# CONVEX HULLS

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**Based on [Berg] and [Mount]**

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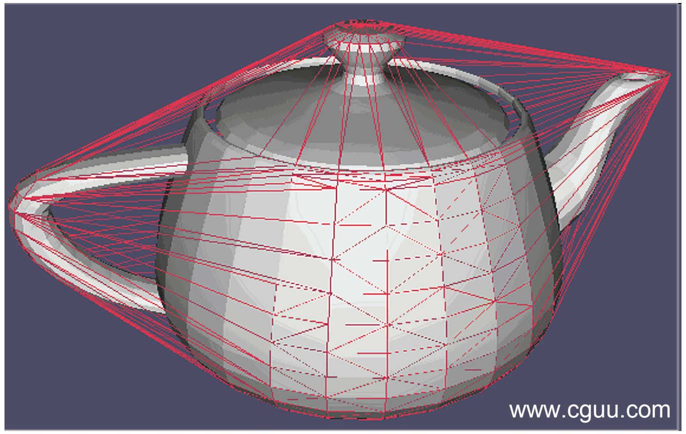
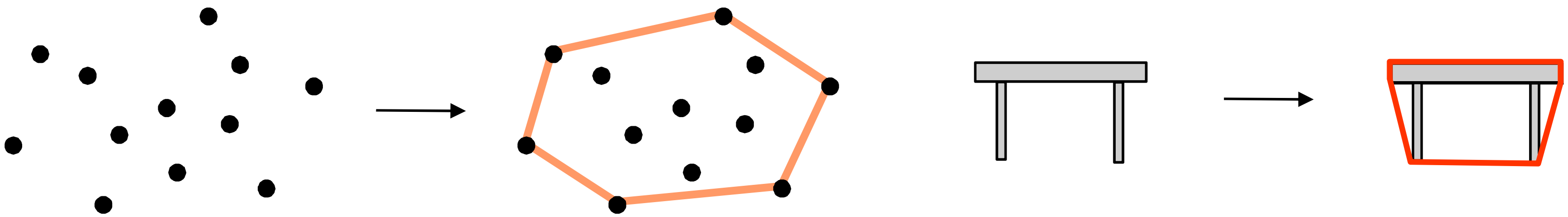
# Talk overview

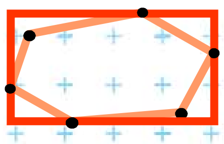
- Motivation and Definitions
- Graham's scan – incremental algorithm
- Divide & Conquer
- Quick hull
- Jarvis's March – selection by gift wrapping
- Chan's algorithm – optimal algorithm



# Convex hull (CH) – why to deal with it?

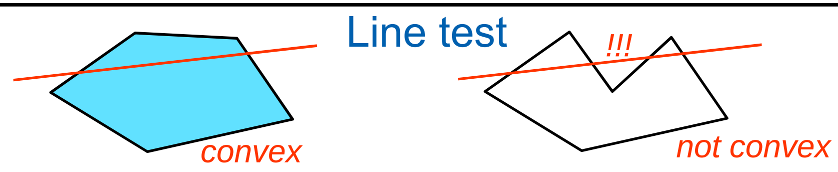


- *Shape approximation* of a point set or complex shapes (other common approximations include: minimal area enclosing rectangle, circle, and ellipse,…) – e.g., for collision detection
- *Initial stage* of many algorithms to filter out irrelevant points, e.g.:
  - diameter of a point set
  - minimum enclosing convex shapes (such as rectangle, circle, and ellipse) depend only on points on CH

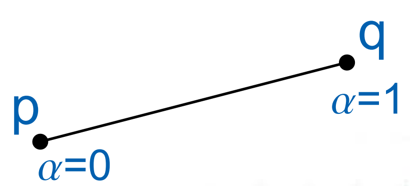


# Convexity

- A set $S$ is *convex*



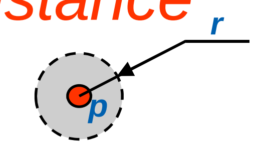

  – if for any points p,q $\in S$ the lines segment $\overline{pq} \subseteq S$, or
  – if any convex combination of *p* and *q* is in *S*
- Convex combination of points *p, q* is any point that can be expressed as
  $(1 - \alpha)\, p + \alpha q$, where $0 \le \alpha \le 1$



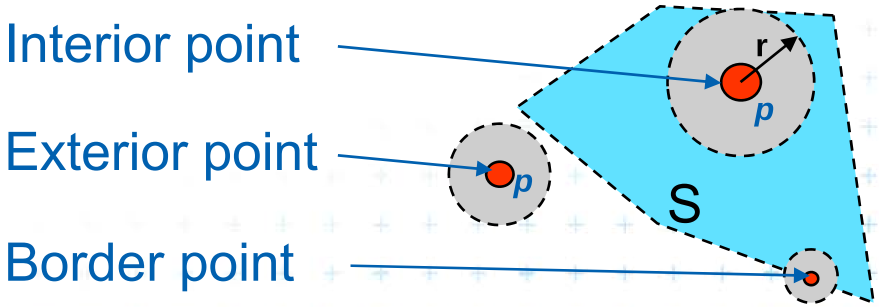

- *Convex hull* CH(S) of set *S* – is (similar definitions)
  – the smallest set that contains *S (convex)*
  – or: intersection of all convex sets that contain *S*
  – Or in 2D for points: the smallest convex polygon containing all given points

# Definitions from topology in metric spaces

- *Metric space* – each two of points have defined a *distance*
- *r-neighborhood* of a point *p* and radius *r > 0*
  = set of points whose distance to *p* is strictly less than *r*
  (open ball of diameter *r* centered about *p*)



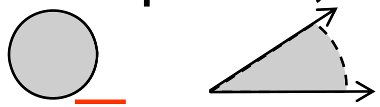

- Given set S, point *p* is
  - Interior point of S – if $\exists r, r > 0$, (r-neighborhood about *p*) $\subset$ S
  - Exterior point – if it lies in interior of the complement of S
  - Border point – is neither interior neither exterior

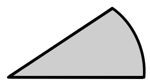


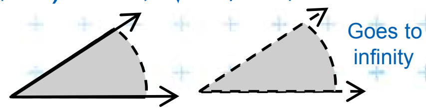
# Definitions from topology in metric spaces

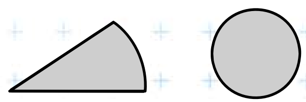
Are border points $p \in S$?

