AE4M33RZN, Fuzzy logic: Tutorial examples

Radomír Černoch

radomir.cernoch@fel.cvut.cz

Faculty of Electrical Engineering, CTU in Prague

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Task 2

Assignment

On the universe $\Delta = \{a, b, c, d\}$ there is a fuzzy set

$$\mu_A = \Big\{(a; 0.3), (b; 1), (c; 0.5)\Big\}.$$

Find its horizontal representation.

$$\mathrm{R}_A(\alpha) = \begin{cases} X & \alpha = 0, \\ \{a, b, c\} & \alpha \in (0; 0.3\rangle, \\ \{b, c\} & \alpha \in (0.3; 0.5\rangle, \\ \{b\} & \alpha \in (0.5; 1\rangle. \end{cases}$$

Task 3

Assignment

The fuzzy set A has a horizontal representation

$$\mathrm{R}_A(\alpha) = \begin{cases} \{a, b, c, d\} & \alpha \in \langle 0, 1/3\rangle \\ \{a, d\} & \alpha \in (1/3, 1/2\rangle \\ \{d\} & \alpha \in (1/2, 2/3\rangle \\ \emptyset & \alpha \in (2/3, 1\rangle \end{cases}$$

Find the vertical representation.

Solution

$$\mu_A(a) = \sup\left\{\alpha \in \langle 0, 1\rangle : \; a \in \mathrm{R}_A(\alpha)\right\} = \sup\langle 0, 1/2\rangle = 1/2$$

$$\mu_A(b) = 1/3, \quad \mu_A(c) = 1/3, \quad \mu_A(d) = 2/3, \quad \text{therefore}$$

$$\mu_A = \left\{(a, 1/2), (b, 1/3), (c, 1/3), (d, 2/3)\right\}$$

Task 5

Assignment

On the universe $\Delta = \mathbb{R}$ there is a fuzzy set A:

$$\mu_A(x) = \begin{cases} x & x \in \langle 0; 1 \rangle \\ 2 - x & x \in (1; 1.5\rangle \\ 0 & \text{otherwise} \end{cases}$$

Find its horizontal represenation.

Solution

$$\mathtt{R}_A(\alpha) = \begin{cases} \mathbb{R} & \alpha = 0 \\ \langle \alpha; 1.5 \rangle & \alpha \in (0; 0.5\rangle \\ \langle \alpha; 2 - \alpha \rangle & \alpha \in (0.5; 1\rangle \end{cases}$$

Task 7

Assignment

The fuzzy set A has a horizontal representation

$$\mathrm{R}_A(\alpha) = \begin{cases} \mathbb{R} & \alpha = 0 \\ \langle \alpha^2, 1) & \text{otherwise.} \end{cases}$$

Find the vertical representation.

$$\mu_A(x) = \begin{cases} \sqrt{x} & x \in (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Task 8

Assignment

Decide if the following function is a fuzzy conjunction.

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \alpha & \beta = 1, \\ \beta & \alpha = 1, \\ \alpha\beta & \alpha\beta \geqslant 1/10,\ \max(\alpha, \beta) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution

The first two possibilities ensure the boundary condition. Comutativity and monotonicity are trivially satisfied. If one of the arguments is 1, the associativity as well. For $\alpha, \beta, \gamma < 1$ the associativity follows from

$$\alpha \underset{\circ}{\wedge} \left(\beta \underset{\circ}{\wedge} \gamma\right) = \begin{cases} \alpha\beta\gamma & \alpha\beta\gamma \geqslant 1/10. \\ 0 & \text{otherwise,} \end{cases}$$

We get the same for $\left(\alpha \underset{\circ}{\wedge} \beta\right) \underset{\circ}{\wedge} \gamma$.

It is always a fuzzy conjunction (interpretable as an algebraic conjunction, in which we ignore small values, e.g. for filtering small values).

Task 10

Assignment

Prove that the Sugeno class of negations are all fuzzy negations

$$\underset{s_\lambda}{\neg}\alpha = \frac{1-\alpha}{1+\lambda\alpha} \text{ for } \lambda \in (-1, \infty,) .$$

Solution

When canceling out terms in a fraction, we use $1 + \lambda\alpha > 0$.

- **Involutivity:** Assuming $\lambda > -1$:

$$\neg s_\lambda \neg s_\lambda \alpha = \frac{1 - \frac{1-\alpha}{1+\lambda\alpha}}{1 + \lambda \frac{1-\alpha}{1+\lambda\alpha}} = \frac{1 + \lambda\alpha - 1 + \alpha}{1 + \lambda\alpha + \lambda - \lambda\alpha} = \frac{\alpha(\lambda + 1)}{\lambda + 1} = \alpha$$

- **Non-increasing:** If $\alpha \leqslant \beta$, then

$$\frac{1 - \alpha}{1 + \lambda\alpha} \geqslant \frac{1 - \beta}{1 + \lambda\beta}$$

$$(1 + \lambda\beta)(1 - \alpha) \geqslant (1 + \lambda\alpha)(1 - \beta)$$

$$\beta \geqslant \alpha$$

Task 12

Assignment

Decide if the following function is a fuzzy conjunction.

$$\alpha \diamond \beta = \begin{cases} \alpha\beta & \alpha\beta \geqslant 0{,}01 \text{ or } \max(\alpha, \beta) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Task 13

Assignment

Decide if the following function is a fuzzy conjunction.

