AE4M33RZN, Fuzzy logic: Fuzzy description logic

Radomír Černoch

radomir.cernoch@fel.cvut.cz

Faculty of Electrical Engineering, CTU in Prague

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Crisp description logic

Our treatment of fuzzy description logic is based on a family of crisp description logic $\mathcal{SHIF(D)}$ [Baader, 2003]:

- $\mathcal{AL}$
  - atomic negation
  - intersection
  - universal restrictions
  - limited existential quantification
- $\mathcal{C}$ = full concept negation
- $\mathcal{S}$ = $\mathcal{ALC}$ + transitive roles
- $\mathcal{H}$ = role hierarchies
- $\mathcal{I}$ = inverse properties
- $\mathcal{F}$
  - concept intersection
  - universal restrictions
  - limited existential quantification
  - role restriction
- $\mathcal{D}$ = data types

$\mathcal{SHIF}$ concepts

Let A and R be the sets of *atomic concepts* and *atomic roles*.

Concept constructors

$$
\begin{aligned}
C, D := {} & \top \mid \bot && \text{top and bottom concepts} && (1)\\
& \mid A && \text{atomic concept} && (2)\\
& \mid \neg C && \text{concept negation} && (3)\\
& \mid C \sqcap D && \text{intersection} && (4)\\
& \mid C \sqcup D && \text{concept union} && (5)\\
& \mid \forall R \cdot C && \text{full universal quantification} && (6)\\
& \mid \exists R \cdot C && \text{full existential quantification} && (7)
\end{aligned}
$$

Crisp description logic ontology

Ontology consists of $\mathscr{A}\textit{Box}$ and $\mathscr{T}\textit{Box}$. We use the set of individuals $\boldsymbol{I}$:

$\mathscr{A}\textit{Box}$ (Assertion Box)

Contains *concept assertions* $\langle \boldsymbol{i} \in \boldsymbol{I} : \mathrm{p} \in \boldsymbol{P} \rangle$
and *role assertions* $\langle (\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}) : \mathrm{r} \in \boldsymbol{R} \rangle$.

$\mathscr{T}\textit{Box}$ (Terminology Box)

Contains *general concept inclusion* (GCI) axioms $\langle \mathrm{C} \sqsubseteq \mathrm{D} \rangle$ and role axioms (role hierarchy $\langle \mathrm{R}_1 \sqsubseteq \mathrm{R}_2 \rangle$, transitivity, ...).

Crisp description logic interpretation

Interpretation $\mathscr{I}$ is a tuple $(\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}})$ (*interpretation domain*, *interpretation function*), which maps

an individual to domain object	$\boldsymbol{i}^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to domain subsets	$\mathsf{C}^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}}$
an atomic role to subset of domain tuples	$\mathsf{R}^{\mathscr{I}} \subseteq \Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}}$

Crisp description logic interpretation

The non-atomic concepts are interpreted as follows:

non-atomic concept	its interpretation
$\top$	$\Delta^{\mathcal{I}}$
$\bot$	$\emptyset$
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\forall R \cdot C$	$\{x \mid \forall y \in \Delta^{\mathcal{I}} . ((x, y) \in R^{\mathcal{I}}) \Rightarrow (y \in C^{\mathcal{I}})\}$
$\exists R \cdot C$	$\{x \mid \exists y \in \Delta^{\mathcal{I}} . ((x, y) \in R^{\mathcal{I}}) \wedge (y \in C^{\mathcal{I}})\}$

Crisp notion of truth

Axiom satisfaction

axiom	satisfied when
$\langle \boldsymbol{i} : \mathsf{C} \rangle$	$\boldsymbol{i}^{\mathscr{I}} \in \mathsf{C}^{\mathscr{I}}$
$\langle (\boldsymbol{i},\boldsymbol{j}) : \mathsf{R} \rangle$	$(\boldsymbol{i}^{\mathscr{I}}, \boldsymbol{j}^{\mathscr{I}}) \in \mathsf{R}^{\mathscr{I}}$
$\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$	$\mathsf{C}^{\mathscr{I}} \sqsubseteq \mathsf{D}^{\mathscr{I}}$
transitive(R)	$\mathsf{R}^{\mathscr{I}}$ is transitive
...	