Goes to infinity?

- Set S is Open (otevřená) 
  – $\forall p \in S \ \exists$ (*r*-neighborhood about *p* of radius *r*) $\subseteq$ S
  – it contains only interior points, none of its border points
- Closed (uzavřená) 
  – If it is equal to its closure $\overline{S}$ (uzávěr = smallest closed set containing S in topol. space)
  $\forall$(*r*-neighborhood about *p* of radius *r*) $\cap$ S $\neq \emptyset$)
- Clopen (otevřená i uzavřená) – Ex. Empty set $\emptyset$, finite set of disjoint components
  – if it is both closed and open
  (S= all positive rational numbers whose square is bigger than 2)

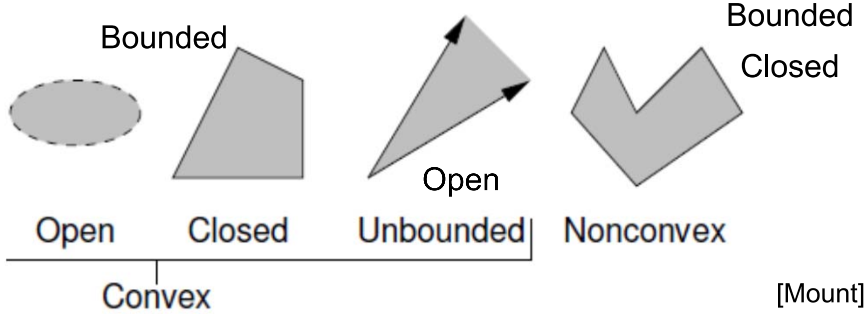
  space Q = rational numbers
  S = ($\sqrt{2}$, $\infty$) in Q, $\sqrt{2} \notin$ Q, S = $\overline{S}$
- Bounded (ohraničená) 

  Unbounded 


  – if it can be enclosed in a ball of finite radius
- Compact (kompaktní) 
  – if it is both closed and bounded

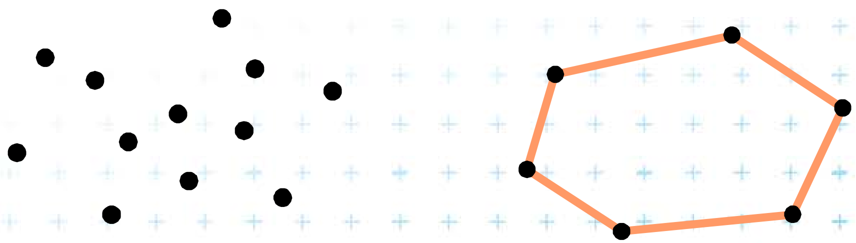
# Definitions from topology in metric spaces

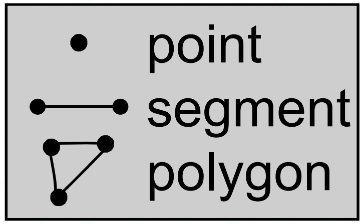
- *Convex set S may be bounded or unbounded*




- *Convex hull* CH(S) of a finite set *S* of points in the plane

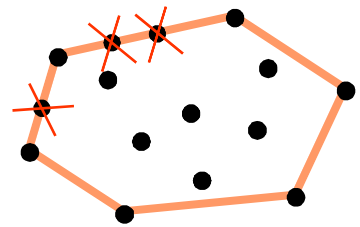
  = Bounded, closed, (= compact) convex polygon





# Convex hull representation

- CCW enumeration of vertices
- Contains only the extreme points ("endpoints" of collinear points)



- Simplification for this semester
  Assume the input points are in general position,
  - no two points have the same *x*-coordinates and
  - no three points are collinear

  -> We avoid problem with non-extreme points on *x*
  (solution may be simple – e.g. lexicographic ordering)

# Online x offline algorithms

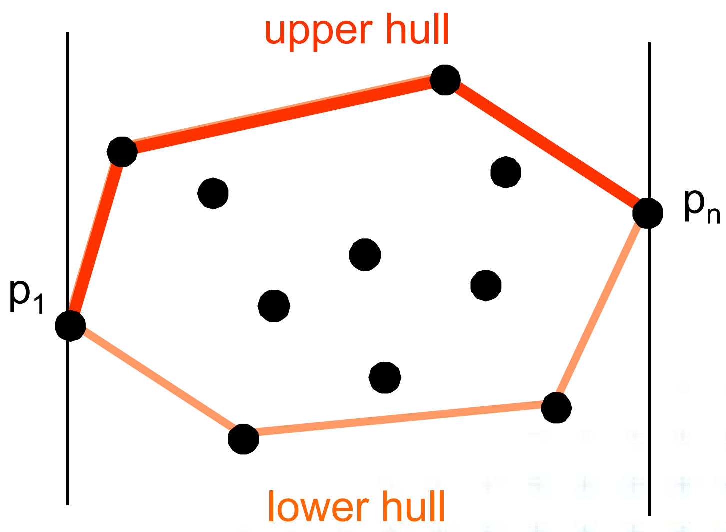
- Incremental algorithm
  - Proceeds one element at a time (step-by-step)
- Online algorithm (must be incremental)
  - is started on a partial (or empty) input and
  - continues its processing as additional input data becomes available (comes online, thus the name).
  - Ex.: insertion sort
- Offline algorithm (may be incremental)
  - requires the entire input data from the beginning
  - than it can start
  - Ex.: selection sort