$\underset{\circ}{\wedge} : \langle 0, 1\rangle^2 \longrightarrow \langle 0, 1\rangle$

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \min(\alpha, \beta) & \alpha + \beta \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Task 14

Assignment

Decide, if for all $\alpha, \beta \in [0, 1]$ holds $(\alpha \overset{\circ}{\vee} \beta) \underset{L}{\wedge} (\alpha \overset{\circ}{\vee} \underset{S}{\neg} \beta) = \alpha$, where the disjunction $\overset{\circ}{\vee}$ is

1. standard, $\overset{S}{\vee}$,
2. algebraic, $\overset{A}{\vee}$.
3. Łukasiewicz , $\overset{L}{\vee}$,

Solution

1. No. Counterexample: $\alpha = 0.5, \beta = 0.1$.
2. Yes: $\max(0, (\alpha + \beta - \alpha\beta) + (\alpha + (1 - \beta) - \alpha(1 - \beta)) - 1) =$

$$= \max(0, \alpha) = \alpha$$

3. No. Counterexample: $\alpha = 0.9, \beta = 0.8$.

Task 16

Assignment

Decide if the function $\underset{\circ}{\wedge} : \langle 0, 1 \rangle^2 \longrightarrow \langle 0, 1 \rangle$

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \alpha\beta & \alpha + \beta \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy conjunction.

Task 17

Assignment

Decide if the $(\alpha \wedge \alpha) \vee (\alpha \wedge \alpha) \leqslant \alpha$ holds for

1. standard, $\overset{S}{\vee}$
2. algebraic, $\overset{A}{\vee}$
3. Łukasiewicz , $\overset{L}{\vee}$

Task 18

Assignment

Decide, which equalities hold:

1. $(\alpha \underset{S}{\wedge} \alpha) \overset{L}{\vee} (\alpha \underset{S}{\wedge} \beta) = \alpha \underset{S}{\wedge} (\alpha \overset{L}{\vee} \beta)$
2. $(\alpha \underset{L}{\wedge} \alpha) \overset{S}{\vee} (\alpha \underset{L}{\wedge} \beta) = \alpha \underset{L}{\wedge} (\alpha \overset{S}{\vee} \beta)$
3. $\alpha \overset{S}{\vee} (\alpha \underset{L}{\wedge} \beta) = \alpha \underset{L}{\wedge} (\alpha \overset{S}{\vee} \beta)$

Justify your conclusions.

Task 19

Assignment

Decide, which equalities hold:

1. $(\alpha \underset{S}{\wedge} \beta) \underset{L}{\wedge} \gamma = \alpha \underset{S}{\wedge} (\beta \underset{L}{\wedge} \gamma)$
2. $\underset{S}{\neg}(\alpha \overset{A}{\vee} \beta) = \underset{S}{\neg}\alpha \underset{L}{\wedge} \underset{S}{\neg}\beta$
3. $(\alpha \underset{L}{\wedge} \alpha) \overset{L}{\vee} \underset{S}{\neg}\alpha = (\underset{S}{\neg}\alpha \underset{L}{\wedge} \underset{S}{\neg}\alpha) \overset{L}{\vee} \alpha$

Justify your conclusions.

Task 20

Assignment

Verify that $\alpha \underset{\circ}{\wedge} (\alpha \overset{\mathrm{R}}{\underset{\circ}{\Rightarrow}} \beta) = \alpha \underset{\mathrm{S}}{\wedge} \beta$ holds for

1. algebraic ops.
2. standard ops.

Solution

We will show the solution for algebraic operations. The other ones are similar.

$$\alpha \underset{A}{\wedge} (\alpha \overset{R}{\underset{A}{\Rightarrow}} \beta) = \left\{ \begin{matrix} \alpha \cdot 1 = \alpha \text{ for } \alpha \leqslant \beta \\ \alpha \cdot \dfrac{\beta}{\alpha} = \beta \text{ for } \alpha > \beta \end{matrix} \right\} = \mathbf{min}(\alpha, \beta) = \alpha \underset{S}{\wedge} \beta$$

Task 22

Assignment

Complete the table, so that R is a $\mathbb{S}$-partial order.

R	a	b	c	d
a				
b	0.5			
c		0.3		
d		0.2		

Solution

Reflexivity implies 1's on the diagonal. $\mathbb{S}$-partial order implies 0's to non-zero elements symmetric over the main diagonal.

R	a	b	c	d
a	1	0	x'	y'
b	0.5	1	0	0
c	x	0.3	1	z'
d	y	0.2	z	1

The transitivity implies e.g. $R(3,2) \underset{\mathbb{S}}{\wedge} R(2,1) \leqslant R(3,1)$, which translates into a condition $\min(0.3, 0.5) \leqslant x$

Using this and similar conditions, we derive the subspace of all solutions: $z \leqslant 0.2$, $x \geqslant 0.3$, $y \geqslant 0.2$, $\min(y, z') \leqslant a$, $x' = y' = 0$.