DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

CONVEX HULLS

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Based on [Berg] and [Mount]

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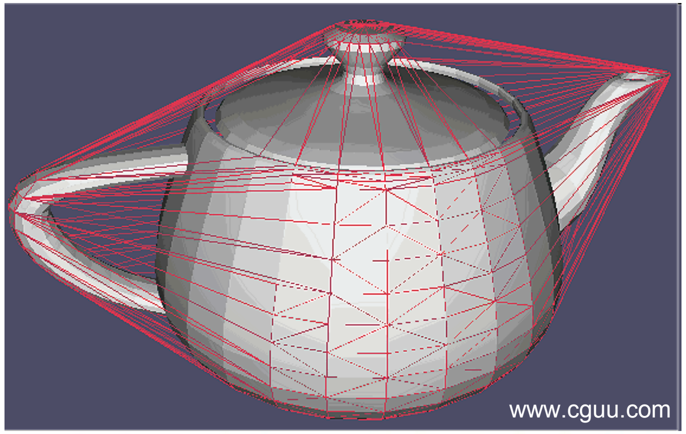
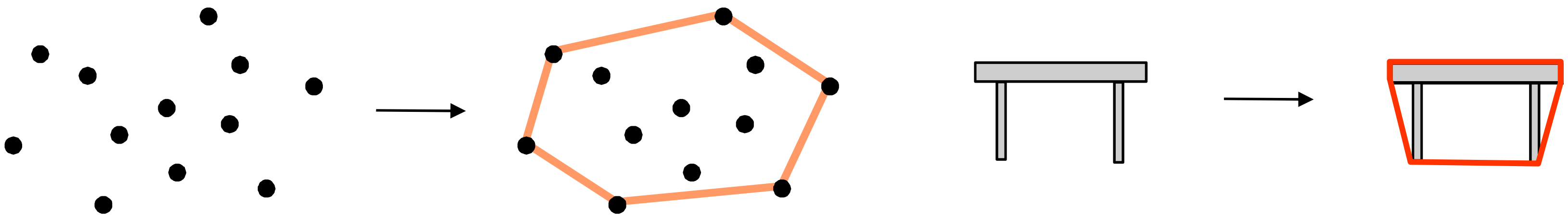
Talk overview

- Motivation and Definitions
- Graham’s scan – incremental algorithm
- Divide & Conquer
- Quick hull
- Jarvis’s March – selection by gift wrapping
- Chan’s algorithm – optimal algorithm

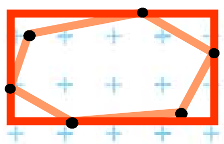


Convex hull (CH) – why to deal with it?



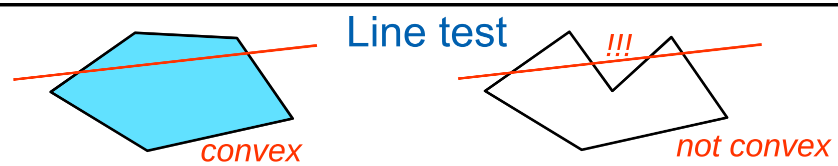
- *Shape approximation* of a point set or complex shapes (other common approximations include: minimal area enclosing rectangle, circle, and ellipse,…) – e.g., for collision detection
- *Initial stage* of many algorithms to filter out irrelevant points, e.g.:
  - diameter of a point set

  - minimum enclosing convex shapes (such as rectangle, circle, and ellipse) depend only on points on CH

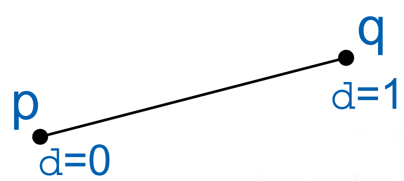


Convexity

- A set *S* is *convex*
  - if for any points p,q $\in$ *S* the lines segment $\overline{pq} \subset S$, or
  - if any convex combination of *p* and *q* is in *S*



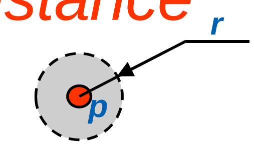
- Convex combination of points *p, q* is any point that can be expressed as
  $(1 - \alpha)\, p + \alpha q$, where $0 \le \alpha \le 1$



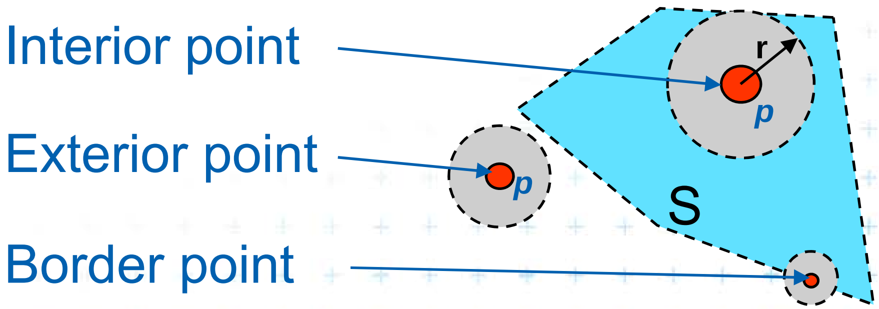
- *Convex hull* CH(S) of set *S* – is (similar definitions)
  - the smallest set that contains *S (convex)*
  - or: intersection of all convex sets that contain *S*
  - Or in 2D for points: the smallest convex polygon containing all given points

Definitions from topology in metric spaces

- *Metric space* – each two of points have defined a *distance*
- *r-neighborhood* of a point *p* and radius *r > 0*
  = set of points whose distance to *p* is strictly less than *r*
  (open ball of diameter *r* centered about *p*)



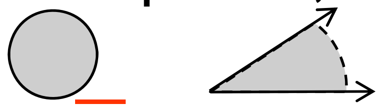
- Given set S, point *p* is
  – Interior point of S – if $\exists r, r > 0$, (r-neighborhood about *p*) ð S
  – Exterior point – if it lies in interior of the complement of S
  – Border point – is neither interior neither exterior



Definitions from topology in metric spaces

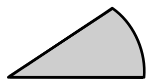
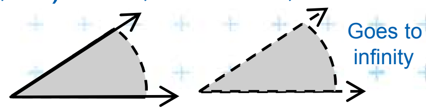
Are border points p ∈ S?