- Concept C is ***satisfiable*** iff there is an interpretation $\mathscr{I}$ s.t. $\mathscr{I} \vDash < i : C >$ for some i.
- Interpretation $\mathscr{I}$ ***satisfies*** a knowledgebase $\mathscr{K}$ = $\mathscr{A}$***Box*** + $\mathscr{T}$***Box*** (or $\mathscr{I}$ is a ***model*** of $\mathscr{K}$) iff $\mathscr{I}$ satisfies all its axioms.
- Axiom T is a ***logical consequence*** of $\mathscr{K}$ iff every model of $\mathscr{K}$ satisfies T. We write $\mathscr{K} \vDash T$.

Fuzzy description logic

Basic idea

1. Keep the the previous slides intact.
2. Add $\circ$ below and above every operation.
3. Watch the semantic change.

Overview

We will show the **fuzzyDL** reasoner [Bobillo and Straccia, 2008] capabilities, which extends the $\mathcal{SHIF}(\mathcal{D})$ family with fuzzy capabilities.

Concept constructors

We start with atomic concepts A. Derived concepts are on the next slide together with their interpretation. (Each concept is interpreted as a fuzzy subset of the domain.)

Fuzzy DL interpretation

Fuzzy interpretation $\mathscr{I}$ is a tuple $\Delta^{\mathscr{I}}, \cdot^{\mathscr{I}}$ which maps

an individual to a domain object	$\boldsymbol{i}^{\mathscr{I}} \in \Delta^{\mathscr{I}}$
an atomic concept to a domain subsets	$\mathsf{C}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}})$
an atomic role to a relation on the domain	$\mathsf{R}^{\mathscr{I}} \in \mathbb{F}(\Delta^{\mathscr{I}} \times \Delta^{\mathscr{I}})$

$\mathsf{C}, \mathsf{D} :=$	interpretation of x
$\bot$	$\mathbf{0}$
$\top$	$\mathbf{1}$
A	$A^{\mathscr{I}}(x)$
$\neg\, \mathsf{C}$	$\underset{\mathrm{S}}{\neg}\, \mathsf{C}^{\mathscr{I}}(x)$
$\mathsf{C} \underset{\mathrm{S}}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \underset{\mathrm{S}}{\wedge} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \underset{\mathrm{L}}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \underset{\mathrm{L}}{\wedge} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{\mathrm{S}}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{\mathrm{S}}{\vee} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \overset{\mathrm{L}}{\sqcap} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{\mathrm{L}}{\vee} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \xmapsto[\mathrm{S}]{\mathrm{R}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{\mathrm{R}}{\underset{\mathrm{S}}{\Longrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \xmapsto[\mathrm{L}]{\mathrm{R}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{\mathrm{R}}{\underset{\mathrm{L}}{\Longrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$
$\mathsf{C} \xmapsto[\mathrm{S}]{\mathrm{S}} \mathsf{D}$	$\mathsf{C}^{\mathscr{I}}(x) \overset{\mathrm{S}}{\underset{\mathrm{S}}{\Longrightarrow}} \mathsf{D}^{\mathscr{I}}(x)$

$\mathsf{C}, \mathsf{D} :=$	interpretation of $\boldsymbol{x}$
$\exists \mathsf{R} \cdot \mathsf{C}$	$\sup_y \mathsf{R}^{\mathcal{I}}(\boldsymbol{x}, \boldsymbol{y}) \mathbin{\overset{}{\underset{\circ}{\wedge}}} \mathsf{C}^{\mathcal{I}}(\boldsymbol{y})$
$\forall \mathsf{R} \cdot \mathsf{C}$	$\inf_y \mathsf{R}^{\mathcal{I}}(\boldsymbol{x}, \boldsymbol{y}) \overset{\circ}{\underset{\circ}{\Rightarrow}} \mathsf{C}^{\mathcal{I}}(\boldsymbol{y})$
$(n\ \mathsf{C})$	$n \cdot \mathsf{C}(\boldsymbol{x})$
$mod(\mathsf{C})$	$mod(\mathsf{C}^{\mathcal{I}}(\boldsymbol{x}))$
$w_1 \mathsf{C}_1 + \ldots + w_k \mathsf{C}_k$	$w_1 \mathsf{C}_1^{\mathcal{I}}(\boldsymbol{x}) + \ldots + w_k \mathsf{C}_k^{\mathcal{I}}(\boldsymbol{x})$
$\mathsf{C} \lesseqqgtr n$	$\begin{cases} \mathsf{C}^{\mathcal{I}}(\boldsymbol{x}) & \mathsf{C}^{\mathcal{I}}(\boldsymbol{x}) \lesseqqgtr n \\ 0 & \text{otherwise} \end{cases}$

Male $\sqcap$ Female $\neq \perp$

Modifiers

Modifier is a function that alters the membership function.