# Graham's scan

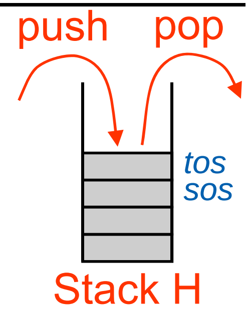
- Incremental O(*n* log *n*) algorithm
- Objects (points) are added one at a time
- Order of insertion is important
  - Random insertion
    –> we need to test: *is-point-inside-the-hull(p)* 
  - Ordered insertion
    Find the point $p$ with the smallest *y* coordinate first
    Sort points $p_i$ according to *increasing angles* around the point $p$ (angle of $pp_i$ and $x$ axis)
  - Andrew's modification: sort points $p_i$ according to *x* and add them left to right (construct upper & lower hull)
    - Sorting *x-coordinates* is simpler to implement than sorting of angles

# Graham's scan – modification by Andrew

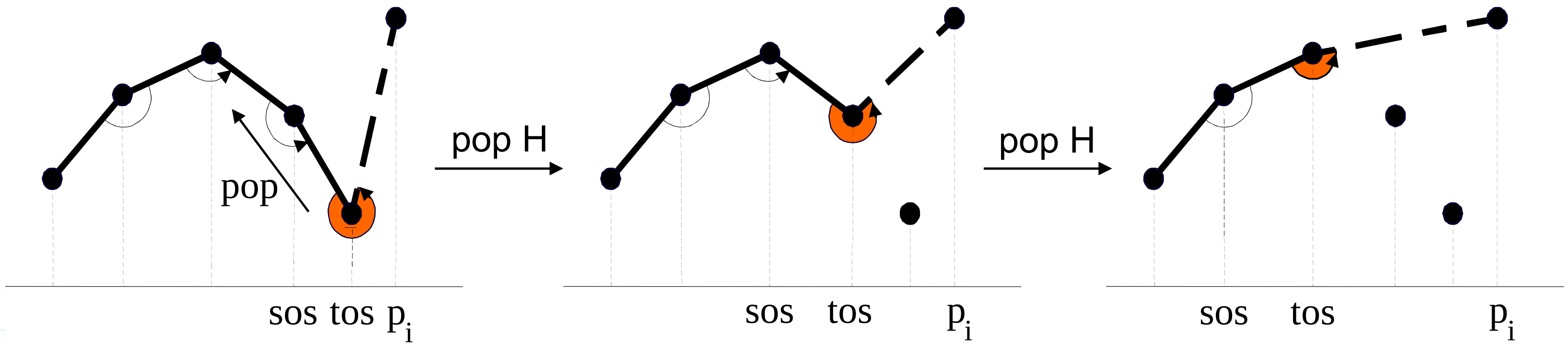
- O($n$ log $n$) for unsorted points, O($n$) for sorted pts.
- Upper hull, then lower hull. Merge.
- Minimum and maximum on $x$ belong to CH



# Graham's scan – incremental algorithm



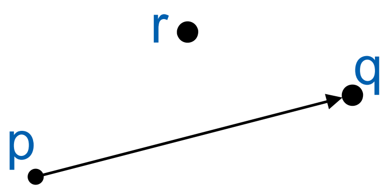

```
GrahamsScan(points p)
Input:    points p
Output:   CCW points on the convex hull
sort points according to increasing x-coord -> {p_1, p_2, …, p_n}
push( p_1, H), push( p_2, H )                                  upper hull
for i = 3 to n do
    while( size(H) ≥ 2 and orient( sos, tos, p_i ) ≥ 0 )  // skip left turns
        pop H                                             // (back-tracking)
    push( p_i, H )                                 // store right turn
store H to the output (in reverse order)   // upper hull
Symmetrically the lower hull
```

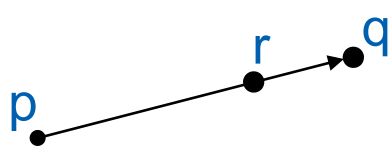


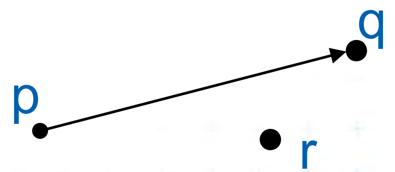
# Position of point in relation to segment

$$\text{orient}(p, q, r) \begin{cases} > 0 & r \text{ is left from } pq\text{, CCW orient} \\ = 0 & \text{if } (p, q, r) \text{ are collinear} \\ < 0 & r \text{ is right from } pq\text{, CW orient} \end{cases}$$

**Point *r* is:** **left from *pq*** **on segment *pq*** **right from *pq***

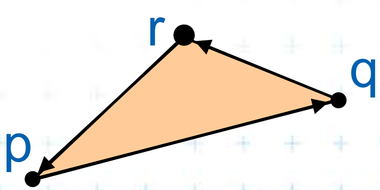


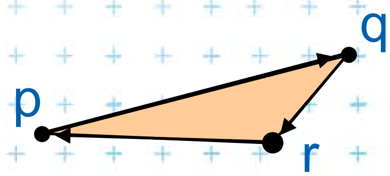




**Convex polygon with edges pq and qr or**

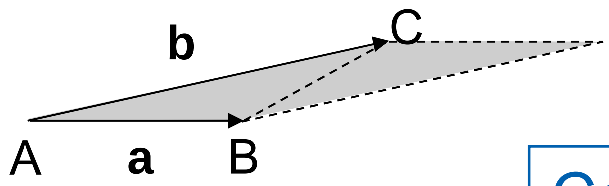
**Triangle *pqr*:** **is CCW oriented** **degenerated to line** **is CW oriented**