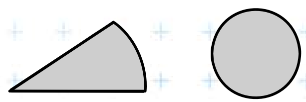
Goes to infinity?

- Set S is Open (otevřená) 
  - %p µ S ′ (*r*-neighborhood about *p* of radius *r*) Å S
  - it contains only interior points, none of its border points
- Closed (uzavřená) 
  - If it is equal to its closure $\overline{S}$ (uzávěr = smallest closed set containing S in topol. space)

    %(*r*-neighborhood about *p* of radius *r*) ý S û þ)
- Clopen (otevřená i uzavřená) – Ex. Empty set þ, finite set of disjoint components
  - if it is both closed and open

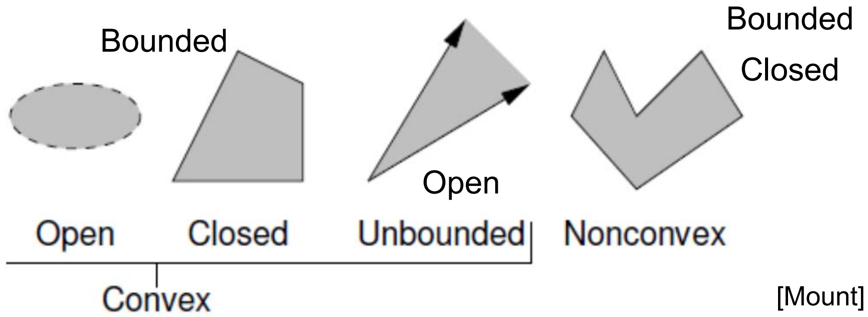
    space Q = rational numbers

    (S= all positive rational numbers whose square is bigger than 2) S = (ô 2, –) in Q, ô 2 Ù Q, S = $\overline{S}$
- Bounded (ohraničená)  Unbounded 

  - if it can be enclosed in a ball of finite radius
- Compact (kompaktní) 
  - if it is both closed and bounded

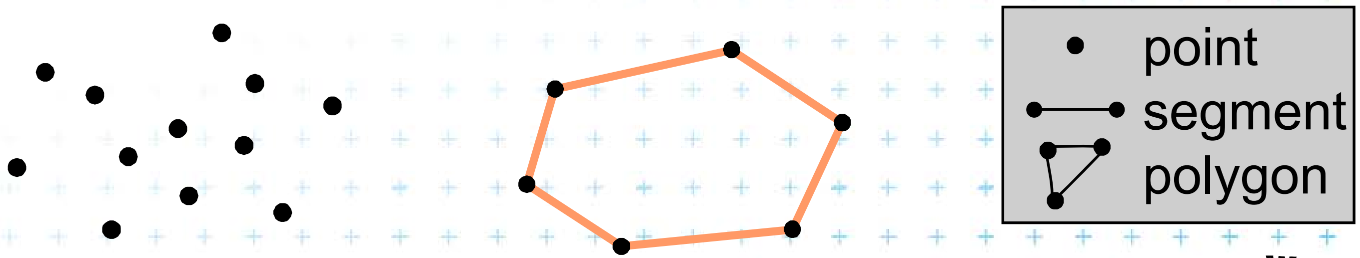
Definitions from topology in metric spaces

- *Convex set S may be bounded or unbounded*



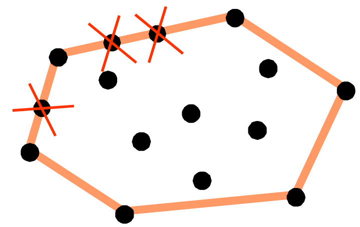
- *Convex hull* CH(S) of a finite set *S* of points in the plane

  = Bounded, closed, (= compact) convex polygon



Convex hull representation

- CCW enumeration of vertices
- Contains only the extreme points ("endpoints" of collinear points)



- Simplification for this semester
  Assume the input points are in general position,
  – no two points have the same *x*-coordinates and
  – no three points are collinear

  -> We avoid problem with non-extreme points on *x*
  (solution may be simple – e.g. lexicographic ordering)

Online x offline algorithms

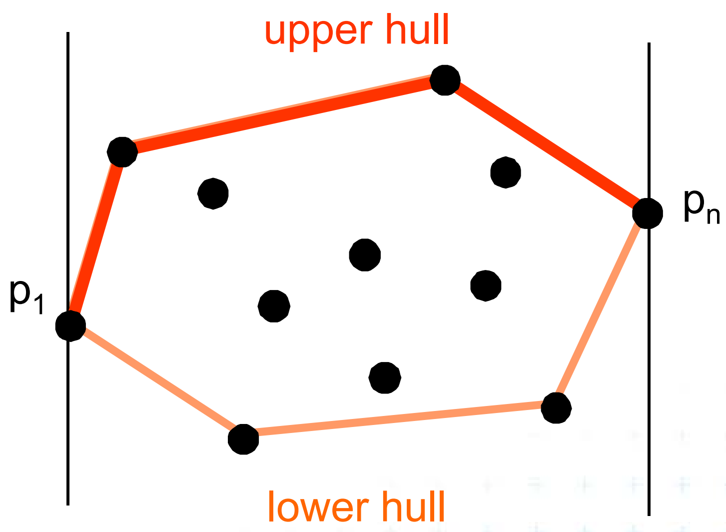
- Incremental algorithm
  - Proceeds one element at a time (step-by-step)
- Online algorithm (must be incremental)
  - is started on a partial (or empty) input and
  - continues its processing as additional input data becomes available (comes online, thus the name).
  - Ex.: insertion sort
- Offline algorithm (may be incremental)
  - requires the entire input data from the beginning
  - than it can start
  - Ex.: selection sort

