GEOMETRIC SEARCHING
PART 2: RANGE SEARCH

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Based on [Berg] and [Mount]

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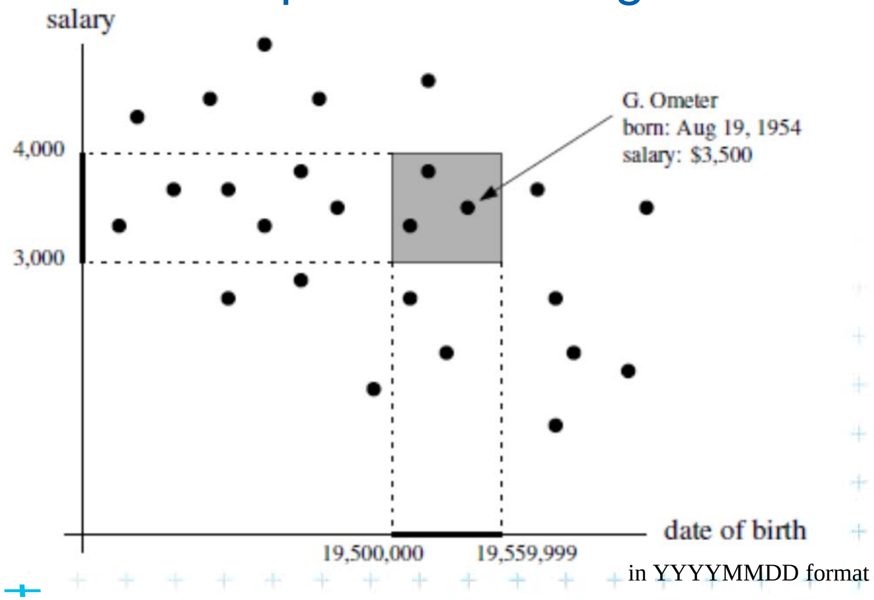
Range search

- Orthogonal range searching
- Canonical subsets
- 1D range tree
- Kd-tree
- 2-nD Range tree
  - With fractional cascading (Layered tree)

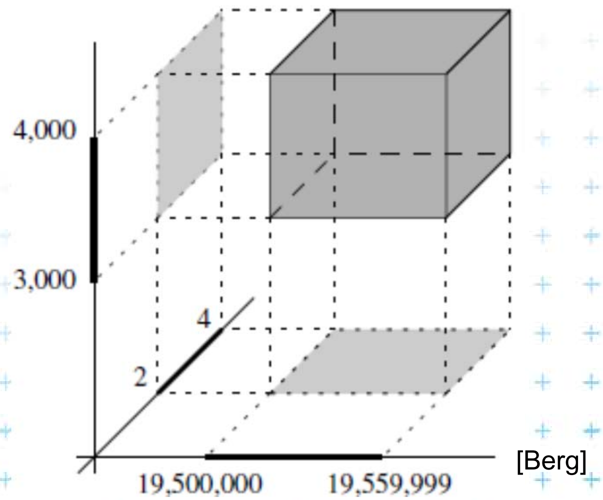
Orthogonal range searching

– Given a set of points P, find the points in the region Q
  - Search space: a set of points P (somehow represented)
  - Query: intervals Q (axis parallel rectangle)
  - Answer: points contained in Q
– Example: Databases (records->points)
  - Find the people with given range of salary, date of birth, kids, …

2D: axis parallel rectangle



3D: axis parallel box



[Berg]

Orthogonal range searching

- Query region = axis parallel rectangle
  - nDimensional search can be decomposed into set of 1D searches

Other range searching variants

- Search space: set of
  - line segments,
  - rectangles, …
- Query region: any other region
  - disc,
  - polygon,
  - halfspace, …
- Answer: subset of P laying in Q

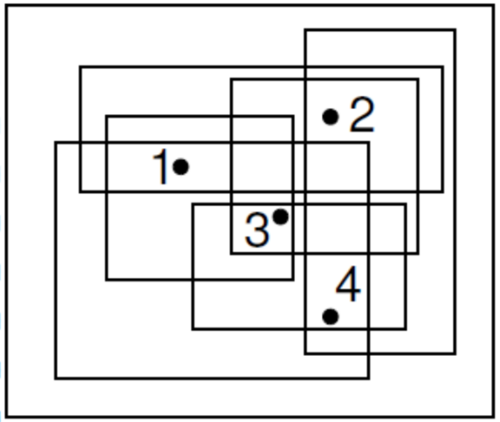
- We concentrate on points in orthogonal ranges

How to represent the search space?

- Not all possible combination can be in the output (not the whole power set)
- => Represent only the “selectable” things (a well selected subset –> one of the canonical subsets)

Subsets selectable by given range class

- The number of subsets that can be selected by simple ranges Q is limited
- It is usually much smaller than the power set of P
  - Power set of P where P = {1,2,3,4} (potenční množina) is {{ }, {1},{2},{3},{4}, {1,2},{1,3},{1,4}, {2,3},…,{2,3,4}, {1,2,3,4} } … $O(2^n)$
    i.e. of all possible subsets
  - Simple rectangular queries are limited
    - Defined by max 4 points along 4 sides => $O(n^4)$ of $O(2^n)$ power set
    - Moreover – not all sets can be formed by □ query



[Mount]

e.g. sets {1,4} and {1,2,4} cannot be formed

Canonical subsets S_i

- Search space S=(P,Q) represented as a collection of canonical subsets $\{S_1, S_2, \ldots, S_k\}$, each $S_i \subseteq S$,
  – S_i may overlap each other
  – Any set can be represented as disjoint union disjunktní sjednocení of canonical subsets S_i (elements can be multiple times)
  – Elements of disjoint union are ordered pairs (*x*, *i*) (every element *x* with index *i* of the subset S_i)
- Can be selected in many ways
    - from *n* singletons $\{p_i\}$ … O(n)
    - to power set of P … $O(2^n)$
  – Good DS balances between total number of canonical subsets and number of CS needed to answer the query

1D range queries (interval queries)

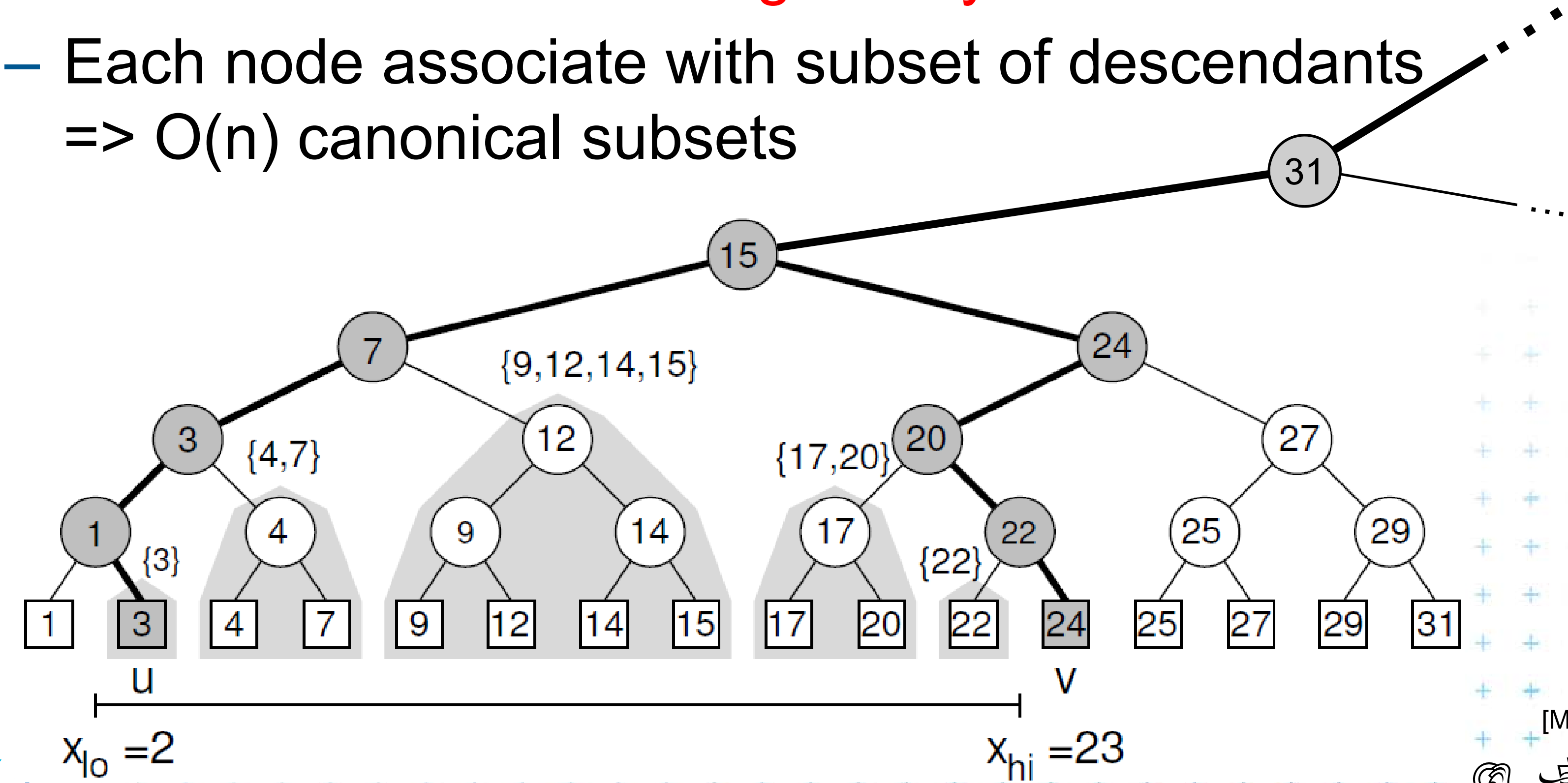
- Search the interval $[x_{lo}, x_{hi}]$ in
- Points $P= \{p_1, p_2, \ldots, p_n\}$ on the line