# Geometric meaning: Area of Triangle ABC

- Position of point C in relation to segment AB is given by the sign of the triangle ABC area  2x Oriented area
- $T = ½ |\overrightarrow{AB} \times \overrightarrow{AC}|$
- $\mathbf{a} = B - A$
- $\mathbf{b} = C – A$
- $T = ½ (\mathbf{a}_x \mathbf{b}_y - \mathbf{a}_y \mathbf{b}_x)$



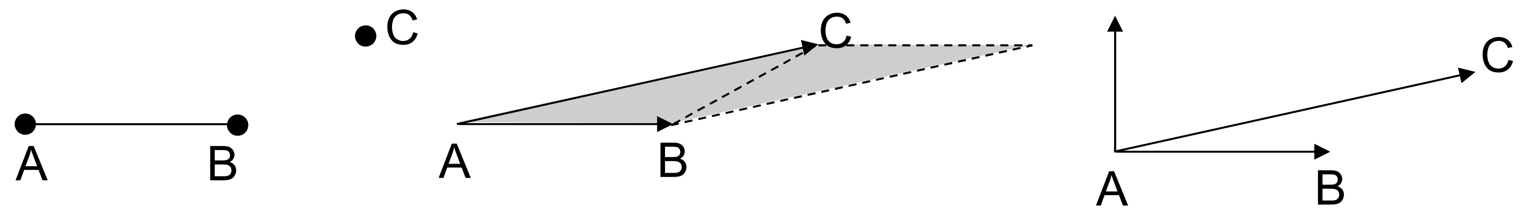

Can be computed as vector product

$$\Rightarrow\ 2T = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$

Or directly as determinant

$$2T = \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$

# Geometric meaning: Area of Triangle ABC



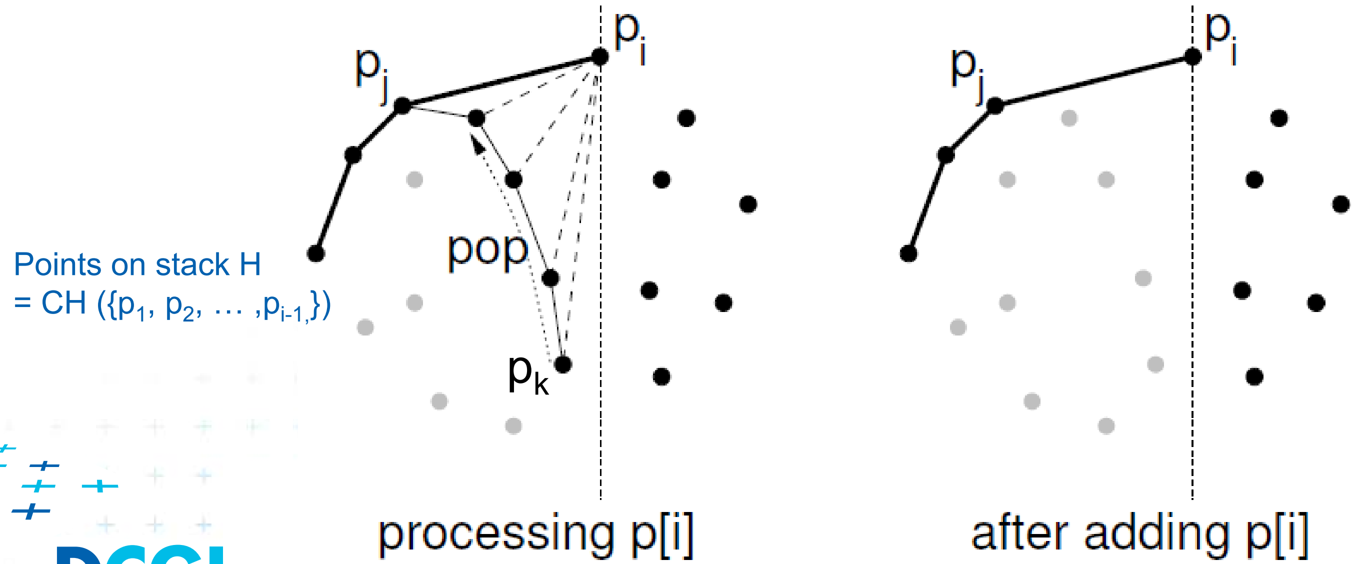

Equal to size of Vector product of vectors AB x AC

= Vector perpendicular to both vectors AB and AC

- If vectors in plane
  - it is perpendicular to the plane (*normal vector of the plane*)
  - only *z*-coordinate is non-zero
- $|\overrightarrow{AB} \times \overrightarrow{AC}|$ = z-coordinate of the normal vector
  = area of parallelopid
  = 2x area *T* of triangle ABC

# Is Graham's scan correct?

- Stack H at any stage contains upper hull of the points $\{p_1,\ldots,p_j, p_i\}$, processed so far
  - For induction basis $H=\{p_1, p_2\}$ … true
  - $p_i$ = last added point to CH, $p_j$ = its predecessor on CH
  - Each point $p_k$ that lies between $p_j$ and $p_i$ lies below $p_jp_i$ and should not be part of UH after addition of $p_i$ => is removed before push $p_i$. [ orient($p_j$, $p_k$, $p_i$) > 0, $p_i$ is left from $p_jp_k$ => $p_k$ is removed from UH]
  - Stop if 2 points in the stack or after construction of the upper hull




[Mount]