Graham's scan

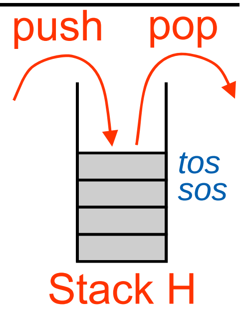
- Incremental O(*n* log *n*) algorithm
- Objects (points) are added one at a time
- Order of insertion is important
  - Random insertion
    –> we need to test: *is-point-inside-the-hull(p)* 
  - Ordered insertion
    Find the point p with the smallest *y* coordinate first
    Sort points p_i according to *increasing angles* around the point p (angle of pp_i and x axis)
  - Andrew's modification: sort points p_i according to *x* and add them left to right (construct upper & lower hull)
    - Sorting *x-coordinates* is simpler to implement than sorting of angles

Graham's scan – modification by Andrew

- O(*n* log *n*) for unsorted points, O(*n*) for sorted pts.
- Upper hull, then lower hull. Merge.
- Minimum and maximum on *x* belong to CH



Graham's scan – incremental algorithm



GrahamsScan(points p)

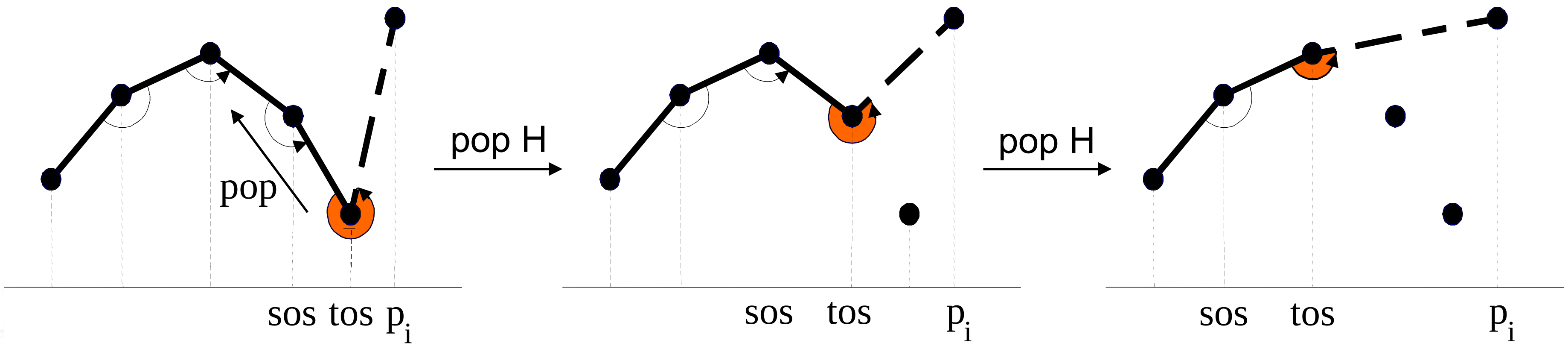
Input: points p

Output: CCW points on the convex hull

1. sort points according to increasing x-coord -> $\{p_1, p_2, \ldots, p_n\}$

upper hull

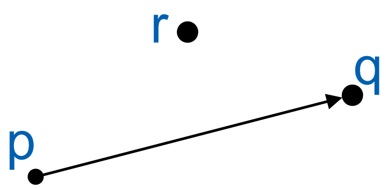
2. push(p_1, H), push(p_2, H)
3. **for** *i* = *3* ***to*** *n* **do**
4. **while**(size(H) – 2 and orient(*sos*, *tos*, p_i) – 0) // skip left turns
5. pop H // (back-tracking)
6. push(p_i, H) // store right turn
7. store H to the output (in reverse order) // upper hull
8. Symmetrically the lower hull



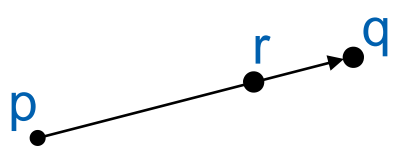
Position of point in relation to segment

$$\text{orient}(\,p, q, r\,) \begin{cases} > 0 & r \text{ is left from } pq\text{, CCW orient} \\ = 0 & \text{if } (\,p, q, r\,) \text{ are collinear} \\ < 0 & r \text{ is right from } pq\text{, CW orient} \end{cases}$$

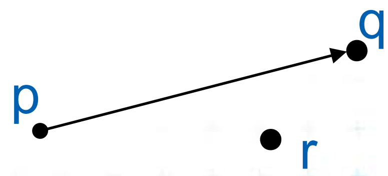
Point *r* is: **left from *pq***



on segment *pq*

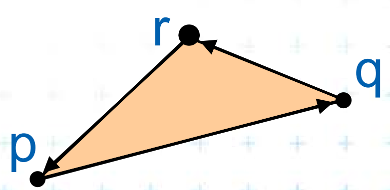


right from *pq*



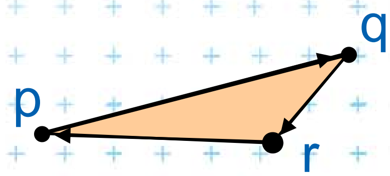
Convex polygon with edges pq and qr or

Triangle *pqr*: **is CCW oriented**



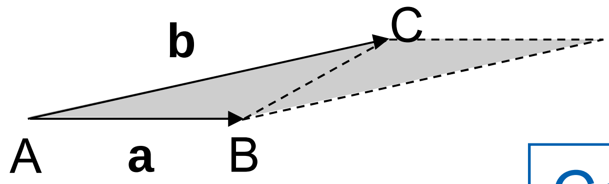
degenerated to line

is CW oriented



Geometric meaning: Area of Triangle ABC

- Position of point C in relation to segment AB is given by the sign of the triangle ABC area   2x Oriented area
- $T = ½\ |\overrightarrow{AB} \times \overrightarrow{AC}|$
- $\mathbf{a} = B - A$
- $\mathbf{b} = C – A$
- $T = ½\ (\mathbf{a}_x\, \mathbf{b}_y\ - \mathbf{a}_y\, \mathbf{b}_x)$