  a) Binary search in an array
  - Simple, but
  - not generalize to any higher dimensions
    (values in inner nodes are not reachable in particular level below,
    to get them, we must traverse back to root)

  b) Balanced binary search tree
  - 1D range tree
  - maintains canonical subsets
  - generalize to higher dimensions

1D range tree definition

- Balanced binary search tree
  - leaves – sorted points
  - inner node label – the largest key in its left child
  - Each node associate with subset of descendants => O(n) canonical subsets



Canonical subsets and <2,23> search

- Canonical subsets for this subtree are — #

{ {1}, {3}, …, {31}, — 16

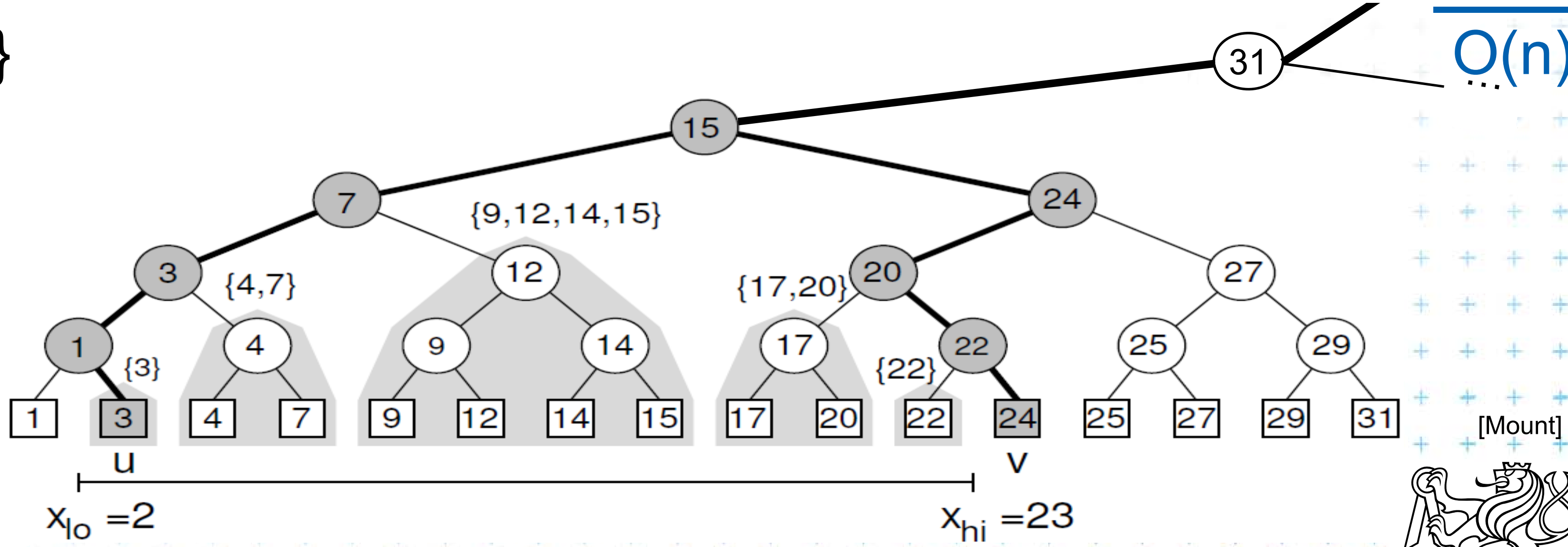
{1, 3}, {4, 7}, …, {29, 31} — 8

{1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31} — 4

{1, 3, 4, 7, 9, 12, 14, 15}, {17, 20, 22, 24, 25, 27, 29, 31} — 2

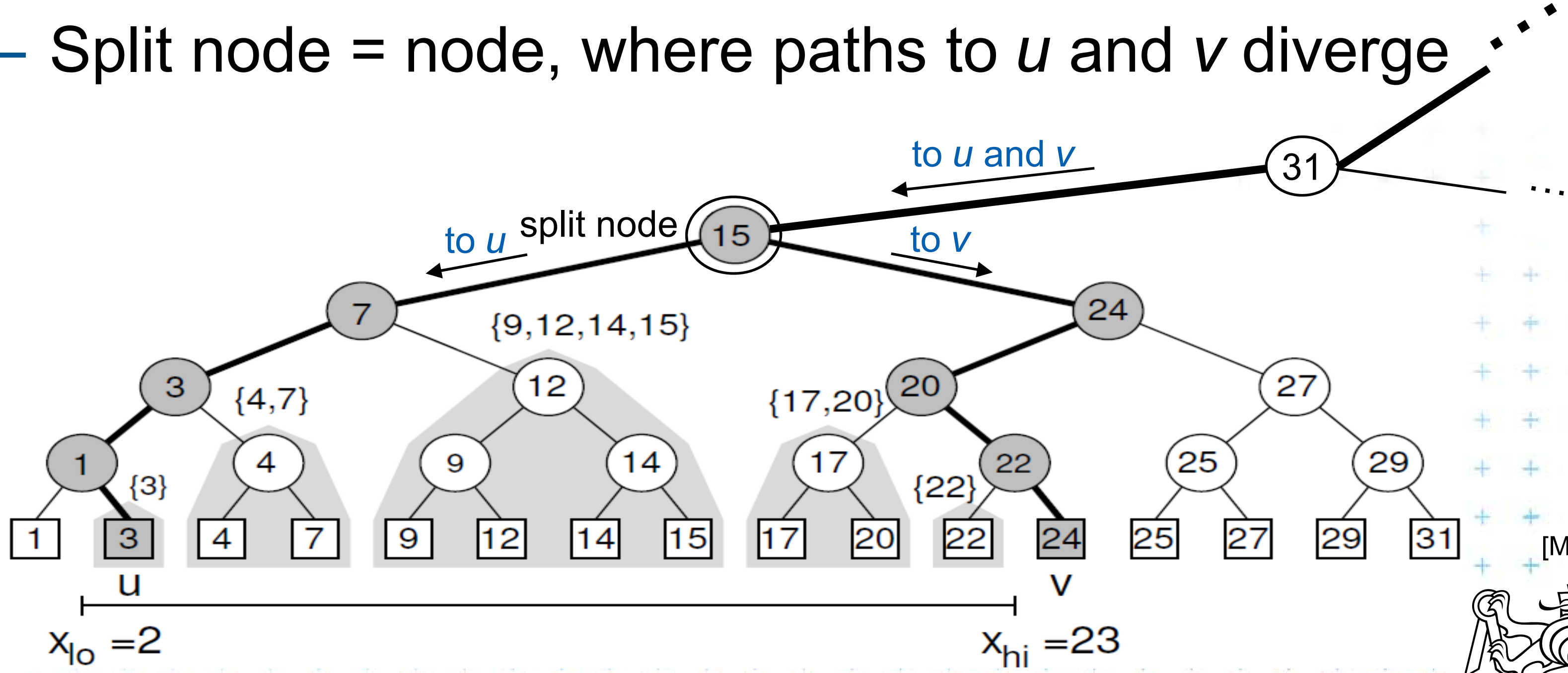
{1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31} — 1

} — O(n)