# Complexity of Graham's scan

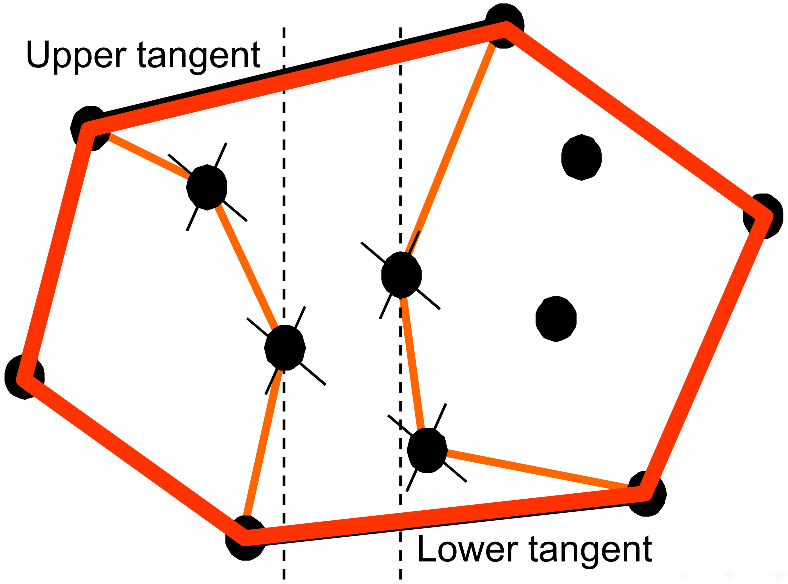
- Sorting according x – $O(n \log n)$
- Each point pushed once – $O(n)$
- Some ($d_i \le n$) points deleted while processing $p_i$ – $O(n)$
- The same for lower hull – $O(n)$

- Total $O(n \log n)$ for unsorted points
  $O(n)$ for sorted points

# Divide & Conquer

- $\Theta(n \log(n))$ algorithm
- Extension of mergesort
- Principle
  - Sort points according to $x$-coordinate,
  - recursively partition the points and solve CH.

# Convex hull by D&C




**ConvexHullD&C(** points **P )**

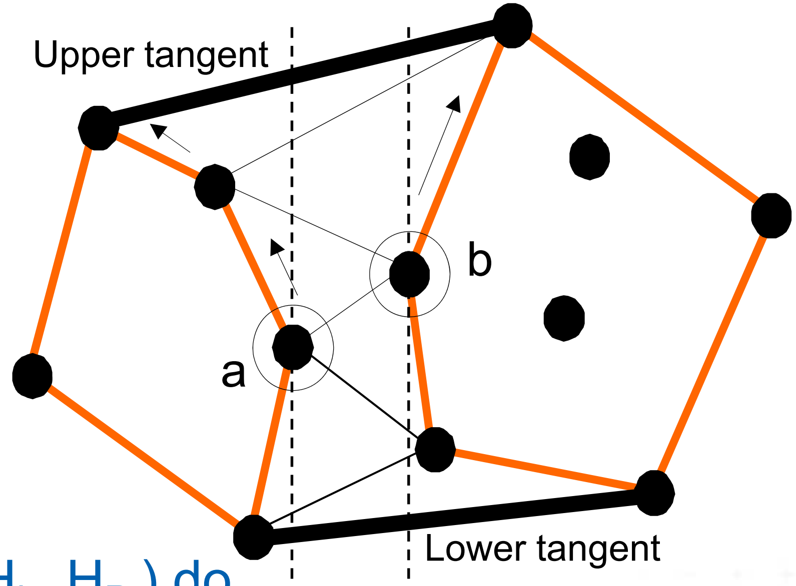
*Input:* points p

*Output:* CCW points on the convex hull

```
Sort points P according to x
return hull( P )

hull( points P )
    if |P| ≤ 3 then
        compute CH by brute force,
        return
    Partition P into two sets L and R (with lower & higher coords x)
    Recursively compute H_L = hull(L), H_R = hull(R)
    H = Merge hulls(H_L, H_R) by computing
        Upper_tangent( H_L, H_R) // find nearest points, H_L CCW, H_R CW
        Lower_tangent( H_L, H_R) // (H_L CW, H_R CCW)
        discard points between these two tangents
    return H
```

# Search for upper tangent (lower is symmetrical)




**Upper_tangent**( $H_L$, $H_R$)
*Input:* two non-overlapping CH's
*Output:* upper tangent *ab*

1. a = rightmost $H_L$
2. b = leftmost $H_R$
3. while( ab is not the upper tangent for $H_L$, $H_R$ ) do
4. while( ab is not the upper tangent for $H_L$) *a = a.succ* // move CCW
5. while( ab is not the upper tangent for $H_R$) *b = b.pred* // move CW
6. Return *ab*

Where: (ab is not the upper tangent for $H_L$) => orient$(a, b, a.succ) \geq 0$
which means *a.succ* is left from line *ab*

$m = |H_L| + |H_R| \leq |L| + |R|$ => Upper Tangent: $O(m) = O(n)$

# Convex hull by D&C complexity

- Initial sort O($n$ log($n$))
- Function hull()
  - Upper and lower tangent O($n$)
  - Merge hulls O(1)
  - Discard points between tangents O($n$)

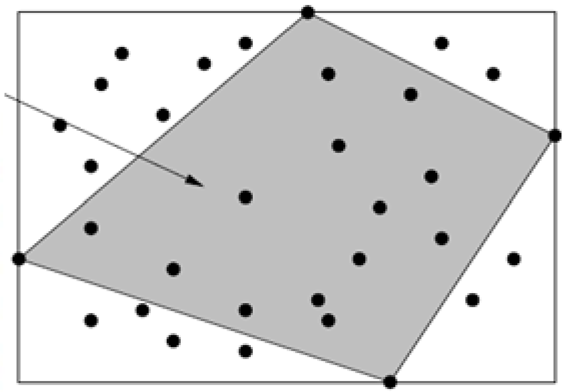
  (together: O($n$))
- Overall complexity
  - Recursion

$$T(n) = \begin{cases} 1 & \dots \text{ if } n \le 3 \\ 2T(n/2) + O(n) & \dots \text{ otherwise} \end{cases}$$

