AE4M33RZN, Fuzzy logic: Fuzzy description logic

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Plan of the lecture

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  - concept intersection
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  - role restriction
- $\mathcal{D}$ = data types

$\mathcal{SHIF}$ concepts

Let A and R be the sets of *atomic concepts* and *atomic roles*.

Concept constructors

$C, D := \top \mid \bot$	top and bottom concepts	(1)
$\mid A$	atomic concept	(2)
$\mid \neg C$	concept negation	(3)
$\mid C \sqcap D$	intersection	(4)
$\mid C \sqcup D$	concept union	(5)
$\mid \forall R \cdot C$	full universal quantification	(6)
$\mid \exists R \cdot C$	full existential quantification	(7)

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$\mathscr{A}Box$ (Assertion Box)

Contains *concept assertions* $\langle i \in I : \mathrm{C} \rangle$
and *role assertions* $\langle (i, j \in I) : \mathrm{R} \rangle$.

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$\mathscr{T}$*Box* (Terminology Box)

Contains *general concept inclusion* (GCI) axioms $\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$ and role axioms (role hierarchy $\langle \mathsf{R}_1 \sqsubseteq \mathsf{R}_2 \rangle$, transitivity, ...).

Crisp description logic interpretation

Interpretation $\mathscr{I}$ is a tuple $(\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}})$ (*interpretation domain*, *interpretation function*), which maps

an individual to domain object	$i^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to domain subsets	$C^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}}$
an atomic role to subset of domain tuples	$R^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}}$

Crisp description logic interpretation

The non-atomic concepts are interpreted as follows:

non-atomic concept	its interpretation
$\top$	$\Delta^{\mathcal{I}}$
$\bot$	$\emptyset$
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\forall R \cdot C$	$\{x \mid \forall y \in \Delta^{\mathcal{I}}. ((x, y) \in R^{\mathcal{I}}) \Rightarrow (y \in C^{\mathcal{I}})\}$
$\exists R \cdot C$	$\{x \mid \exists y \in \Delta^{\mathcal{I}}. ((x, y) \in R^{\mathcal{I}}) \wedge (y \in C^{\mathcal{I}})\}$

Crisp notion of truth

Axiom satisfaction

axiom	satisfied when
$\langle i : C \rangle$	$i^{\mathcal{I}} \in C^{\mathcal{I}}$
$\langle (i,j) : R \rangle$	$(i^{\mathcal{I}}, j^{\mathcal{I}}) \in R^{\mathcal{I}}$
$\langle C \sqsubseteq D \rangle$	$C^{\mathcal{I}} \sqsubseteq D^{\mathcal{I}}$
transitive(R)	$R^{\mathcal{I}}$ is transitive
...	

- Concept C is *satisfiable*

- Concept C is ***satisfiable*** iff there is an interpretation $\mathscr{I}$ s.t. $\mathscr{I} \models < i : C >$ for some i.
- Interpretation $\mathscr{I}$ ***satisfies*** a knowledgebase $\mathscr{K} = \mathscr{A}$***Box*** + $\mathscr{T}$***Box*** (or $\mathscr{I}$ is a ***model*** of $\mathscr{K}$)

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- Axiom T is a ***logical consequence*** of $\mathcal{K}$

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- Axiom T is a ***logical consequence*** of $\mathcal{K}$ iff every model of $\mathcal{K}$ satisfies T. We write $\mathcal{K} \models T$.

Basic idea

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1. Keep the the previous slides intact.

Fuzzy description logic

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1. Keep the the previous slides intact.
2. Add $\circ$ below and above every operation.

Fuzzy description logic

Basic idea

1. Keep the the previous slides intact.
2. Add ∘ below and above every operation.
3. Watch the semantic change.

Overview

We will show the **fuzzyDL** reasoner [Bobillo and Straccia, 2008] capabilities, which extends the $\mathcal{SHIF}(\mathcal{D})$ family with fuzzy capabilities.

Concept constructors

We start with atomic concepts A. Derived concepts are on the next slide together with their interpretation. (Each concept is interpreted as a fuzzy subset of the domain.)

Fuzzy DL interpretation

Fuzzy interpretation $\mathscr{I}$ is a tuple $\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}}$ which maps

an individual to a domain object	$\boldsymbol{i}^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to a domain subsets	$\mathsf{C}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}})$
an atomic role to a relation on the domain	$\mathsf{R}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}})$

$C, D :=$	interpretation of x
$\bot$	0
$\top$	1
A	$A^{\mathcal{I}}(x)$
$\neg C$	$\underset{S}{\neg} C^{\mathcal{I}}(x)$

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$C \underset{S}{\sqcap} D$	$C^{\mathscr{I}}(x) \underset{S}{\wedge} D^{\mathscr{I}}(x)$
$C \underset{L}{\sqcap} D$	$C^{\mathscr{I}}(x) \underset{L}{\wedge} D^{\mathscr{I}}(x)$

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$C \overset{S}{\sqcup} D$	$C^{\mathcal{I}}(x) \overset{S}{\vee} D^{\mathcal{I}}(x)$
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$C \overset{L}{\sqcup} D$	$C^{\mathcal{I}}(x) \overset{L}{\vee} D^{\mathcal{I}}(x)$
$C \overset{R}{\underset{S}{\mapsto}} D$	$C^{\mathcal{I}}(x) \overset{R}{\underset{S}{\Rightarrow}} D^{\mathcal{I}}(x)$
$C \overset{R}{\underset{L}{\mapsto}} D$	$C^{\mathcal{I}}(x) \overset{R}{\underset{L}{\Rightarrow}} D^{\mathcal{I}}(x)$
$C \overset{S}{\underset{S}{\mapsto}} D$	$C^{\mathcal{I}}(x) \overset{S}{\underset{S}{\Rightarrow}} D^{\mathcal{I}}(x)$

