AE4M33RZN, Fuzzy logic: Tutorial examples

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Task 2

Assignment

On the universe $\Delta = \{a, b, c, d\}$ there is a fuzzy set

$$\mu_A = \Big\{(a; 0.3), (b; 1), (c; 0.5)\Big\}.$$

Find its horizontal representation.

$$\mathrm{R}_A(\alpha) = \begin{cases} X & \alpha = 0, \\ \{a, b, c\} & \alpha \in (0; 0.3\rangle, \\ \{b, c\} & \alpha \in (0.3; 0.5\rangle, \\ \{b\} & \alpha \in (0.5; 1\rangle. \end{cases}$$

Task 3

Assignment

The fuzzy set A has a horizontal representation

$$\mathrm{R}_A(\alpha) = \begin{cases} \{a, b, c, d\} & \alpha \in \langle 0, 1/3 \rangle \\ \{a, d\} & \alpha \in (1/3, 1/2 \rangle \\ \{d\} & \alpha \in (1/2, 2/3 \rangle \\ \emptyset & \alpha \in (2/3, 1 \rangle \end{cases}$$

Find the vertical representation.

Solution

$$\mu_A(a) = \sup\left\{\alpha \in \langle 0, 1\rangle : \; a \in \mathtt{R}_A(\alpha)\right\} = \sup\langle 0, 1/2\rangle = 1/2$$

$$\mu_A(b) = 1/3, \quad \mu_A(c) = 1/3, \quad \mu_A(d) = 2/3, \quad \text{therefore}$$

$$\mu_A = \left\{(a, 1/2), (b, 1/3), (c, 1/3), (d, 2/3)\right\}$$

Task 5

Assignment

On the universe $\Delta = \mathbb{R}$ there is a fuzzy set A:

$$\mu_A(x) = \begin{cases} x & x \in \langle 0; 1 \rangle \\ 2 - x & x \in (1; 1.5 \rangle \\ 0 & \text{otherwise} \end{cases}$$

Find its horizontal represenation.

Solution

$$\mathtt{R}_A(\alpha) = \begin{cases} \mathbb{R} & \alpha = 0 \\ \langle \alpha; 1.5 \rangle & \alpha \in (0; 0.5\rangle \\ \langle \alpha; 2 - \alpha \rangle & \alpha \in (0.5; 1\rangle \end{cases}$$

Task 7

Assignment

The fuzzy set A has a horizontal representation

$$\mathrm{R}_A(\alpha) = \begin{cases} \mathbb{R} & \alpha = 0 \\ \langle \alpha^2, 1) & \text{otherwise.} \end{cases}$$

Find the vertical representation.

$$\mu_A(x) = \begin{cases} \sqrt{x} & x \in (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Task 8

Assignment

Decide if the following function is a fuzzy conjunction.

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \alpha & \beta = 1, \\ \beta & \alpha = 1, \\ \alpha\beta & \alpha\beta \geq 1/10,\ \max(\alpha, \beta) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution

The first two possibilities ensure the boundary condition. Comutativity and monotonicity are trivially satisfied. If one of the arguments is 1, the associativity as well. For $\alpha, \beta, \gamma < 1$ the associativity follows from

$$\alpha \underset{\circ}{\wedge} \left(\beta \underset{\circ}{\wedge} \gamma\right) = \begin{cases} \alpha\beta\gamma & \alpha\beta\gamma \geq 1/10. \\ 0 & \text{otherwise,} \end{cases}$$

We get the same for $\left(\alpha \underset{\circ}{\wedge} \beta\right) \underset{\circ}{\wedge} \gamma$.

It is always a fuzzy conjunction (interpretable as an algebraic conjunction, in which we ignore small values, e.g. for filtering small values).

Task 10

Assignment

Prove that the Sugeno class of negations are all fuzzy negations

$$\underset{s_\lambda}{\neg}\alpha = \frac{1-\alpha}{1+\lambda\alpha} \text{ for } \lambda \in (-1, \infty,) .$$

Solution

When canceling out terms in a fraction, we use $1 + \lambda\alpha > 0$.