  - Overall complexity of CH by D&C: => O($n$ log($n$))

# Quick hull

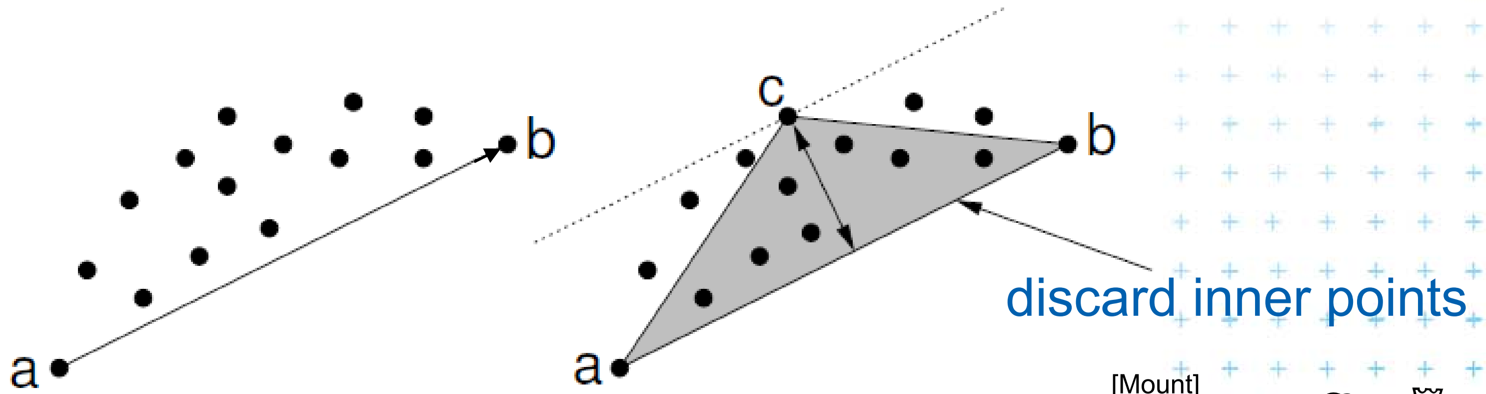
- A variant of Quick Sort
- O($n$ log $n$) expected time, max O($n^2$)
- Principle
  - in praxis, most of the points lie in the interior of CH
  - E.g., for uniformly distributed points in unit square, we expect only O(log $n$) points on CH
- Find extreme points (parts of CH) quadrilateral, discard inner points
  - Add 4 edges to temp hull T
  - Process points outside 4 edges



[Mount]

# Process each of four groups of points outside

- For points outside *ab* (left from *ab* for clockwise CH)
  - Find point *c* on the hull – max. perpend. distance to *ab*
  - Discard points inside triangle *abc* (right from the edges)
  - Split points into two subsets
    - outside *ac* (left from *ac*) and outside *cb* (left from *cb*)
  - Process points outside *ac* and *cb* recursively
  - Replace edge *ab* in *T* by edges *ac* and *cb*

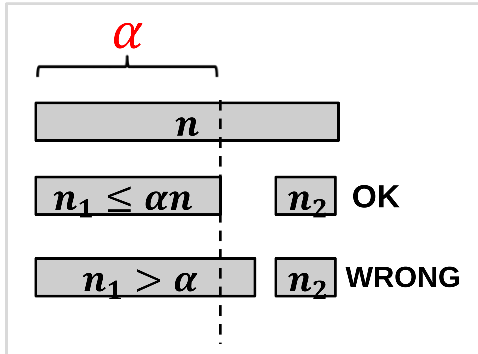



[Mount]

# Quick hull complexity

- *n* points remain outside the hull
- $T(n)$ = running time for such *n* points outside
  - O(*n*) - selection of splitting point *c*
  - O(*n*) - point classification to inside & $(n_1+n_2)$ outside
  - $n_1+n_2 \leq n$
  - The running time is given by recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n_1) + T(n_2) & \text{where } n_1+n_2 \leq n \end{cases}$$

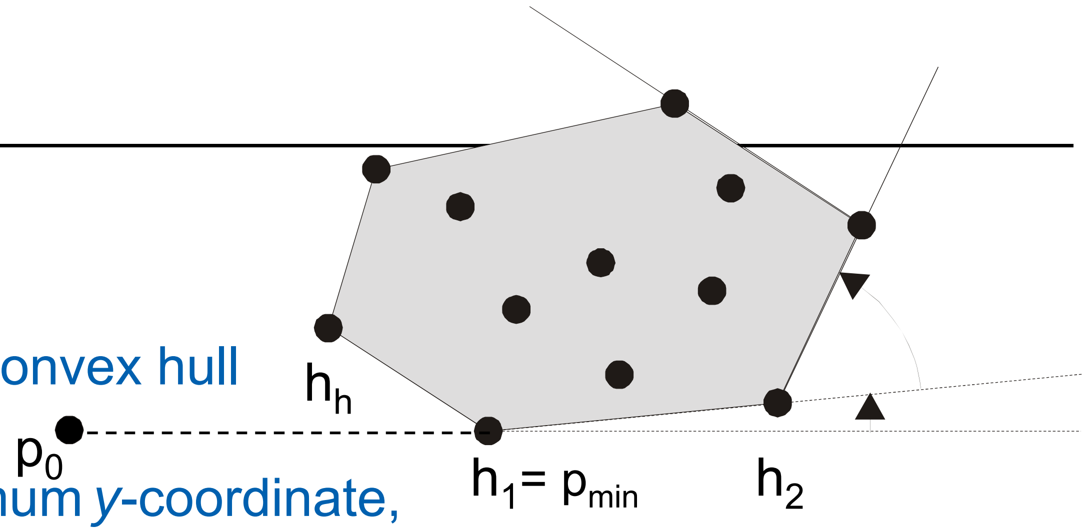



  - If evenly distributed that $\max(n_1, n_2) \leq \alpha n, 0 < \alpha < 1$ then solves as QuickSort to O(*cn* log *n*) where c=f($\alpha$) else O($n^2$) for unbalanced splits