$C, D :=$	interpretation of x
$\exists R \cdot C$	$\sup_y R^{\mathcal{I}}(x, y) \mathbin{\underset{\circ}{\wedge}} C^{\mathcal{I}}(y)$
$\forall R \cdot C$	$\inf_y R^{\mathcal{I}}(x, y) \mathbin{\overset{\circ}{\underset{\circ}{\Rightarrow}}} C^{\mathcal{I}}(y)$

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$(n\ C)$	$n \cdot C(x)$
$mod(C)$	$mod(C^{\mathscr{I}}(x))$

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$w_1 C_1 + \ldots + w_k C_k$	$w_1 C_1^{\mathcal{I}}(x) + \ldots + w_k C_k^{\mathcal{I}}(x)$

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$w_1 C_1 + \ldots + w_k C_k$	$w_1 C_1^{\mathcal{I}}(x) + \ldots + w_k C_k^{\mathcal{I}}(x)$
$C \lesseqqgtr n$	$\begin{cases} C^{\mathcal{I}}(x) & C^{\mathcal{I}}(x) \lesseqqgtr n \\ 0 & \text{otherwise} \end{cases}$

$\text{Male} \sqcap \text{Female} \neq \bot$

Modifiers

Modifier is a function that alters the membership function.

Example

Linear modifier of degree c is

$$a = \frac{c}{c+1}$$
$$b = \frac{1}{c+1}$$

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$\mathscr{T}$*Box* (Terminology Box)

GCI axioms $\langle \mathsf{C} \sqsubseteq \mathsf{D} \,|\, \alpha \rangle$ state that "C is D **at least** by α".
Besides GCI, there are role hierarchy axioms $\langle \mathsf{R}_1 \sqsubseteq \mathsf{R}_2 \rangle$, transitivity axioms and definitions of inverse relations.

Notion of a fuzzy truth

Fuzzy axioms

axiom	satisfied if
$\langle i : \mathsf{C} \mid \alpha \rangle$	$\mathsf{C}^{\mathcal{I}}(i^{\mathcal{I}}) \geq \alpha$

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axiom	satisfied if
$\langle i : C \mid \alpha \rangle$	$C^{\mathcal{I}}(i^{\mathcal{I}}) \geq \alpha$
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$\langle R_1 \sqsubseteq R_2 \rangle$	$R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$
$\langle$transitive $R\rangle$	R is $\circ$-transitive

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$\langle$transitive $R\rangle$	R is $\circ$-transitive
$\langle R_1 = R_2^{-1} \rangle$	$R_1^{\mathcal{I}} = (R_2^{\mathcal{I}})^{-1}$

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Using these definitions, the notions of *logical consequence* and *satisfiability* (of both concepts and axioms) remains the same.
More on slide 317.

What can you ask the reasoner?

Best/Worst Degree Bound

What is the minimal degree of an axiom that $\mathcal{K}$ ensures?

$$\mathrm{glb}(\mathcal{K}, \tau) = \mathbf{sup}\{\alpha \mid \mathcal{K} \models \langle \tau \geq \alpha \rangle\}$$
$$\mathrm{lub}(\mathcal{K}, \tau) = \mathbf{inf}\{\alpha \mid \mathcal{K} \models \langle \tau \leq \alpha \rangle\}$$

where τ is an axiom of type $\langle i : \mathsf{C} \rangle$ or $\langle (i, j) : \mathsf{R} \rangle$ or $\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$.

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- From an empty $\mathcal{K}$, you cannot infer anything and therefore $\mathrm{glb}(\mathcal{K}, \tau) = 0$ and $\mathrm{lub}(\mathcal{K}, \tau) = 1$ (if using atomic concepts only). Only by adding new axioms into $\mathcal{K}$, the bounds “tighten up”.

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- What happens if $\mathrm{glb}(\mathcal{K}, \tau) \geq \mathrm{lub}(\mathcal{K}, \tau)$ for some axiom τ?

What can you ask the reasoner?

Best Satisfiability Bound

What is the maximal degree of satisfiability of C?

$$\mathrm{glb}(\mathcal{K}, \mathsf{C}) = \sup_{\mathcal{I}} \sup_{x \in \Delta} \{\mathsf{C}^{\mathcal{I}}(x) \mid \mathcal{I} \models \mathcal{K}\} \,.$$

What can you ask the reasoner?

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$$\mathrm{glb}(\mathcal{K}, C) = \sup_{\mathcal{I}} \sup_{x \in \Delta} \{C^{\mathcal{I}}(x) \mid \mathcal{I} \models \mathcal{K}\}\,.$$

This is a generalization of *concept satisfiability*.

Homework

Next time we will see a reasoning algorithm for fuzzy DL. Please read [Straccia and Bobillo, 2008]:

Basic idea of the fuzzyDL solver:

Straccia, Umberto and Fernando Bobillo. **"Mixed integer programming, general concept inclusions and fuzzy description logics."** Mathware & Soft Computing 14, no. 3 (2008): 247-259.

Where can you find the article? Google scholar is a place to start.

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