- **Involutivity:** Assuming $\lambda > -1$:

$$\overline{s}_\lambda \overline{s}_\lambda \alpha = \frac{1 - \frac{1-\alpha}{1+\lambda\alpha}}{1 + \lambda \frac{1-\alpha}{1+\lambda\alpha}} = \frac{1 + \lambda\alpha - 1 + \alpha}{1 + \lambda\alpha + \lambda - \lambda\alpha} = \frac{\alpha(\lambda + 1)}{\lambda + 1} = \alpha$$

- **Non-increasing:** If $\alpha \leq \beta$, then

$$\frac{1 - \alpha}{1 + \lambda\alpha} \geq \frac{1 - \beta}{1 + \lambda\beta}$$

$$(1 + \lambda\beta)(1 - \alpha) \geq (1 + \lambda\alpha)(1 - \beta)$$

$$\beta \geq \alpha$$

Task 12

Assignment

Decide if the following function is a fuzzy conjunction.

$$\alpha \diamond \beta = \begin{cases} \alpha\beta & \alpha\beta \geq 0{,}01 \text{ or } \max(\alpha, \beta) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Task 13

Assignment

Decide if the following function is a fuzzy conjunction.

$\underset{\circ}{\wedge} : \langle 0, 1\rangle^2 \longrightarrow \langle 0, 1\rangle$

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \min(\alpha, \beta) & \alpha + \beta \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Task 14

Assignment

Decide, if for all $\alpha, \beta \in [0, 1]$ holds $(\alpha \overset{\circ}{\vee} \beta) \underset{L}{\wedge} (\alpha \overset{\circ}{\vee} \underset{S}{\neg} \beta) = \alpha$, where the disjunction $\overset{\circ}{\vee}$ is

1. standard, $\overset{S}{\vee}$,
2. algebraic, $\overset{A}{\vee}$.
3. Łukasiewicz , $\overset{L}{\vee}$,

Solution

1. No. Counterexample: $\alpha = 0.5, \beta = 0.1$.
2. Yes: $\max(0, (\alpha + \beta - \alpha\beta) + (\alpha + (1 - \beta) - \alpha(1 - \beta)) - 1) =$

$$= \max(0, \alpha) = \alpha$$

3. No. Counterexample: $\alpha = 0.9, \beta = 0.8$.

Task 16

Assignment

Decide if the function $\underset{\circ}{\wedge} : \langle 0, 1\rangle^2 \longrightarrow \langle 0, 1\rangle$

$$\alpha \underset{\circ}{\wedge} \beta = \begin{cases} \alpha\beta & \alpha + \beta \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy conjunction.

Task 17

Assignment

Decide if the $(\alpha \wedge \alpha) \vee (\alpha \wedge \alpha) \leq \alpha$ holds for

1. standard, $\overset{S}{\vee}$
2. algebraic, $\overset{A}{\vee}$
3. Łukasiewicz , $\overset{L}{\vee}$

Task 18

Assignment

Decide, which equalities hold:

1. $(\alpha \underset{S}{\wedge} \alpha) \overset{L}{\vee} (\alpha \underset{S}{\wedge} \beta) = \alpha \underset{S}{\wedge} (\alpha \overset{L}{\vee} \beta)$
2. $(\alpha \underset{L}{\wedge} \alpha) \overset{S}{\vee} (\alpha \underset{L}{\wedge} \beta) = \alpha \underset{L}{\wedge} (\alpha \overset{S}{\vee} \beta)$
3. $\alpha \overset{S}{\vee} (\alpha \underset{L}{\wedge} \beta) = \alpha \underset{L}{\wedge} (\alpha \overset{S}{\vee} \beta)$

Justify your conclusions.