1D range tree search interval <2,23>

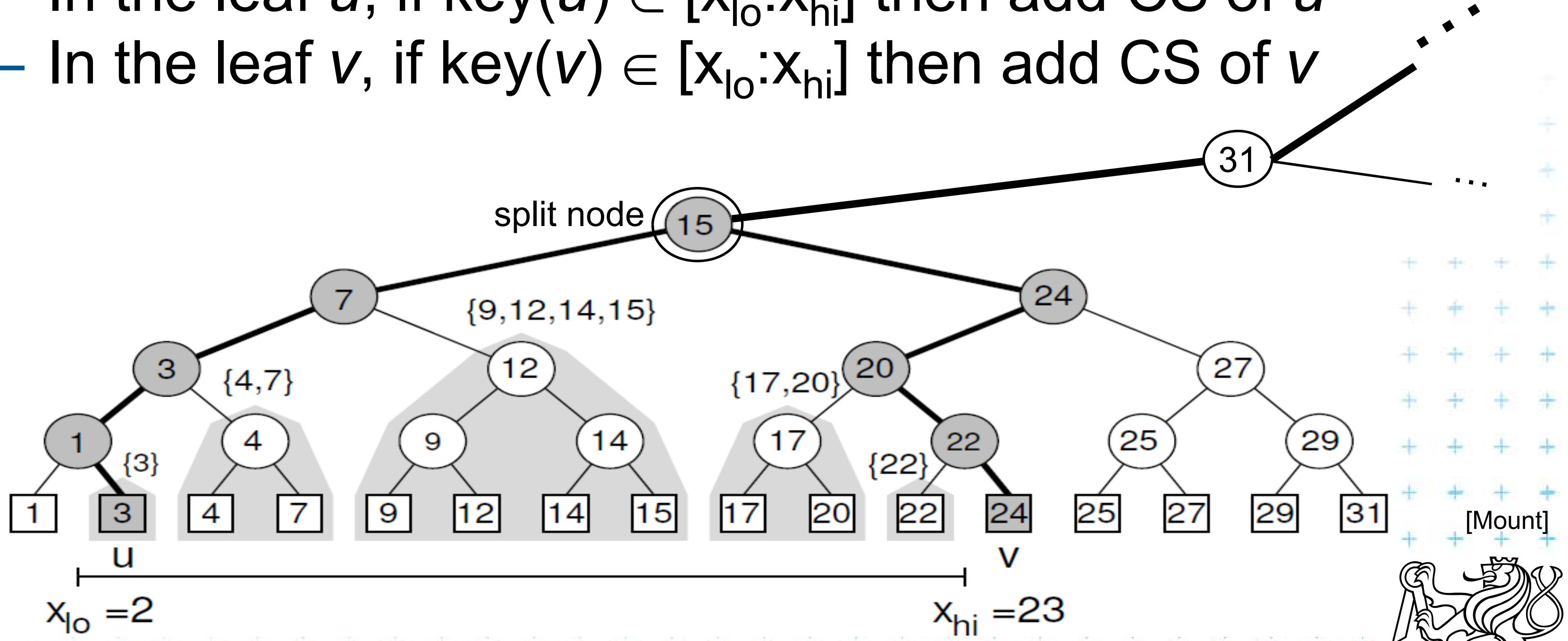
- Canonical subsets for any range found in O(log *n*)
  - Search x_{lo}: Find leftmost leaf *u* with key(*u*) ≥ x_{lo} 2 -> 3
  - Search x_{hi}: Find leftmost leaf *v* with key(*v*) ≥ x_{hi} 23 -> 24
  - Points between *u* and *v* lie within the range => report canon. subsets of maximal subtrees between *u* and *v*
  - Split node = node, where paths to *u* and *v* diverge



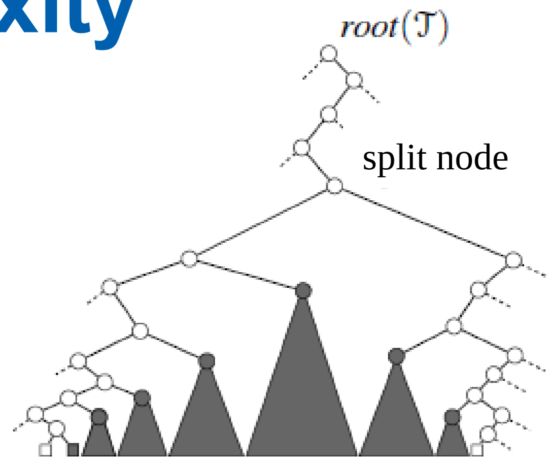
[Mount]

1D range tree search

- Reporting the subtrees (below the split node)
  - On the path to *u* whenever the path goes left, add the canonical subset associated to right child
  - On the path to *v* whenever the path goes right, add the canonical subset associated to left child
  - In the leaf *u*, if key(*u*) $\in [x_{lo}:x_{hi}]$ then add CS of *u*
  - In the leaf *v*, if key(*v*) $\in [x_{lo}:x_{hi}]$ then add CS of *v*



1D range tree search complexity



[Berg]

- Path lengths O(log n)

  => O(log n) canonical subsets
  (subtrees)

- Range counting queries
  – Return just the number of points in given range
  – Sum the total numbers of leaves stored in maximal subtree roots … O(log *n*) time

- Range reporting queries
  – Return all *k* points in given range
  – Traverse the canonical subtrees … O(log *n* + *k*) time

- O(*n*) storage, O(*n* log *n*) preprocessing (sort P)

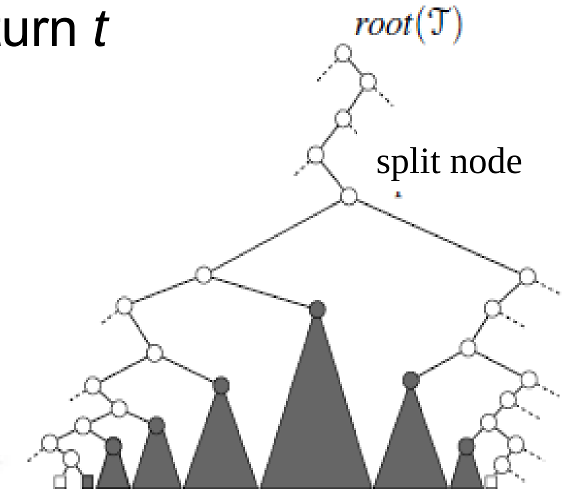
Find split node

FindSplitNode(*T*, [x:x’])

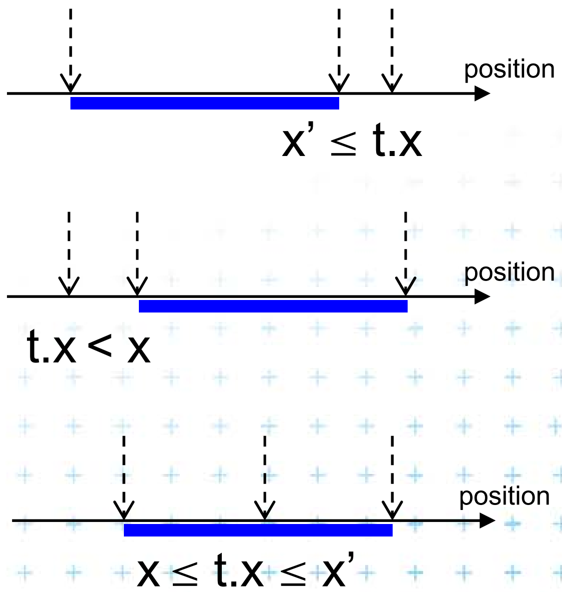
Input: Tree *T* and Query range [x:x’], x ≤ x’

Output: The node, where the paths to x and x’ split or the leaf, where both paths end

1. *t = root(T)*
2. while(*t* is not a leaf **and** x’ ≤ t.x **or** t.x < x) // out of the range [x:x’]
3. if(*x’* ≤ t.x) *t = t.left*
4. else *t = t.right*
5. return *t*



[Berg]



1D range search

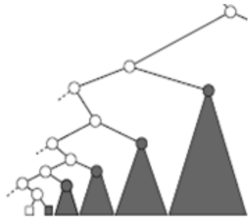
(2D on slide 30)

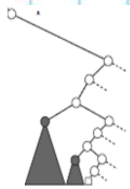
1dRangeQuery(*t*, [x:x’])

Input: 1d range tree *t* and Query range

Output: All points in *t* liying in the range

```
t_split = FindSplitNode( t, x, x’ )          // find interval point t ∈ [x:x’]
if( t_split is leaf )
    check if the point in t_split must be reported
else // follow the path to x, reporting points in subtrees right of the path
    t = t_split.left
    while( t is not a leaf )
        if( x ≤ t.x)
            ReportSubtree( t( t.right ) )    // any kind of tree traversal
            t = t.left
        else t = t.right
    check if the point in leaf t must be reported
    // Symmetrically follow the path to x’ reporting points left of the path
    t = t_split.right …
```





Multidimensional range searching

- Equal principle – find the largest subtrees contained within the range
- Separate one *n*-dimensional search into *n* 1-dimensional searches
- Different tree organization
  - Kd tree
  - Orthogonal (Multilevel) search tree – range tree