# Jarvis's March – selection by gift wrapping

- Variant of $O(n^2)$ selection sort
- Output sensitive algorithm
- $O(nh)$ … *h = number of points on convex hull*

# Jarvis's March




**JarvisCH**(points P)
*Input:* points p
*Output:* CCW points on the convex hull

1. Take point $p_{min}$ with minimum *y*-coordinate,
   // $p_{min}$ will be the first point in the hull – append it to the hull as $h_1$
2. Take a horizontal line, i.e., create temporary point $p_0 = (-\infty, h_1.y)$
3. j = 1
4. repeat
5. Rotate the line around $h_j$ until it bounces to the nearest point q = $p_q$
   // compute the smallest angle by the "smallest orient($h_{j-1}$ , $h_j$, $q$)"
6. j++
   append the bounced nearest point q to the hull as next $h_j$
7. until ($q \neq p_{min}$)

Output sensitive algorithm

Complexity: O( *n* ) + O( *n* ) * *h* => O( *h***n* )
good for low number of points on convex hull

# Output sensitive algorithm

- Worst case complexity analysis analyzes the worst case data
  - Presumes, that all (const fraction of) points lie **on** the CH
  - The points are ordered along CH

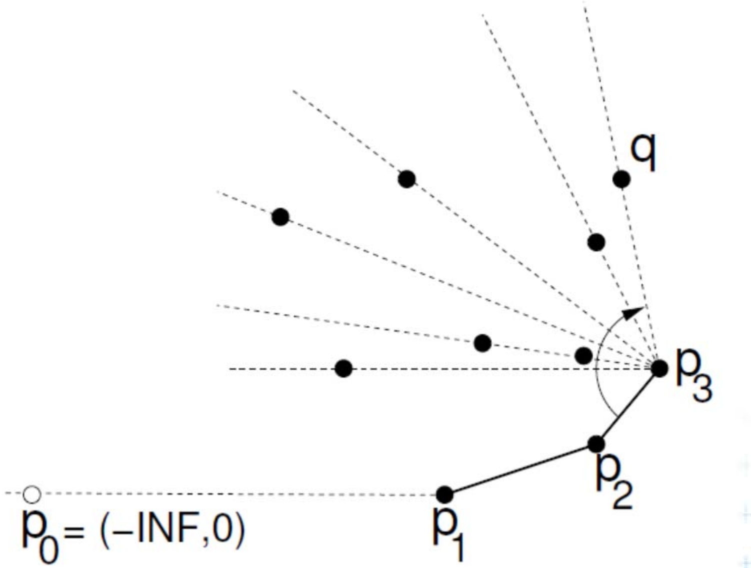
    => We need sorting => $\Omega(n \log n)$ of CH algorithm
- Such assumption is rare
  - usually only much less of points are on CH
- Output sensitive algorithms
  - Depend on: input size $n$ and the size of the output $h$
  - Are more efficient for small output sizes
  - Reasonable time for CH is $O(n \log h)$, *$h$ = Number of points on the CH*

# Chan's algorithm

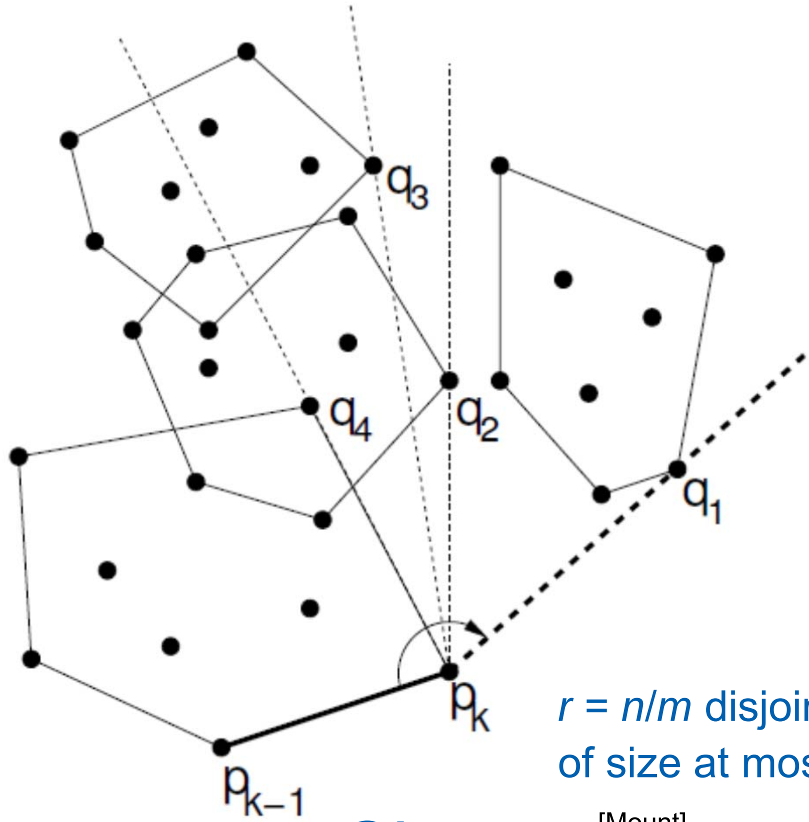
- Cleverly combines Graham's scan and Jarvis's march algorithms
- Goal is O($n$ log $h$) running time
  – We cannot afford sorting of all points - $\Omega(n \log n)$
  => Idea: work on parts, limit the part sizes to polynomial $h^c$
  the complexity does not change => $\log h^c = \log h$
  – $h$ is unknown – we get the estimation later
  – Use estimation $m$, better not too high => $h \leq m \leq h^2$
- 1. Partition points $P$ into $r$-groups of size $m$, $r = n/m$
  – Each group take O($m$ log $m$) time - sort + Graham
  – r-groups take O($r$ $m$ log $m$) = O($n$ log $m$) - Jarvis