Task 19

Assignment

Decide, which equalities hold:

1. $(\alpha \underset{S}{\wedge} \beta) \underset{L}{\wedge} \gamma = \alpha \underset{S}{\wedge} (\beta \underset{L}{\wedge} \gamma)$
2. $\underset{S}{\neg}(\alpha \overset{A}{\vee} \beta) = \underset{S}{\neg}\alpha \underset{L}{\wedge} \underset{S}{\neg}\beta$
3. $(\alpha \underset{L}{\wedge} \alpha) \overset{L}{\vee} \underset{S}{\neg}\alpha = (\underset{S}{\neg}\alpha \underset{L}{\wedge} \underset{S}{\neg}\alpha) \overset{L}{\vee} \alpha$

Justify your conclusions.

Task 20

Assignment

Verify that $\alpha \underset{\circ}{\wedge} (\alpha \overset{R}{\underset{\circ}{\Rightarrow}} \beta) = \alpha \underset{S}{\wedge} \beta$ holds for

1. algebraic ops.
2. standard ops.

Solution

We will show the solution for algebraic operations. The other ones are similar.

$$\alpha \underset{\text{A}}{\wedge} (\alpha \overset{\text{R}}{\underset{\text{A}}{\Rightarrow}} \beta) = \left\{ \begin{array}{c} \alpha \cdot 1 = \alpha \text{ for } \alpha \leq \beta \\ \alpha \cdot \dfrac{\beta}{\alpha} = \beta \text{ for } \alpha > \beta \end{array} \right\} = \mathbf{min}(\alpha, \beta) = \alpha \underset{\text{S}}{\wedge} \beta$$

Task 22

Assignment

Complete the table, so that R is a $\mathbb{S}$-partial order.

R	a	b	c	d
a				
b	0.5			
c		0.3		
d		0.2		

Solution

Reflexivity implies 1's on the diagonal. S-partial order implies 0's to non-zero elements symmetric over the main diagonal.

R	a	b	c	d
a	1	0	x'	y'
b	0.5	1	0	0
c	x	0.3	1	z'
d	y	0.2	z	1

The transitivity implies e.g. $R(3,2) \underset{S}{\wedge} R(2,1) \leq R(3,1)$, which translates into a condition $\min(0.3, 0.5) \leq x$

Using this and similar conditions, we derive the subspace of all solutions: $z \leq 0.2$, $x \geq 0.3$, $y \geq 0.2$, $\min(y, z') \leq a$, $x' = y' = 0$.

Ex: Jim revisited

We will use the Lukasiewicz logic in the following examples ($\sqcap = \sqcap_L$, ...).

$$\langle jim : \text{Male} \mid 0.9 \rangle \quad (1)$$

$$\langle jim : \text{Female} \mid 0.2 \rangle \quad (2)$$

$$\langle \text{Male} \sqcap \text{Female} \sqsubseteq \bot \mid 1 \rangle \quad (3)$$

The interpretation domain is $\Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2} = \{j\}, jim^{\mathcal{I}_1} = jim^{\mathcal{I}_2} = j$.

$\text{Male}^{\mathcal{I}_1} = \{(j; 0.9)\}$ $\text{Male}^{\mathcal{I}_2} = \{(j; 0.9)\}$

$\text{Female}^{\mathcal{I}_1} = \{(j; 0)\}$ $\text{Female}^{\mathcal{I}_2} = \{(j; 0.2)\}$

Ex: Jim revisited (check your knowledge)

Let's check the interpretation against the definitions...

$\mathscr{I} \models \tau$	$\tau_{(1)}$	$\tau_{(2)}$	$\tau_{(3)}$
$\mathscr{I}_1$	səʎ	ou	səʎ
$\mathscr{I}_2$	əsʎ	səʎ	ou

Ex: Jim revisited (in fuzzyDL)

Let's change the weights and encode the example in fuzzyDL:

```
(instance jim Male 0.4)
(instance jim Female 0.2)

(l-implies (and Male Female) *bottom* 0.9)

(min-instance? jim Male)
(max-instance? jim Male)
(min-instance? jim Female)
(max-instance? jim Female)
```

Let $\langle jim : \text{Male} \,|\, \alpha \rangle$ and $\langle jim : \text{Female} \,|\, \beta \rangle$, what are the bounds on α and β? fuzzyDL shows that $0.4 \leq \alpha \leq 0.9$ and $0.2 \leq \beta \leq 0.7$. Why?

