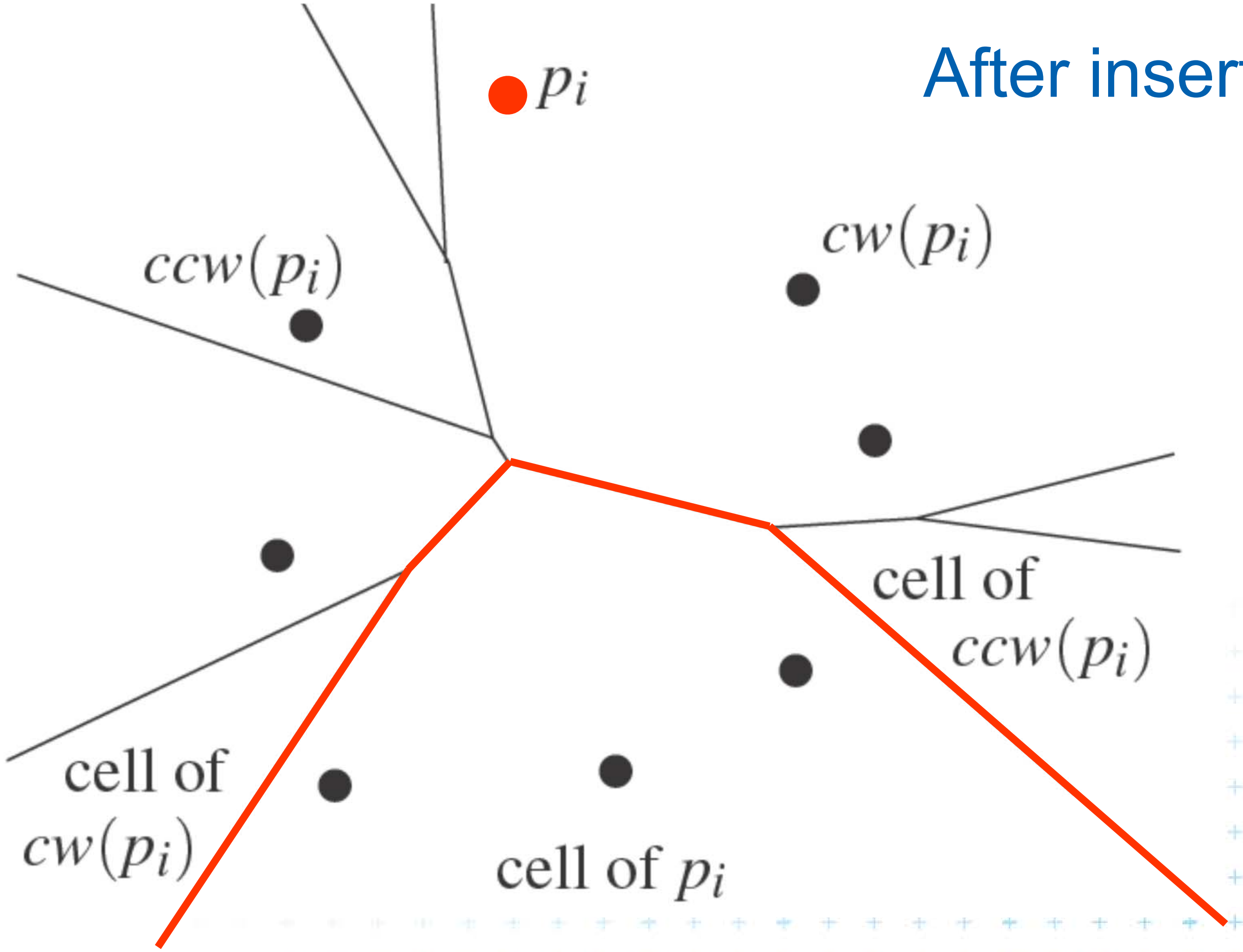
Insertion of site p_i

Start with site ccw(p_i)

and ccw edge of its cell

Farthest-point Voronoi d. construction

After insertion of site p_i



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