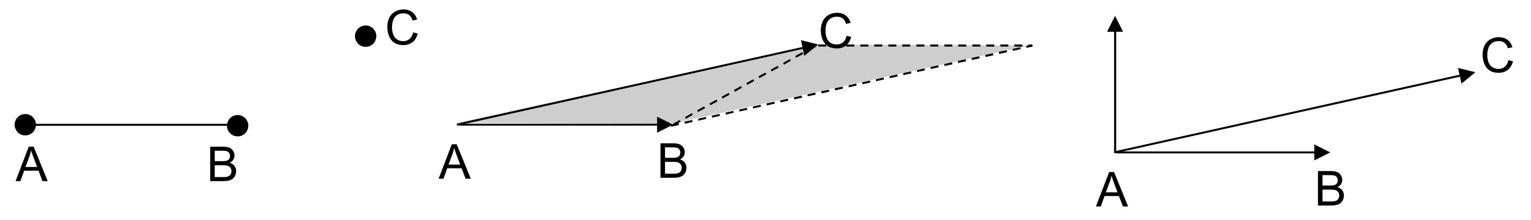
Can be computed as vector product

$$\Rightarrow\quad 2T = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$

Or directly as determinant

$$2T = \begin{vmatrix} A_x & A_y & 1 \\ B_x & B_y & 1 \\ C_x & C_y & 1 \end{vmatrix} = A_x B_y + B_x C_y + C_x A_y - A_x C_y - B_x A_y - C_x B_y$$

Geometric meaning: Area of Triangle ABC



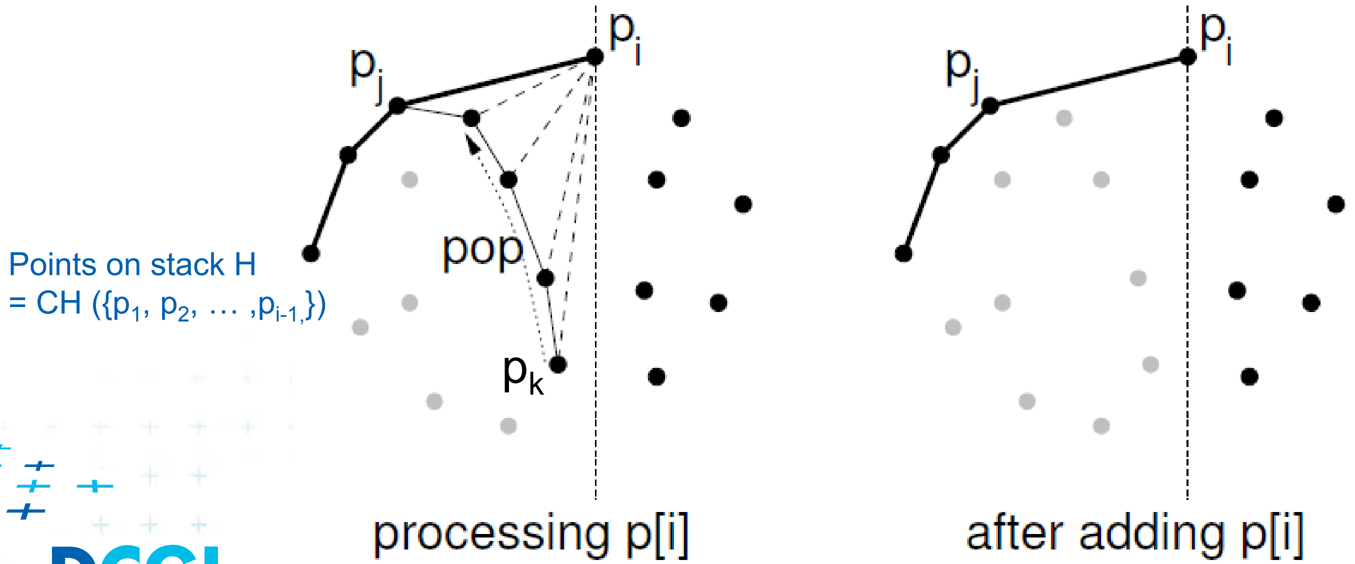
Equal to size of Vector product of vectors AB x AC

= Vector perpendicular to both vectors AB and AC

- If vectors in plane
  - it is perpendicular to the plane (*normal vector of the plane*)
  - only *z*-coordinate is non-zero
- $|\overrightarrow{AB} \times \overrightarrow{AC}|$ = z-coordinate of the normal vector
  = area of parallelopid
  = 2x area *T* of triangle ABC

Is Graham's scan correct?