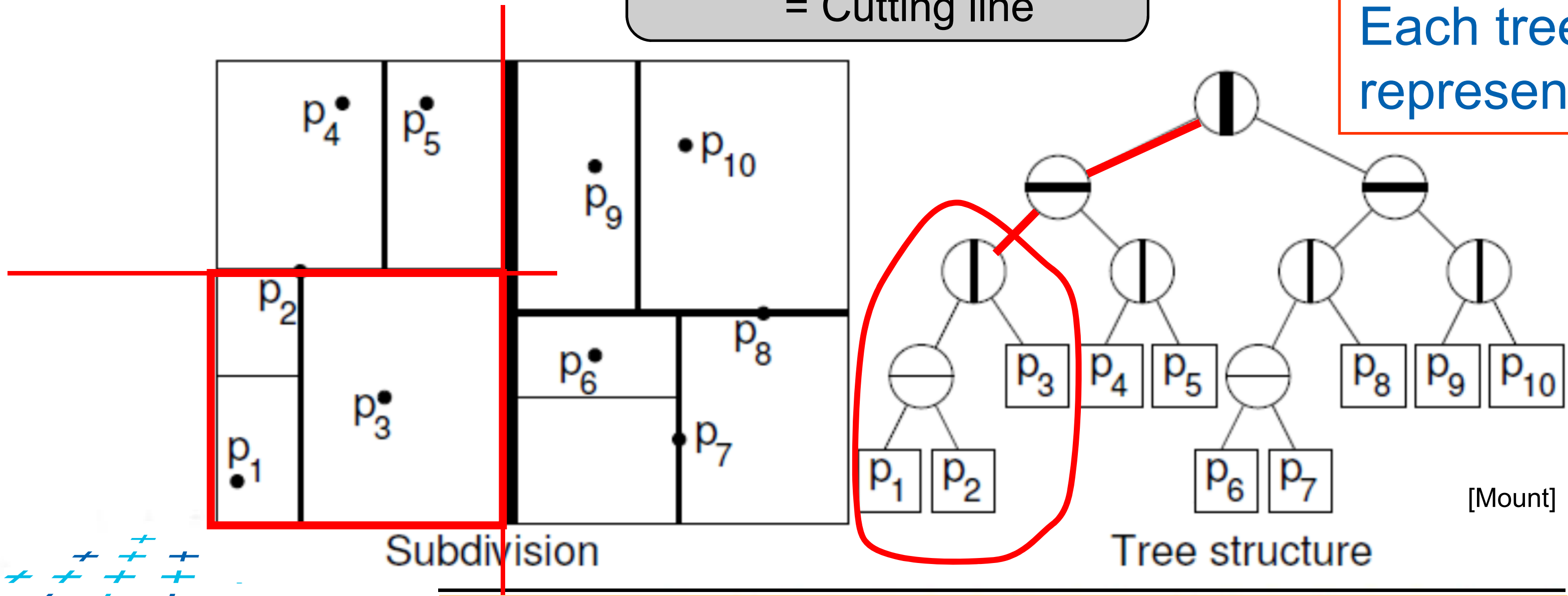
Kd-tree

- Easy to implement
- Good for different searching problems (counting queries, nearest neighbor,…)
- Designed by Jon Bentley as k-dimensional tree (2-dimensional kd-tree was a 2-d tree, …)
- Not the asymptotically best for orthogonal range search (=> range tree is better)
- Types of queries
  - Reporting – points in range
  - Counting – number of points in range

Kd-tree principle

- Subdivide space according to different dimension (x-coord, then y-coord, …)
- This subdivides space into rectangular cells
  => hierarchical decomposition of space
- In node t store: cutDim, cutVal, (size (for counting queries))
  = Cutting line

Each tree node represents a region



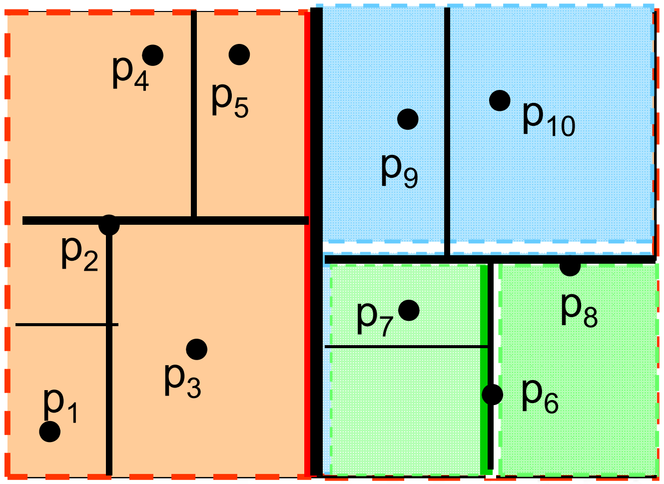
Where is a mistake in the figure?

Kd-tree principle

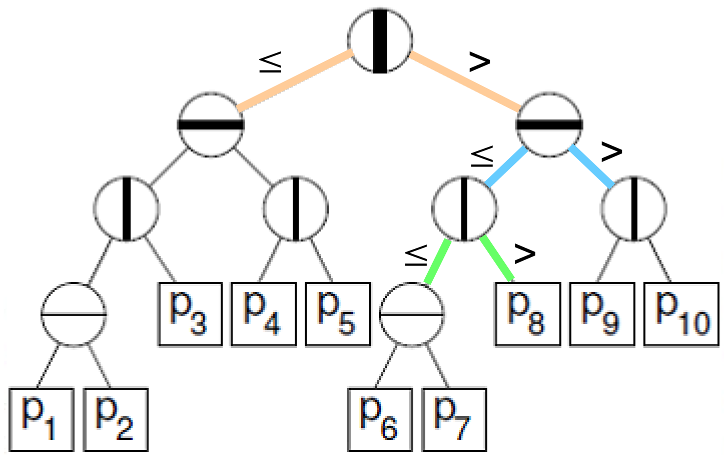
- Which dimension to cut? (cutDim)
  - Cycle through dimensions (round robin)
    - Save storage – cutDim is implicit ~ depth in the tree
    - May produce elongated cells (if uneven data distribution)
  - Greatest spread (the largest difference of coordinates)
    - Adaptive
    - Called “Optimal kd-tree”
- Where to cut? (cutVal)
  - Median, or midpoint between upper and lower median -> $O(n)$
  - Presort coords of points in each dimension (x-, y-,...) for $O(1)$ median – resp. $O(d)$ for all d dimensions

Kd-tree principle

- What about points on the cell boundary?
  - Boundary belongs to the left child
  - Left: $p_{cutDim} \leq cutVal$
  - Right: $p_{cutDim} > cutVal$



Subdivision



Tree structure

[Mount]

Kd-tree construction in 2-dimensions

BuildKdTree(*P, depth*)

Input: A set of points *P* and current *depth.*

Output: The root of a kD tree storing P.

Split according to (*depth%max_dim)* dimension

1. **If (***P* contains only one point) [or small set of (10 to 20) points]
2. **then return** a leaf storing this point
3. **else if (***depth* is even)
4. **then** split *P* with a vertical line *l* through median *x* into two subsets P_1 and P_2 (left and right from median)
5. **else** split *P* with a horiz. line *l* through median y into two subsets P_1 and P_2 (below and above the median)
6. t_{left} = BuildKdTree(P_1, *depth*+1)
7. t_{right} = BuildKdTree(P_2, *depth*+1)
8. create node *t* storing *l*, t_{left} and t_{right} children // l = cutDim, cutVal
9. **return** *t*

If median found in O(1) and array split in O(n)