# Merging of *m* parts in Chan’s algorithm

- 2. Merge *r*-group CHs as “fat points”
  - Tangents to convex *m*-gon can be found in O(log *m*) by binary search




Jarvis [Mount]




Chan [Mount]

$r = n/m$ disjoint subsets of size at most $m$

# Chan's algorithm complexity

- *h* points on the final convex hull

  => at most *h* steps in the Jarvis march algorithm

  – each step computes *r*-tangents, O(log *m*) each

  – merging together O(*hr* log *m*)

*r*-groups of size *m*, $r = n/m$

- Complete algorithm O(*n* log *h*)

  – Graham's scan on partitions O(*r* .*m* log *m*)=O(*n* log *m*)

  – Jarvis Merging: O(*hr* log *m*) = O(*h n*/*m* log *m*), …4a)

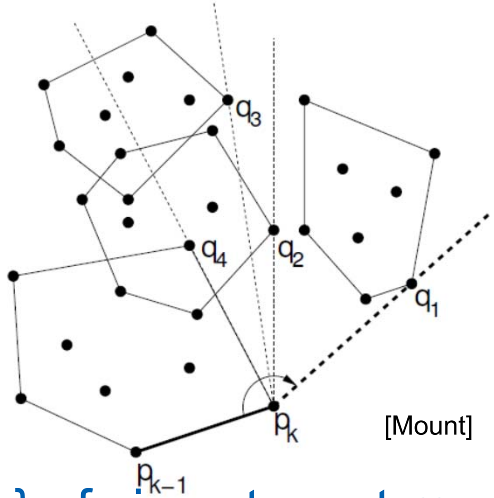
  $h \leq m \leq h^2$ = O(*n* log *m*)

  – Altogether O(*n* log *m*)

  – How to guess *m*? *Wait!*

  *1) use m as an estimation of h 2) if it fails, increase m*

# Chan's algorithm for known *m*




PartialHull( *P*, *m*)
*Input:* points P
*Output:* group of size *m*

1. Partition *P* into $r = \lceil n/m \rceil$ disjoint subsets $\{p_1, p_2, \ldots, p_r\}$ of size at most *m*
2. for *i=1 to r do*
   a) Convex hull by GrahamsScan($P_i$), store vertices in ordered array
3. let $p_1$ = the bottom most point of P and $p_0 = (-\infty, p_1.y)$
4. for *k* = 1 to *m* do // compute merged hull points
   a) for *i* = 1 to *r* do // angle to all *r* subsets => points $q_i$
      Compute the point $q_i \in P$ that maximizes the angle $\angle\, p_{k-1}, p_k, q_i$
   b) let $p_{k+1}$ be the point $q \in \{q_1, q_2, \ldots, q_r\}$ that maximizes $\angle\, p_{k-1}, p_k, q$
      ($p_{k+1}$ is the new point in CH)
   c) if $p_{k+1} = p_1$ then return $\{p_1, p_2, \ldots, p_k\}$
5. return "Fail, *m* was too small"

**O(log *m*)**

Jarvis

# Chan's algorithm – estimation of *m*

ChansHull

*Input:* points P

*Output:* convex hull $p_1 \ldots p_k$

1. for $t$ = 1, 2, … , $\lceil \lg \lg h \rceil$ do {
   a) let $m = \min(2^{2^t}, n)$
   *b)* $L$ = PartialHull( $P$, $m$)
   c) if $L \neq$ "Fail, *m* was too small" then return $L$

   }

Sequence of choices of *m* are { 4, 16, 256,…, $2^{2^t}$ ,…, *n* } … squares

Example: for h = 23 points on convex hull of n = 57 points, the algorithm will try this sequence of choices of *m* { 4, 16, 57 }

1. 4 and 16 will fail
2. 256 will be replaced by *n=57*

# Complexity of Chan's Convex Hull?

- The worst case: Compute all iterations
- $t^{th}$ iteration takes O( $n$ log $2^{2^t}$) = $O(n\ 2^t)$
- Algorithm stops when $2^{2^t} \geq h$ => $t = \lceil \lg \lg h \rceil$
- All $t = \lceil \lg \lg h \rceil$ iterations take:

Using the fact that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

$$\sum_{t=1}^{\lg \lg h} n2^t = n \sum_{t=1}^{\lg \lg h} 2^t \leq n2^{1+\lg \lg h} = 2n \lg h = O(n \log h)$$

2x more work in the worst case

# Conclusion in 2D

- Graham's scan: $O(n \log n)$, $O(n)$ for sorted pts
- Divide & Conquer: $O(n \log n)$
- Quick hull: $O(n \log n)$, max $O(n^2)$ ~ distrib.
- Jarvis's march: $O(hn)$, max $O(n^2)$ ~ pts on CH
- Chan's alg.: $O(n \log h)$ ~ pts on CH

asymptotically optimal

but

constants are too high to be useful

# References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