- Stack H at any stage contains upper hull of the points $\{p_1,\ldots,p_j, p_i\}$, processed so far
  - For induction basis $H=\{p_1, p_2\}$ … true
  - p_i = last added point to CH, p_j = its predecessor on CH
  - Each point p_k that lies between p_j and p_i lies below p_jp_i and should not be part of UH after addition of p_i => is removed before push p_i. [orient(p_j, p_k, p_i) > 0, p_i is left from p_jp_k => p_k is removed from UH]
  - Stop if 2 points in the stack or after construction of the upper hull



Complexity of Graham's scan

- Sorting according x – O(n log n)
- Each point pushed once – O(n)
- Some (d_i ¸ n) points deleted while processing p_i
  – O(n)
- The same for lower hull – O(n)

- Total O(n log n) for unsorted points
  O(n) for sorted points

Divide & Conquer

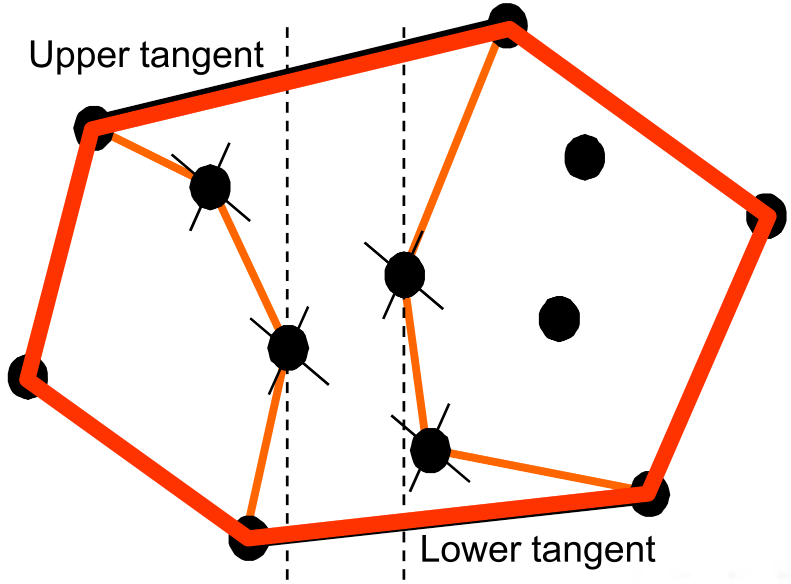
- $\mathrm{T}(n \log(n))$ algorithm
- Extension of mergesort
- Principle
  - Sort points according to *x*-coordinate,
  - recursively partition the points and solve CH.

Convex hull by D&C

ConvexHullD&C(points **P)**

Input: points p

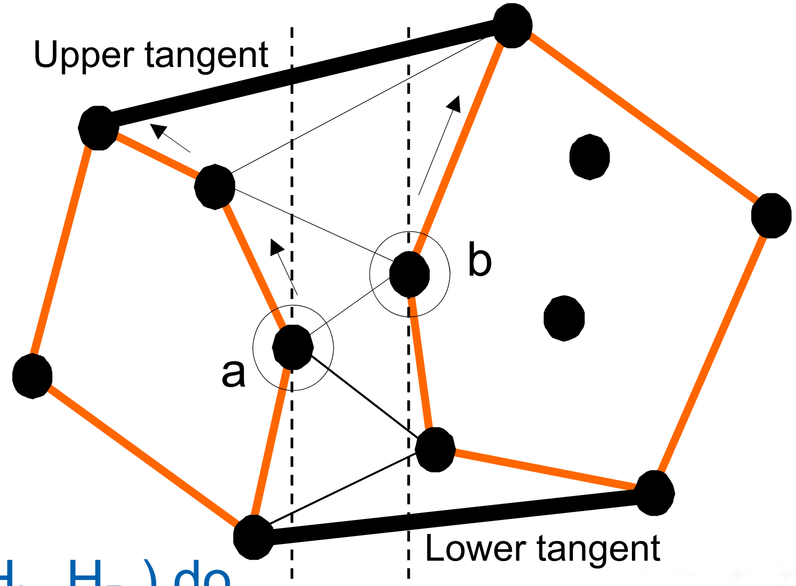
Output: CCW points on the convex hull




```
Sort points P according to x
return hull( P )

hull( points P )
    if |P| ≤ 3 then
            compute CH by brute force,
            return
    Partition P into two sets L and R (with lower & higher coords x)
    Recursively compute H_L = hull(L), H_R = hull(R)
    H = Merge hulls(H_L, H_R) by computing
        Upper_tangent( H_L, H_R) // find nearest points, H_L CCW, H_R CW
        Lower_tangent( H_L, H_R) // (H_L CW, H_R CCW)
        discard points between these two tangents
    return H
```

Search for upper tangent (lower is symmetrical)



Upper_tangent(H_L, H_R)
Input: two non-overlapping CH's
Output: upper tangent *ab*

1. a = rightmost H_L
2. b = leftmost H_R
3. while(ab is not the upper tangent for H_L, H_R) do
4. while(ab is not the upper tangent for H_L) *a = a.succ* // move CCW
5. while(ab is not the upper tangent for H_R) *b = b.pred* // move CW
6. Return *ab*

Where: (ab is not the upper tangent for H_L) => orient(*a, b, a.succ*) – 0
which means *a.succ* is left from line *ab*

m = $|H_L|$+ $|H_R|$ ¸ $|L|$ + $|R|$ => Upper Tangent: O(m) = O(n)

Convex hull by D&C complexity

- Initial sort O(*n* log(*n*))
- Function hull()

$$\left.\begin{array}{ll}\text{– Upper and lower tangent} & O(n)\\ \text{– Merge hulls} & O(1)\\ \text{– Discard points between tangents} & O(n)\end{array}\right\} O(n)$$