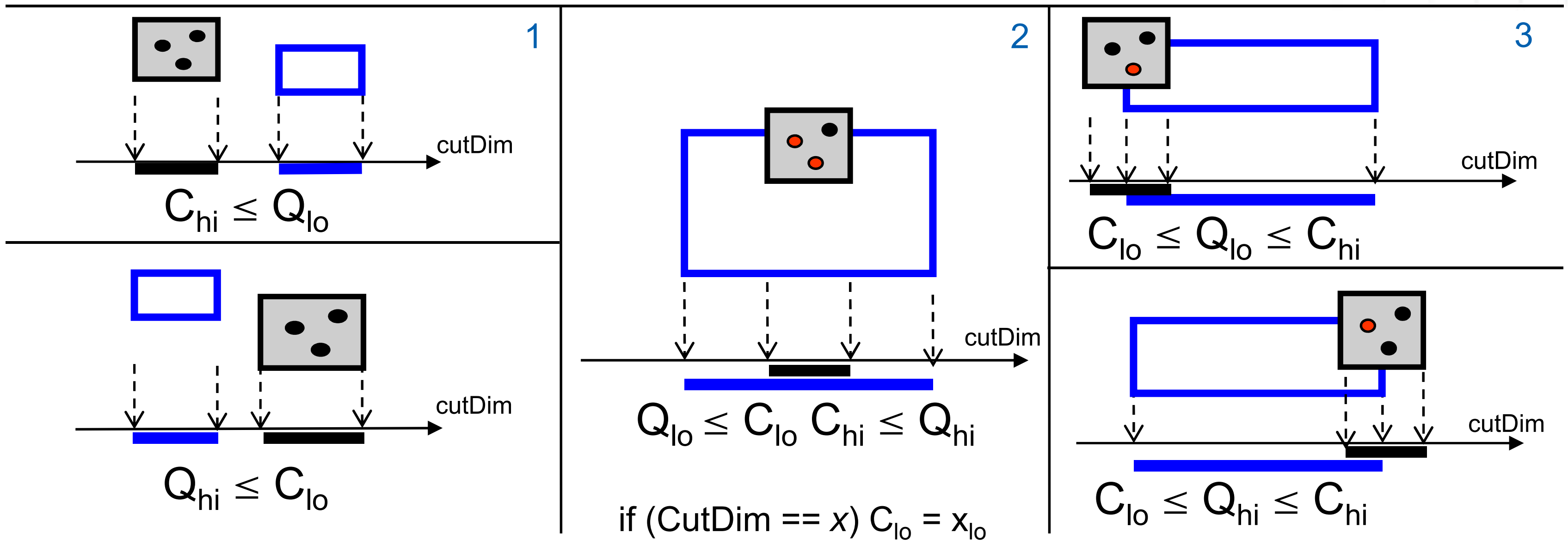
T(n) = 2 T(n/2) + n => O(*n* log *n*) construction

Kd-tree variants

Test interval-interval

a) Compare rectang. Array Q with rectangular cells C

- Rectangle C:[x_{lo}, x_{hi}, y_{lo}, y_{hi}] – computed on the fly
- Test of kD node cell C against query Q (in one cutDim)
  1. if cell is disjoint with Q … $C \cap Q = \emptyset$ … stop
  2. If cell C completely inside Q … $C \subseteq Q$ … stop and report cell points
  3. else cell C overlaps Q … recurse on both children
- Recursion stops on the largest subtree (in/out)



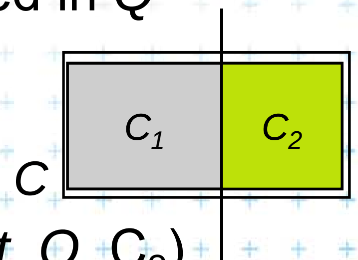
Kd-tree rangeCount (with rectangular cells)

int rangeCount(*t*, *Q*, *C*)

Input: The root *t* of kD tree, query range *Q and t's* cell C.

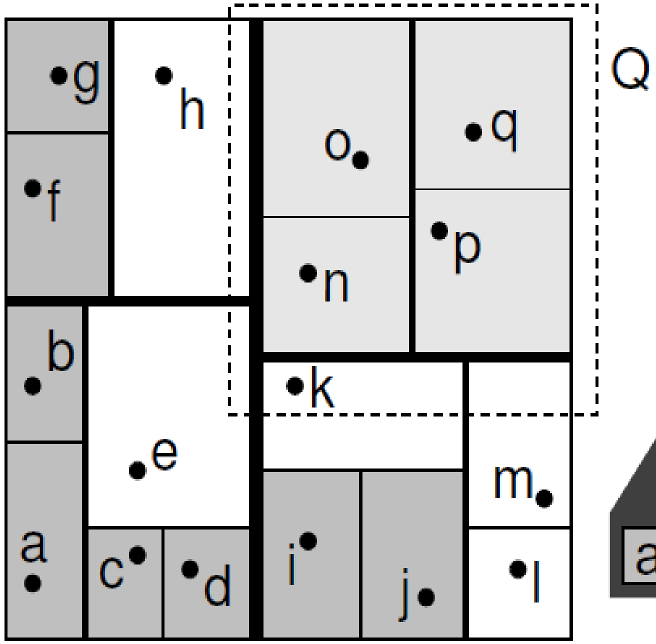
Output: Number of points at leaves below *t* that lie in the range.

```
if (t is a leaf)
    if (t.point lies in Q) return 1     // or loop this test for all points in leaf
    else return 0                       // visited, not counted
else // (t is not a leaf)
    if (C ∩ Q = ∅) return 0             … disjoint
    else if (C ⊆ Q) return t.size       … C is fully contained in Q
    else
        split C along t's cutting value and dimension,
        creating two rectangles C1 and C2.
        return rangeCount(t.left, Q, C1) + rangeCount(t.right, Q, C2)
```



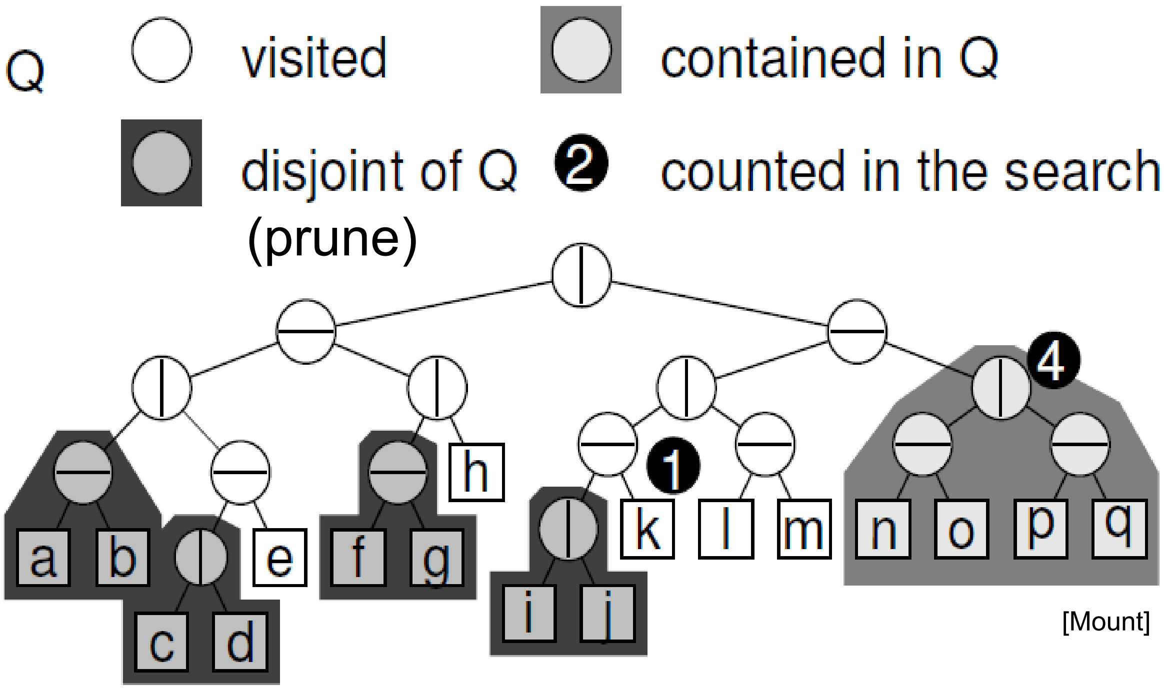
// (pictograms refer to the next slide)

Kd-tree rangeCount example



kd–tree subdivision

Tree node (rectangular region)



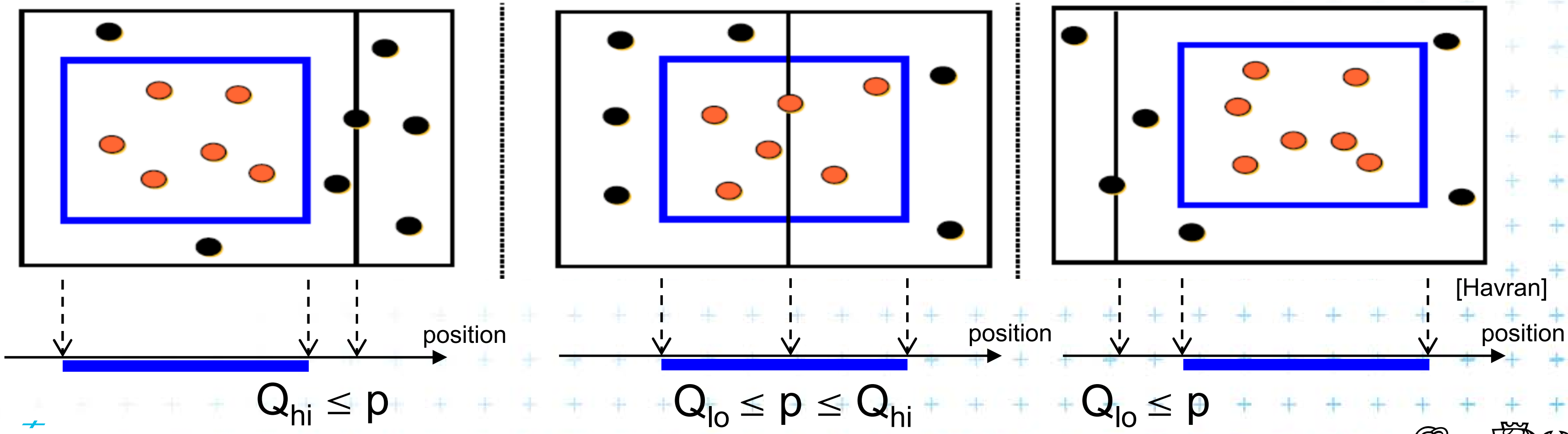
Nodes visited in range search

Kd-tree variants

Test point-interval

b) Compare Q with cutting lines

- Line = Splitting value *p* in one of the dimensions
- Test of single position given by dimension against Q
  1. Line is right from Q … recurse on left child only (prune right child)
  2. Line is left from Q … recurse on right child only (prune left ch.)
  3. Line intersects Q … recurse on both children
- Recursion stops in leaves - traverses the whole tree