Example

Linear modifier of degree c is

$$a = \frac{c}{c+1}$$
$$b = \frac{1}{c+1}$$

Fuzzy DL ontology

Ontology consists of $\mathscr{A}$*Box* and $\mathscr{T}$*Box*:

$\mathscr{A}$*Box* (Assertion Box)

Contains *concept assertions* $\langle \boldsymbol{i} \in \boldsymbol{I} : \mathsf{p} \in \boldsymbol{P} \,|\, \alpha \rangle$ and *role assertions* $\langle (\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}) : \mathsf{r} \in \boldsymbol{R} \,|\, \alpha \rangle$.

$\mathscr{T}$*Box* (Terminology Box)

GCI axioms $\langle \mathsf{C} \sqsubseteq \mathsf{D} \,|\, \alpha \rangle$ state that "C is D **at least** by α".
Besides GCI, there are role hierarchy axioms $\langle \mathsf{R}_1 \sqsubseteq \mathsf{R}_2 \rangle$, transitivity axioms and definitions of inverse relations.

Notion of a fuzzy truth

Fuzzy axioms

axiom	satisfied if
$\langle i : C \mid \alpha \rangle$	$C^{\mathcal{I}}(i^{\mathcal{I}}) \geqslant \alpha$
$\langle (i,j) : R \mid \alpha \rangle$	$R^{\mathcal{I}}(i^{\mathcal{I}}, j^{\mathcal{I}}) \geqslant \alpha$
$\langle C \sqsubseteq D \mid \alpha \rangle$	$C \overset{\circ}{\underset{\circ}{\subseteq}} D \geqslant \alpha$
$\langle R_1 \sqsubseteq R_2 \rangle$	$R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$
$\langle$transitive $R\rangle$	R is $\circ$-transitive
$\langle R_1 = R_2^{-1} \rangle$	$R_1^{\mathcal{I}} = (R_2^{\mathcal{I}})^{-1}$

Using these definitions, the notions of ***logical consequence*** and ***satisfiability*** (of both concepts and axioms) remains the same.

More on slide 307.

What can you ask the reasoner?

Best/Worst Degree Bound

What is the minimal degree of an axiom that $\mathcal{K}$ ensures?

$$\boldsymbol{bdb}(\mathcal{K}, \tau) = \mathbf{sup}\{\alpha \mid \mathcal{K} \models \langle \tau \mid \alpha \rangle\}$$
$$\boldsymbol{wdb}(\mathcal{K}, \tau) = \mathbf{inf}\{\alpha \mid \mathcal{K} \models \langle \tau \mid \alpha \rangle\}$$

where τ is an axiom of type $\langle \boldsymbol{i} : \mathsf{C} \rangle$ or $\langle (\boldsymbol{i}, \boldsymbol{j}) : \mathsf{R} \rangle$ or $\langle \mathsf{C} \sqsubseteq \mathsf{D} \rangle$.

- From an empty $\mathcal{K}$, you cannot infer anything and therefore $\boldsymbol{bdb}(\mathcal{K}, \tau) = 1$ and $\boldsymbol{wdb}(\mathcal{K}, \tau) = 0$ (if using atomic concepts only). Only by adding new axioms into $\mathcal{K}$, the bounds “tighten up”.
- What happens if $\boldsymbol{wdb}(\mathcal{K}, \tau) \geqslant \boldsymbol{bdb}(\mathcal{K}, \tau)$ for some axiom τ?

What can you ask the reasoner?

Best Satisfiability Bound

What is the maximal degree of satisfiability of C?

$$\boldsymbol{bsb}(\mathcal{K}, \mathsf{C}) = \sup_{\mathcal{I}} \sup_{x \in \Delta} \{\mathsf{C}^{\mathcal{I}}(x) \mid \mathcal{I} \models \mathcal{K}\} \, .$$

This is a generalization of *concept satisfiability*.

Bibliography

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