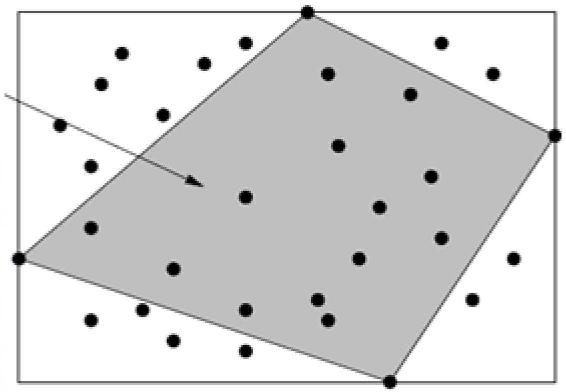
- Overall complexity
  - Recursion

$$T(n) = \begin{cases} 1 & \dots \text{ if } n \text{ ¸ } 3 \\ 2T(n/2) + O(n) & \dots \text{ otherwise} \end{cases}$$

  - Overall complexity of CH by D&C: => O(*n* log(*n*))

Quick hull

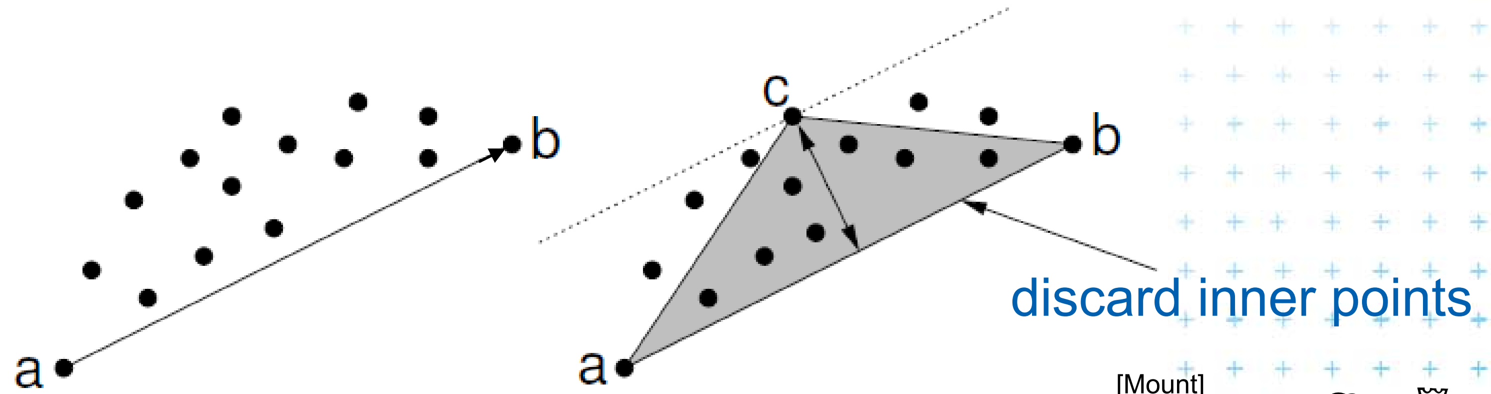
- A variant of Quick Sort
- O(n log n) expected time, max O(n^2)
- Principle
  - in praxis, most of the points lie in the interior of CH
  - E.g., for uniformly distributed points in unit square, we expect only O(log n) points on CH
- Find extreme points (parts of CH) quadrilateral, discard inner points
  - Add 4 edges to temp hull T
  - Process points outside 4 edges



[Mount]

Process each of four groups of points outside

- For points outside *ab* (left from *ab* for clockwise CH)
  - Find point *c* on the hull – max. perpend. distance to *ab*
  - Discard points inside triangle *abc* (right from the edges)
  - Split points into two subsets
    - outside *ac* (left from *ac*) and outside *cb* (left from *cb*)
  - Process points outside *ac* and *cb* recursively
  - Replace edge *ab* in *T* by edges *ac* and *cb*

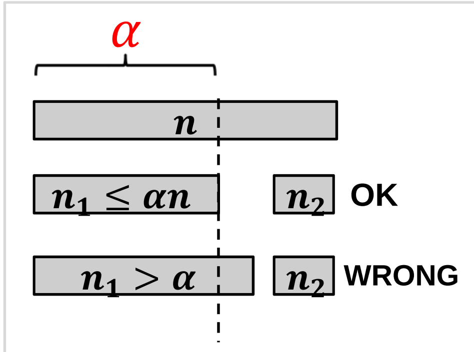


[Mount]

Quick hull complexity

- n points remain outside the hull
- $T(n)$ = running time for such n points outside
  - O(n) - selection of splitting point c
  - O(n) - point classification to inside & (n_1+n_2) outside
  - $n_1+n_2 \le n$
  - The running time is given by recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n_1) + T(n_2) & \text{where } n_1+n_2 \le n \end{cases}$$



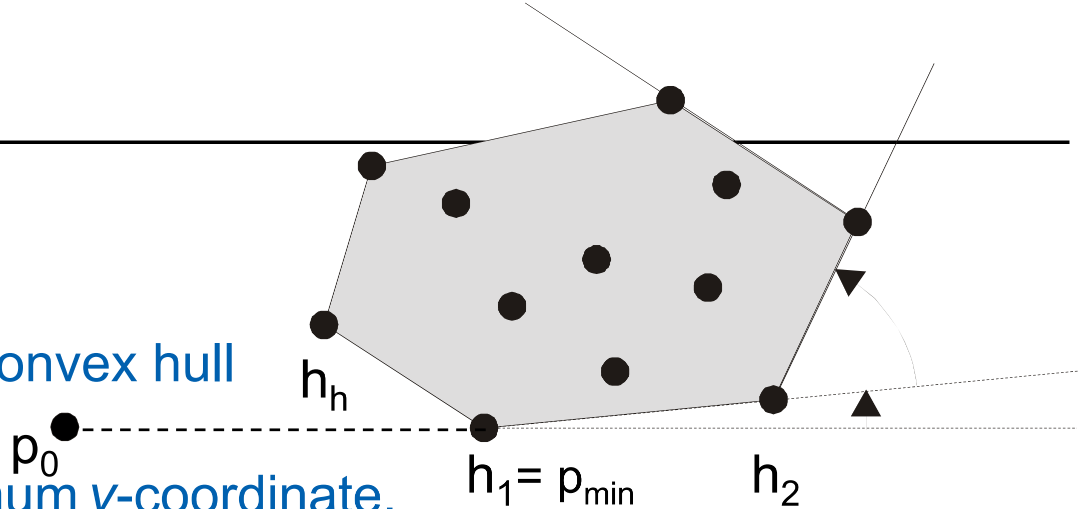
  - If evenly distributed that $\max(n_1, n_2) \le \alpha n, 0 < \alpha < 1$ then solves as QuickSort to O(cn log n) where c=f(d) else O(n^2) for unbalanced splits