Ex: Smokers

Recall the motivational example from the first lecture:

$$\langle \text{symmetric}(\mathsf{friend}) \rangle \tag{4}$$

$$\langle (anna, bill) : \mathsf{friend} \,|\, 1 \rangle \tag{5}$$

$$\langle (bill, cloe) : \mathsf{friend} \,|\, 1 \rangle \tag{6}$$

$$\langle (cloe, dirk) : \mathsf{friend} \,|\, 1 \rangle \tag{7}$$

$$\langle anna : \mathsf{Smoker} \,|\, 1 \rangle \tag{8}$$

$$\langle \exists \mathsf{friend} \cdot \mathsf{Smoker} \sqsubseteq \mathsf{Smoker} \,|\, 0.7 \rangle \tag{9}$$

What are the bounds on $\langle i : \mathsf{Smoker} \rangle$ for $i \in \{anna, bill, cloe, dirk\}$?

Ex: Smokers

What changes if we add

$$\langle dirk : \neg \text{Smoker} \,|\, 0.7 \rangle \tag{10}$$

(11)

What are the bounds on $\langle i : \neg \text{Smoker} \rangle$ for $i \in \{anna, bill, cloe, dirk\}$?

Ex: Smokers (in fuzzyDL)

```
(implies (some friendOf Smoker) Smoker 0.7)

(symmetric friendOf)
(related anna bill friendOf)
(related bill cloe friendOf)
(related cloe dirk friendOf)

(instance anna Smoker)
(instance dirk (not Smoker) 0.7)

(min-instance? anna Smoker)
(min-instance? bill Smoker)
(min-instance? cloe Smoker)
(min-instance? dirk Smoker)

(max-instance? anna Smoker)
(max-instance? bill Smoker)
(max-instance? cloe Smoker)
(max-instance? dirk Smoker)
```

Concrete data types

The domain $\Delta^{\mathscr{I}}$ is an unordered set. This is good for modelling cathegorical data: e.g. colors, people, ...

General idea: Extended interpretation

But we also need to include real numbers $\mathbb{R}$. The *fuzzy description logic with concrete datatypes* $\mathcal{RGHE}$—$\mathcal{C}($ uses “abstract objects” and “concrete objects”:

$$\Delta^{\mathscr{I}} = \Delta_a^{\mathscr{I}} \cup \mathbb{R}$$

Concrete data types

- *Concrete individuals*, are interpreted as objects from $\mathbb{R}$.
- *Concrete concepts*, are interpreted as subsets from $\mathbb{R}$.
- *Concrete roles*, are interpreted as subsets from $(\Delta_a^{\mathscr{I}} \times \mathbb{R})$.

All non-concrete notions are called *abstract*.

Concrete data types: New concepts

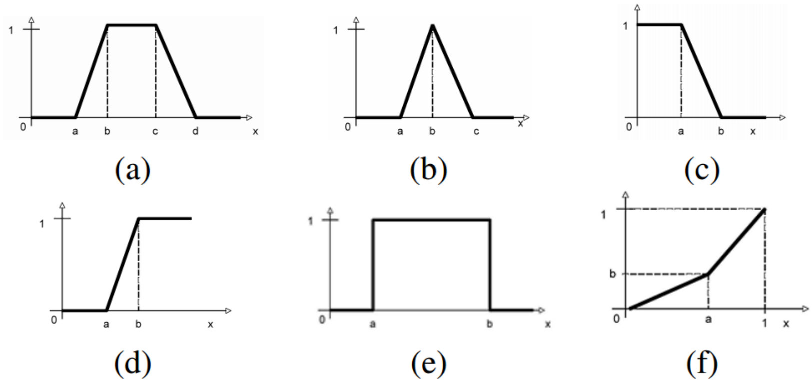

Fig. 1. (a) Trapezoidal function; (b) Triangular function; (c) *L*-function; (d) *R*-function; (e) Crisp interval; (f) Linear function.