Kd-tree rangeSearch (with cutting lines)

int rangeSearch(*t*, *Q*)

Input: The root *t* of (a subtree of a) kD tree and query range *Q.*

Output: Points at leaves below *t* that lie in the range.

```
if (t is a leaf)
    if (t.point lies in Q) report t.point   // or loop test for all points in leaf
    else return
else  (t is not a leaf)
    if (Q_hi ≤ t.cutVal) rangeSearch(t.left, Q)   // go left only
    if (Q_lo > t.cutVal) rangeSearch(t.right, Q) // go right only
    else
        rangeSearch(t.left, Q)                    // go to both
        rangeSearch(t.right, Q)
```

Kd-tree - summary

- Orthogonal range queries in the plane (in balanced 2d-tree)
  - Counting queries O(√n) time
  - Reporting queries O(√n + k) time, where k = No. of reported points
  - Space O(n)
  - Preprocessing: Construction O(n log n) time (Proof: if presorted points to arrays in dimensions. Median in O(1) and split in O(n) per level, log n levels of the tree)
- For d≥2:
  - Construction O(d n log n), space O(dn), Search O(d n^(1-1/d) + k)

Orthogonal range tree (RT)

- DS highly tuned for orthogonal range queries
- Query times in plane

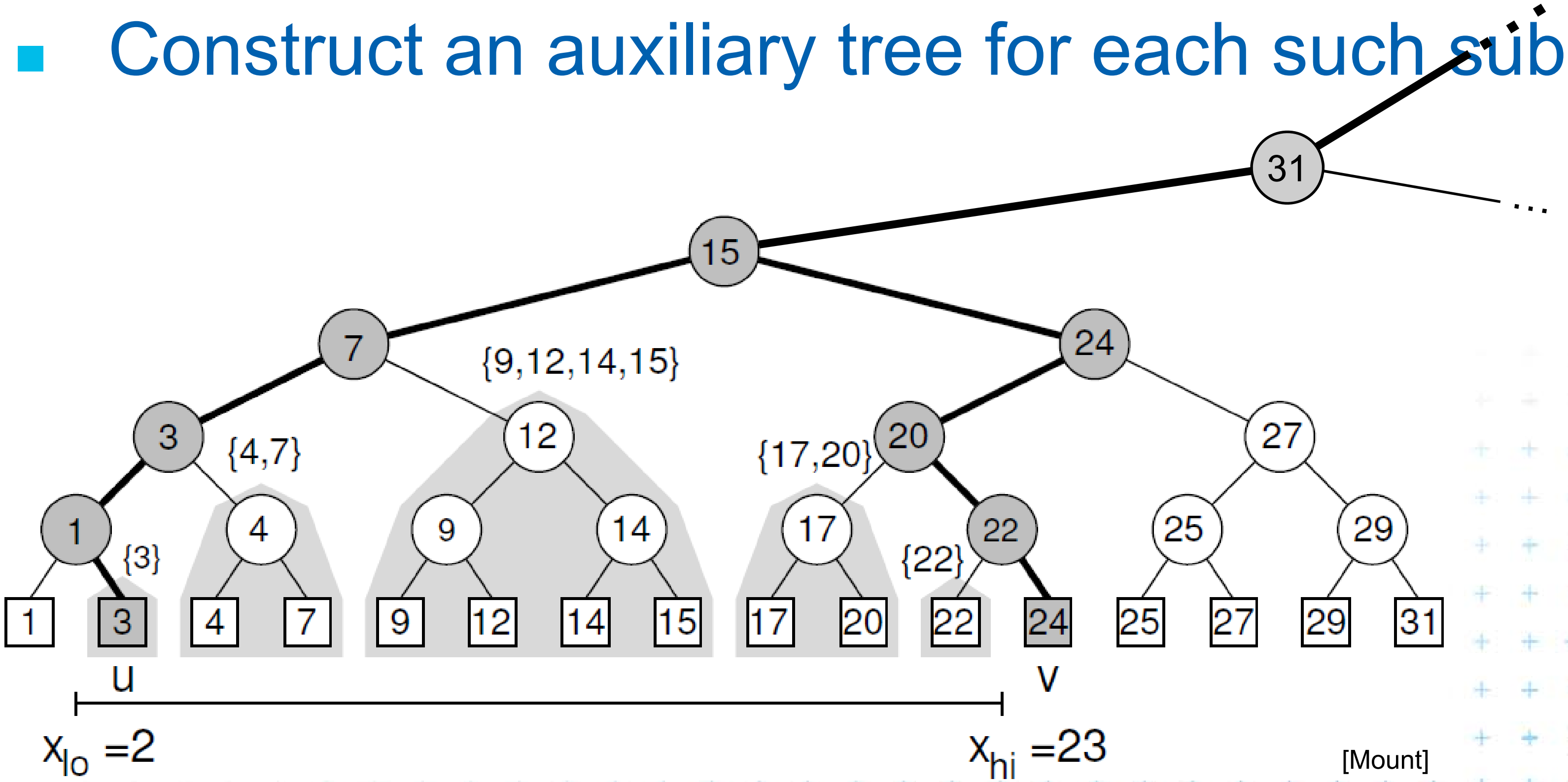
2d tree versus	range tree
O($\sqrt{n}$ + k) time of Kd	O(log n) time query
O(n) space of Kd	O(n log n) space