Jarvis's March – selection by gift wrapping

- Variant of $O(n^2)$ selection sort
- Output sensitive algorithm
- $O(nh)$ … *h = number of points on convex hull*

Jarvis's March

JarvisCH(points P)
Input: points p
Output: CCW points on the convex hull



1. Take point p_{min} with minimum *y*-coordinate,
   // p_{min} will be the first point in the hull – append it to the hull as h_1
2. Take a horizontal line, i.e., create temporary point $p_0 = (—, h_1.y)$
3. j = 1
4. repeat
5. | Rotate the line around h_j until it bounces to the nearest point q = p_q
   | // compute the smallest angle by the "smallest orient(h_{j-1}, h_j, q)"
6. | j++
   | append the bounced nearest point q to the hull as next h_j
7. until ($q \neq p_{min}$)

Output sensitive algorithm

Complexity: $O(n) + O(n) * h \Rightarrow O(h*n)$
<u>good for low number of points on convex hull</u>

DCGI

Output sensitive algorithm

- Worst case complexity analysis analyzes the worst case data
  - Presumes, that all (const fraction of) points lie **on** the CH
  - The points are ordered along CH

    => We need sorting => Z(n log n) of CH algorithm
- Such assumption is rare
  - usually only much less of points are on CH
- Output sensitive algorithms
  - Depend on: input size n and the size of the output h
  - Are more efficient for small output sizes
  - Reasonable time for CH is O(n log h), *h = Number of points on the CH*

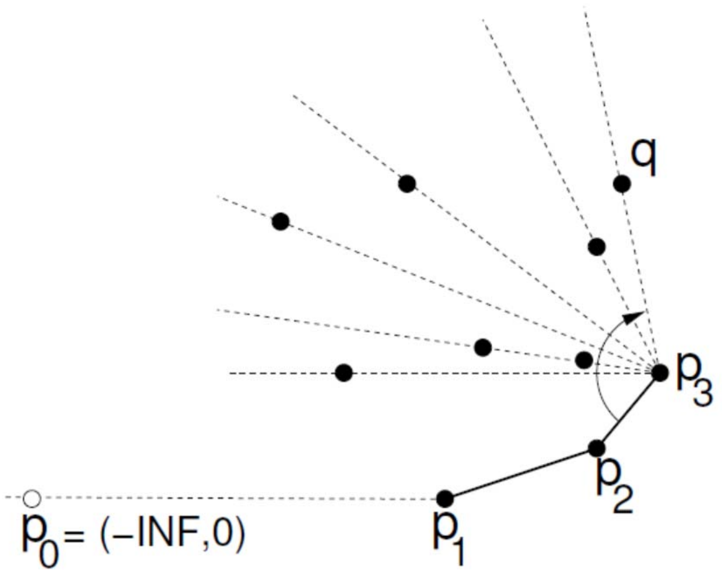
Chan's algorithm

- Cleverly combines Graham's scan and Jarvis's march algorithms
- Goal is $O(n \log h)$ running time
  - We cannot afford sorting of all points - $\Omega(n \log n)$

  => Idea: work on parts, limit the part sizes to polynomial h^c
  the complexity does not change => $\log h^c = \log h$
  - *h* is unknown – we get the estimation later
  - Use estimation *m*, better not too high => $h \le m \le h^2$
- 1. Partition points *P* into *r*-groups of size *m*, $r = n/m$
  - Each group take $O(m \log m)$ time - sort + Graham
  - r-groups take $O(r\, m \log m) = O(n \log m)$ - Jarvis

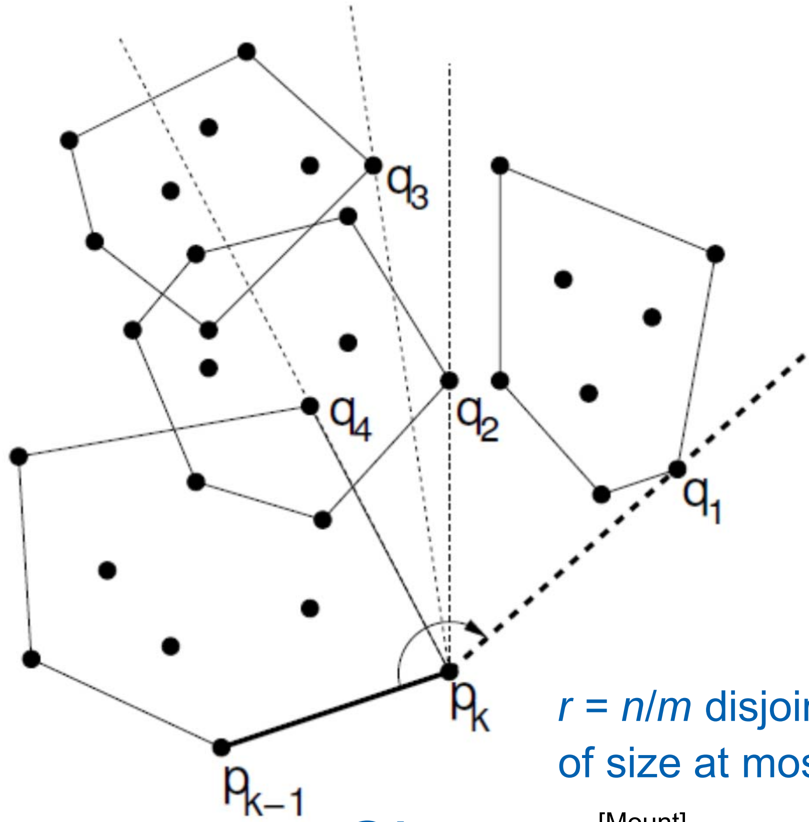
Merging of *m* parts in Chan's algorithm