Ex: Age of parents

```
(related adam bob parent) (related adam eve parent)

(define-fuzzy-concept around23 triangular(0,100, 18,23,26))
(define-fuzzy-concept moreTh17 right-shoulder(0,100, 13,21))
(instance bob (some age around23) 0.9)
(instance eve (some age moreTh17))

(define-fuzzy-concept young left-shoulder(0,100, 17,25))
(define-concept YoungPerson (some age young))

(min-instance? eve YoungPerson) (max-instance? eve YoungPerson)
(min-instance? bob YoungPerson) (max-instance? bob YoungPerson)
(min-instance? adam (all  parent YoungPerson))
(max-instance? adam (all  parent YoungPerson))
(min-instance? adam (some parent YoungPerson))
(max-instance? adam (some parent YoungPerson))
```

Ex: Age of parents

1. What are the bounds on α from $\langle eve : \text{YoungPerson} \mid \alpha \rangle$?

   Start by drawing the concept around23, then construct an interpretation. How much freedom do you have when constructing the interpretation?

2. Let fuzzyDL reasoner give you both bounds on $\langle i : \text{YoungPerson} \mid \beta_i \rangle$ for $i \in \{eve, bob\}$.

   How do you infer the bounds on $\langle adam : \text{YoungPerson} \mid \gamma \rangle$?

Ex: Car dealing

1. The buyer wants a **passenger** that costs **less than €26000**.
2. If there is an **alarm system** in the car, **then** he is satisfied with paying no more than **€22300**, but he can go up to **€22750** with a lesser degree of satisfaction.
3. The **driver insurance**, **air conditioning** and the **black color** are important factors.
4. Preferably the price is no more than **€22000**, but he can go to **€24000** to a lesser degree of satisfaction.

Ex: Car dealing

1. The seller wants to sell no less than **€22000**.
2. Preferably the buyer buys the **insurance plus** package.
3. If the **color is black**, then it is highly possible the car has an **air-conditioning**.

This can be formalized in fuzzy description logic.
We have the background knowledge:

$$\langle \mathsf{Sedan} \sqsubseteq \mathsf{PassengerCar} \,|\, 1 \rangle$$

$$\langle \mathsf{InsurancePlus} = \mathsf{DriverInsurance} \sqcap \mathsf{TheftInsurance} \,|\, 1 \rangle$$

Ex: Car dealing

The buyer's preferences:

1. $B = \text{PassengerCar} \sqcap \exists \text{price} \cdot \leq 26000$
2. $B_1 = \text{AlarmSystem} \mapsto \exists \text{price} \cdot l.sh.(22300, 22750)$
3. $B_2 = \text{DriverInsurance}, B_3 = \text{AirCondition}, B_4 = \exists \text{color} \cdot \text{Black}$
4. $B_5 = \exists \text{price} \cdot l.sh.(22000, 24000)$

The buyer's preferences:

1. $S = \text{PassengerCar} \sqcap \exists \text{price} \cdot \geq 22000$
2. $S_1 = \text{InsurancePlus}$
3. $S_2 = (0.5\ (\exists \text{color} \cdot \text{Black}) \mapsto \text{AirCondition})$

Ex: Car dealing

We know that S and B are hard constraints and $B_{1..5}$ and $S_{1..2}$ are soft preferences. All the concepts can be “summed up”:

$$\text{Buy} = B \sqcap (0.1B_1 + 0.2B_2 + 0.1B_3 + 0.4B_4 + 0.2B_5)$$

and

$$\text{Sell} = S \sqcap (0.6S_1 + 0.4S_2)$$

A good choice of $\sqcap$ can make B a hard constraint.

Ex: Car dealing

Optimal match

$$\boldsymbol{bsb}(K\,, \text{Buy} \sqcap \text{Sell})$$

Finds the optimal match between a seller and a buyer. (Finds an ideal, imaginary car that maximizes satisfaction of both parties.)

Particular car

$$\boldsymbol{bdb}(K\,, \langle \boldsymbol{auditTT} : \text{Buy} \sqcap \text{Sell} \rangle)$$

Finds the degree of satisfaction for a particuklar car *auditTT*.