n = number of points

k = number of reported points

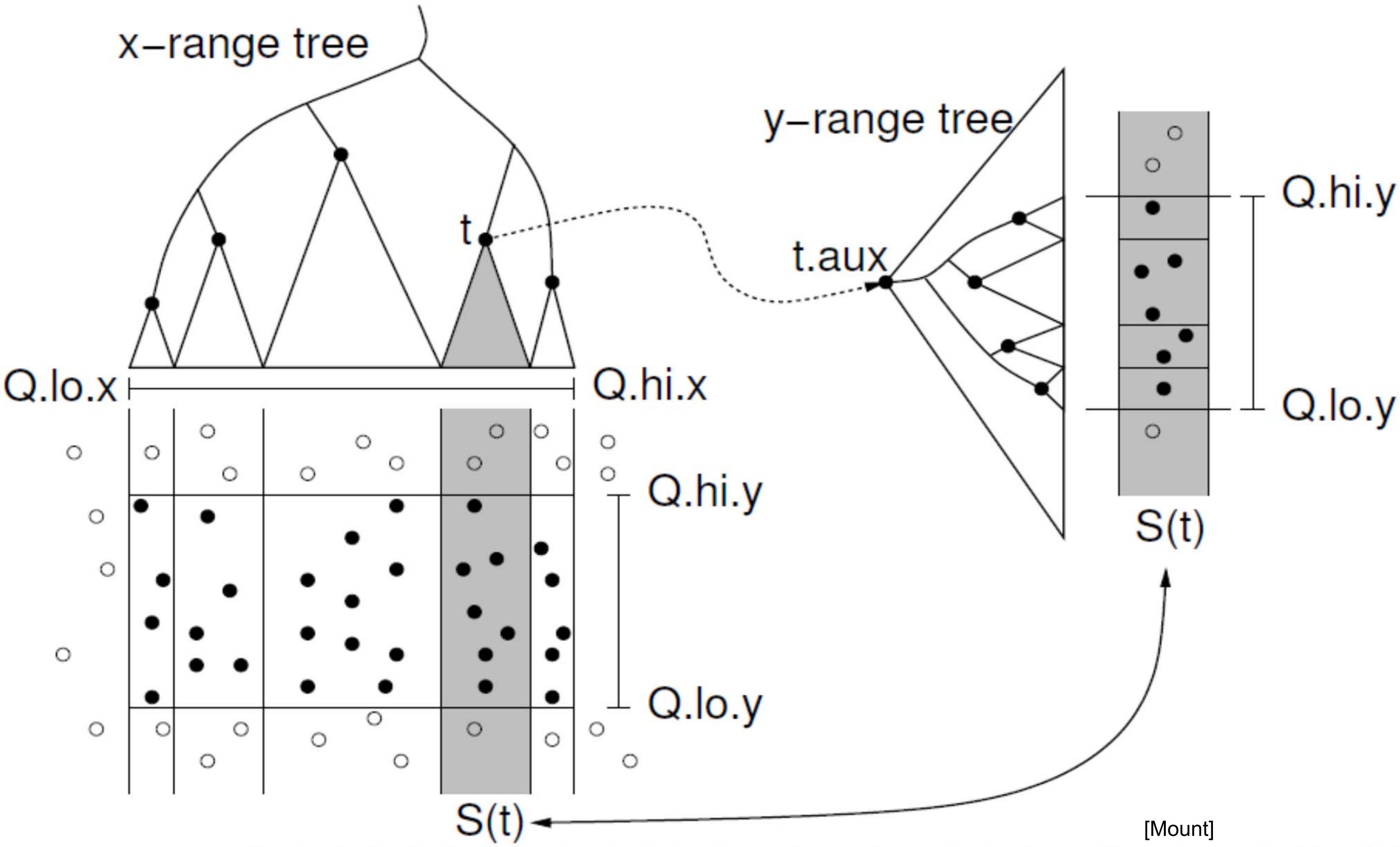
From 1D to 2D range tree

- Search points from [$Q.x_{lo}$, $Q.x_{hi}$] [$Q.y_{lo}$, $Q.y_{hi}$]
- 1d range tree: log n canonical subsets based on x
- Construct an auxiliary tree for each such subset y



[Mount]

2D range tree



[Mount]

2D range search

2dRangeQuery(*t*, [x:x'] × [y:y'])

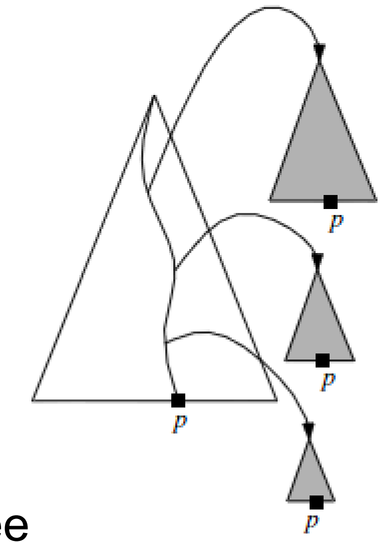
Input: 2d range tree *t* and Query range

Output: All points in *t* laying in the range

```
t_split = FindSplitNode( t, x, x' )
if( t_split is leaf )
    check if the point in t_split must be reported      … t.x ∈ [x:x'], t.y ∈ [y:y']
else // follow the path to x, calling 1dRangeQuery on y
    t = t_split.left    // path to the left
    while( t is not a leaf )
        if( x ≤ t.x)
            1dRangeQuerry( t_assoc( t.right ), [y:y'] ) // check associated subtree
            t = t.left
        else t = t.right
  check if the point in leaf t must be reported     … t.x ≤ x', t.y ∈ [y:y']
  Similarly for the path to x'      … // path to the right
```

2D range tree

- Search $O(\log^2 n + k)$ – log n in x-, log n in y
- Space O(n log n)
  - O(n) the tree for x-coords
  - O(n log n) trees for y-coords
    - Point p is stored in all canonical subsets along the path from root to leaf with p,
    - once for x-tree level
    - each canonical subsets is stored in one auxiliary tree
    - log n levels of x-tree => O(n log n) space for y-trees



[Berg]

- Construction - O(n log n)
  - Sort points (by *x* and by *y*). Bottom up construction

Canonical subsets

- Canonical subsets for this subtree are #

  { {1}, {3}, …, {31}, 16

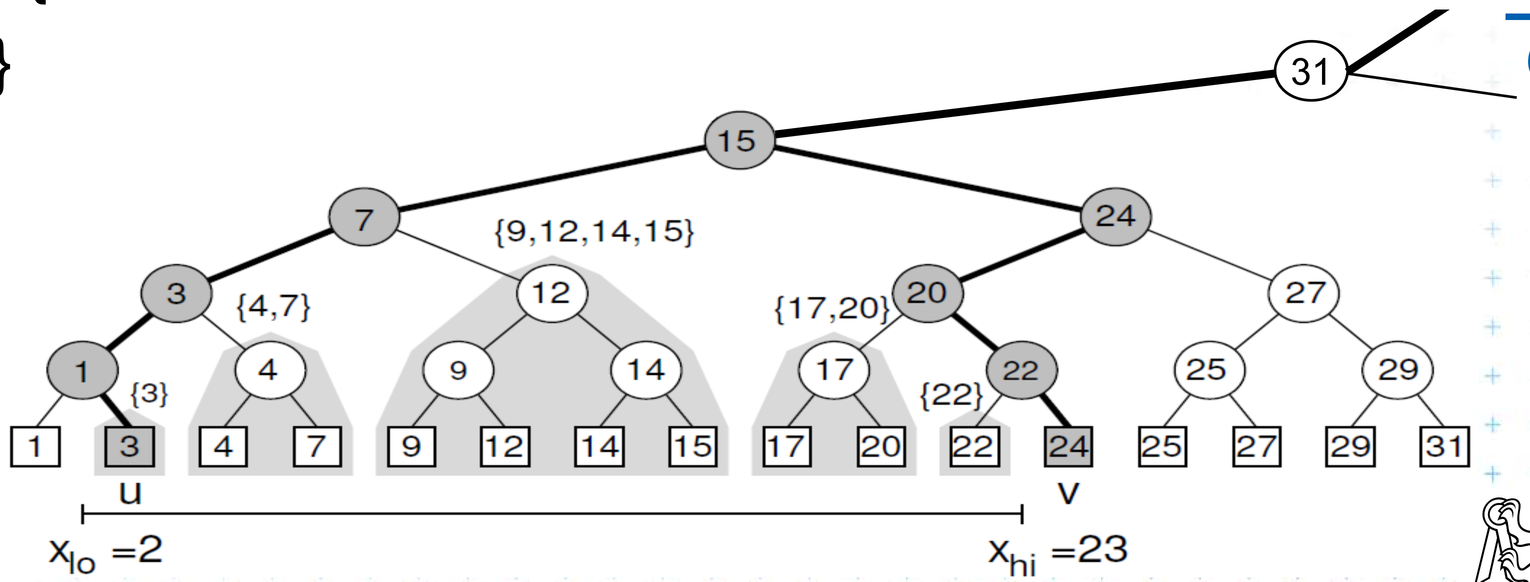
  {1, 3}, {4, 7}, …, {29, 31} 8

  {1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31} 4

  {1, 3, 4, 7, 9, 12, 14, 15}, {17, 20, 22, 24, 25, 27, 29, 31} 2

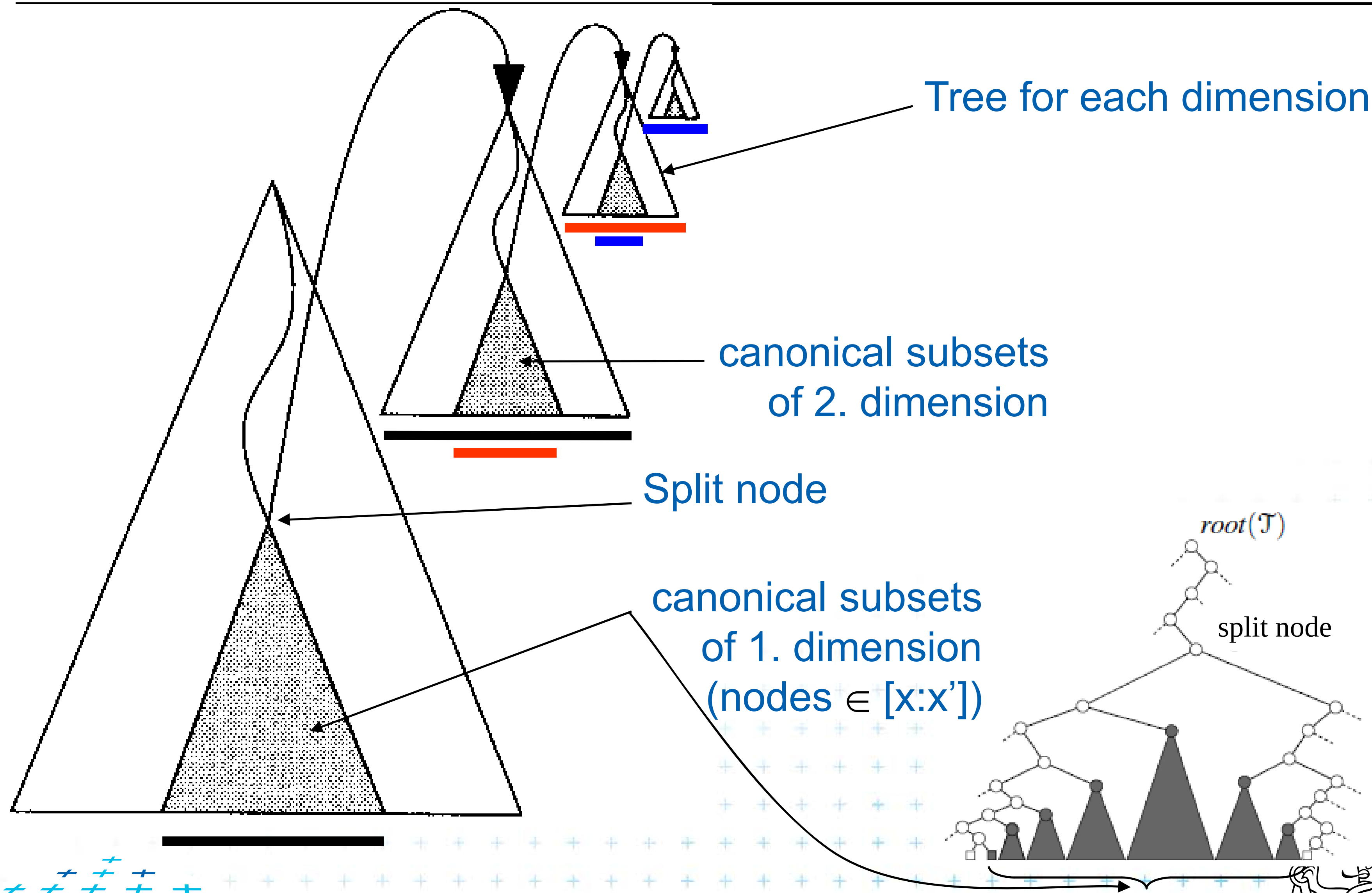
  {1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31} 1