- 2. Merge *r*-group CHs as "fat points"
  - Tangents to convex *m*-gon can be found in O(log *m*) by binary search



Jarvis

[Mount]



r = *n*/*m* disjoint subsets of size at most *m*

Chan

[Mount]

Chan's algorithm complexity

- *h* points on the final convex hull

  => at most *h* steps in the Jarvis march algorithm

  – each step computes *r*-tangents, O(log *m*) each

  – merging together O(*hr* log *m*)

  r-groups of size *m*, *r* = *n*/*m*

- Complete algorithm O(*n* log *h*)

  – Graham's scan on partitions O(*r* .*m* log *m*)=O(*n* log *m*)

  – Jarvis Merging: O(*hr* log *m*) = O(*h n*/*m* log *m*), …4a)

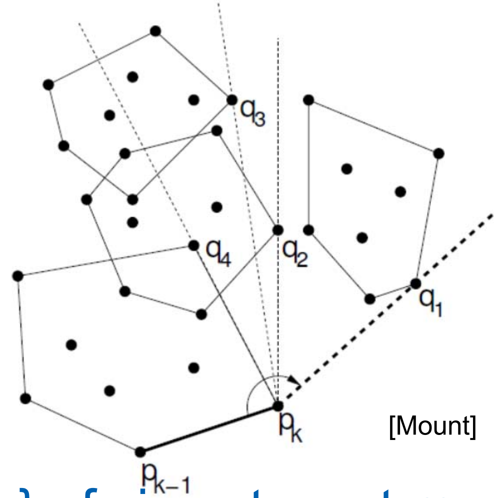
  h ¸ m ¸ h^2 = O(*n* log *m*)

  – Altogether O(*n* log *m*)

  – How to guess *m*? *Wait!*

  1) use m as an estimation of h 2) if it fails, increase m

Chan's algorithm for known *m*



PartialHull(*P*, *m*)
Input: points P
Output: group of size *m*

1. Partition *P* into $r = \lceil n/m \rceil$ disjoint subsets $\{p_1, p_2, \dots, p_r\}$ of size at most *m*
2. for *i=1 to r do*
   a) Convex hull by GrahamsScan(P_i), store vertices in ordered array
3. let p_1 = the bottom most point of P and p_0 = (—— , $p_1.y$)
4. for *k* = 1 to *m* do // compute merged hull points
   a) for *i* = 1 to *r* do // angle to all *r* subsets => points q_i
      Compute the point q_i µ *P* that maximizes the angle È p_{k-1}, p_k, q_i
   b) let p_{k+1} be the point *q* µ $\{q_1, q_2, \dots, q_r\}$ that maximizes È p_{k-1}, p_k, q
   (p_{k+1} is the new point in CH)
   c) if $p_{k+1} = p_1$ then return $\{p_1, p_2, \dots, p_k\}$
5. return "Fail, *m* was too small"

O(log *m*)

Jarvis

Chan's algorithm – estimation of *m*

ChansHull
Input: points P
Output: convex hull $p_1 \dots p_k$

1. for $t = 1, 2, \dots, [\lg \lg h]$ do {
   a) let $m = \min(2^{2^t}, n)$
   b) L = PartialHull(P, m)
   c) if L û "Fail, *m* was too small" then return L

   }

Sequence of choices of *m* are { 4, 16, 256,…, 2^{2^t} ,…, *n* } … squares

Example: for h = 23 points on convex hull of n = 57 points, the algorithm will try this sequence of choices of *m* { 4, 16, 57 }

1. 4 and 16 will fail
2. 256 will be replaced by *n=57*

Complexity of Chan's Convex Hull?

- The worst case: Compute all iterations
- t^{th} iteration takes O(n log $2^{2^\wedge t}$) = O(n 2^t)
- Algorithm stops when $2^{2^\wedge t}$ – h => $t = \lceil \lg \lg h \rceil$
- All $t = \lceil \lg \lg h \rceil$ iterations take:

Using the fact that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

$$\sum_{t=1}^{\lg \lg h} n2^t = n \sum_{t=1}^{\lg \lg h} 2^t \le n2^{1+\lg \lg h} = 2n \lg h = O(n \log h)$$

2x more work in the worst case

Conclusion in 2D

- Graham's scan: $O(n \log n)$, $O(n)$ for sorted pts
- Divide & Conquer: $O(n \log n)$
- Quick hull: $O(n \log n)$, max $O(n^2)$ ~ distrib.
- Jarvis's march: $O(hn)$, max $O(n^2)$ ~ pts on CH
- Chan's alg.: $O(n \log h)$ ~ pts on CH

asymptotically optimal

but

constants are too high to be useful

References

- [Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapter 5, http://www.cs.uu.nl/geobook/
- [Mount] David Mount, - CMSC 754: Computational Geometry, Lecture Notes for Spring 2007, University of Maryland, Lectures 3 and 4. http://www.cs.umd.edu/class/spring2007/cmsc754/lectures.shtml
- [Chan] Timothy M. Chan. Optimal output-sensitive convex hull algorithms in two and three dimensions., *Discrete and Computational Geometry*, 16, 1996, 361-368.

  http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.44.389