  } O(n)



[Mount]

nD range tree (multilevel search tree)

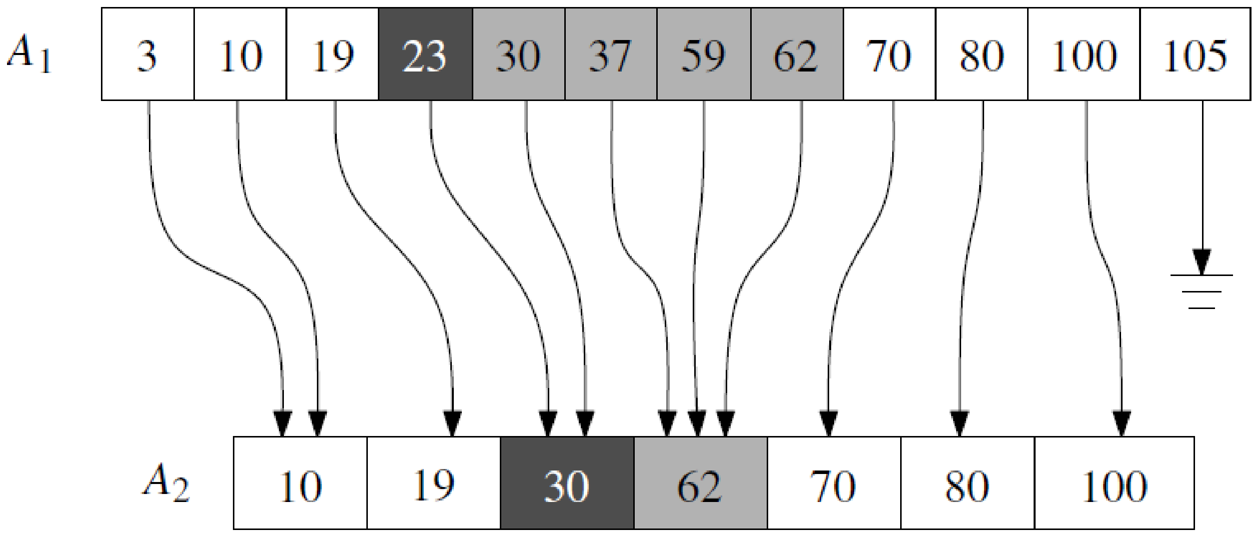


Fractional cascading - principle

- Two sets S_1, S_2 stored in sorted arrays A_1, A_2
- Report objects in both whose keys in [y:y']
- Naïve approach
  - $O(\log n_1 + k_1)$ – search in A_1 + report k_1 elements
  - $O(\log n_2 + k_2)$ – search in A_2 + report k_2 elements
- Fractional cascading – adds pointers from A_1 to A_2
  - $O(\log n_1 + k_1 + 1 + k_2)$ – search in A_1 + report k_1 elements
  - $O(1 + k_2)$ – jump to A_2 + report k_2 elements
  - Saves the $O(\log n_2)$ – search

Fractional cascading – principle for arrays

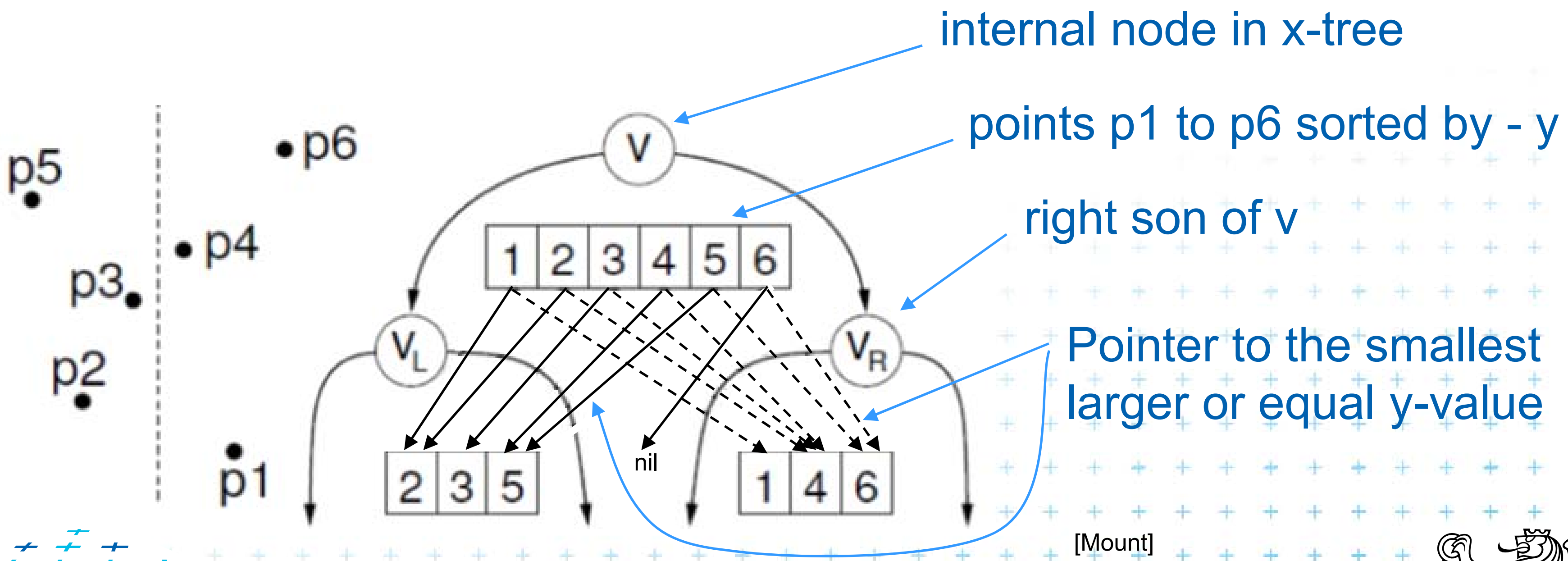
- Add pointers from A_1 to A_2
  - From element in A_1 with a key y_i point to the element in A_2 with the smallest key larger or equal to y_i
- Example query with the range [20 : 65]



[Berg]

Fractional cascading in the 2D range tree

- How to save one log n during last dim. search?
  - Store canonical subsets in arrays sorted by y
  - Pointers to subsets for both child nodes v_L and v_R
  - O(1) search in lower levels => in two dimensional search O($\log^2 n$) time -> O(2 log n)



[Mount]

Orthogonal range tree - summary

- Orthogonal range queries in plane
  - Counting queries O($\log^2 n$) time,
    or with fractional cascading O($\log n$) time
  - Reporting queries plus O(k) time, for k reported points
  - Space O($n \log n$)
  - Construction O($n \log n$)
- Orthogonal range queries in d-dimensions, d≥2
  - Counting queries O($\log^d n$) time,
    or with fractional cascading O($\log^{(d-1)} n$) time
  - Reporting queries plus O(k) time, for k reported points
  - Space O($n \log^{(d-1)} n$)
  - Construction O($n \log^{(d-1)} n$) time

References

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