

AE4M33RZN, Fuzzy logic: Fuzzy relations

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Organizational:

- Next week, there will be a short test (max 5 points) during the tutorials.
- Tutorial slides will be updated today.
- Lecture slides have been updated. No more bugs I know about!
- This week we are having the last **theoretical lecture**.

Fuzzy implication

We already know *fuzzy negation* $\overset{\circ}{\neg}$, *fuzzy conjunction* $\overset{\circ}{\wedge}$ and *fuzzy disjunction* $\overset{\circ}{\vee}$. Unfortunately, there is no nice formula...

Definition

Fuzzy implication is any function

$$\overset{\circ}{\Rightarrow} : [0, 1]^2 \rightarrow [0, 1] \quad (1)$$

which overlaps with the boolean implication on $x, y \in \{0, 1\}$:

$$(x \overset{\circ}{\Rightarrow} y) = (x \Rightarrow y) . \quad (2)$$

Residue implication

Despite the lack of a uniform definition of fuzzy implication, there is a useful class of implications:

Defintion

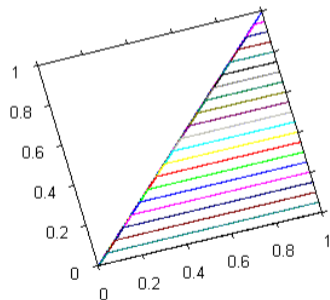
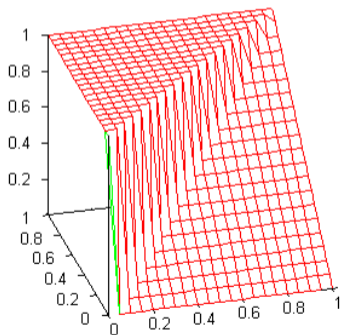
The *R-implication* (residuum, „*reziduovaná implikace*“) is a function obtained from a fuzzy T-norm:

$$\alpha \overset{R}{\underset{\circ}{\Rightarrow}} \beta = \sup\{\gamma \mid \alpha \underset{\circ}{\wedge} \gamma \leq \beta\} \quad (\text{RI})$$

R-implication: Examples (1)

Standard implication (Gödel) is derived from (RI) using the standard conjunction $\hat{\wedge}$:

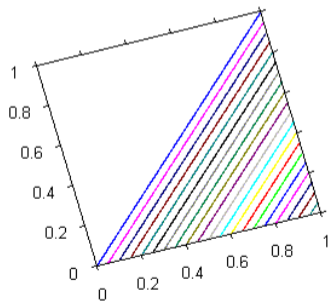
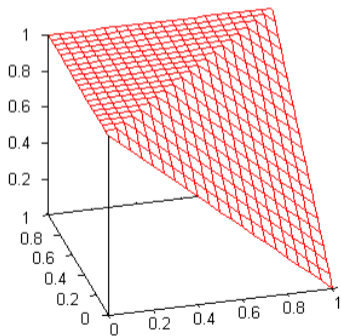
$$\alpha \xrightarrow[\text{S}]{\text{R}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases} \quad (3)$$



R-implication: Examples (2)

Lukasiewicz implication is derived from (RI) using the Łukasiewicz conjunction $\hat{\wedge}$:

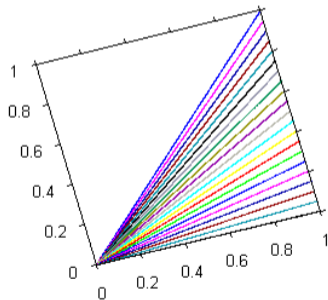
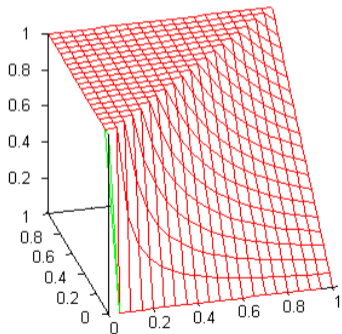
$$\alpha \stackrel{\text{R}}{\underset{\text{L}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha + \beta & \text{otherwise} \end{cases} \quad (4)$$



R-implication: Examples (3)

Algebraic implication (Gougen, Gaines) is derived from (RI) using the algebraic conjunction $\hat{\wedge}_A$:

$$\alpha \xrightarrow[A]{R} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases} \quad (5)$$



R-implication: Properties

Theorem 108.

Let \wedge_{\circ} be a continuous fuzzy conjunction. Then R-implication satisfies:

$$\alpha \xrightarrow[\circ]{R} \beta = \mathbf{1} \text{ iff } \alpha \leq \beta \quad (I1)$$

$$\mathbf{1} \xrightarrow[\circ]{R} \beta = \beta \quad (I2)$$

$$\alpha \xrightarrow[\circ]{R} \beta \text{ is not increasing in } \alpha \text{ and not decreasing in } \beta \quad (I3)$$

R-implication: Properties

Proof of theorem 108: Let's denote $\{\gamma \mid \alpha \wedge_{\circ} \gamma \leq \beta\} = \Gamma$.

- Proving (I3) uses monotonicity: Increasing α can only shrink Γ and increasing β can only enlarge Γ .
- Proving (I2) is easy: $1 \xrightarrow{\circ} \beta = \sup\{\gamma \mid 1 \wedge_{\circ} \gamma \leq \beta\}$. From definition of \wedge_{\circ} , we write $1 \xrightarrow{\circ} \beta = \sup\{\gamma \mid \gamma \leq \beta\} = \beta$.

R-implication: Properties

Proof of theorem 108 (contd.):

- For (I1) one needs to check 2 cases:
 - If $\alpha \leq \beta$, then $1 \in \Gamma$, because $\alpha \wedge 1 = \alpha \leq \beta$ and therefore the condition $\alpha \wedge \gamma \leq \beta$ is true for all possible values of γ .
 - If $\alpha > \beta$, then $1 \notin \Gamma$, because $\alpha \wedge 1 = \alpha > \beta$ and therefore the condition $\alpha \wedge \gamma \leq \beta$ is false for $\gamma = 1$.

S-implication

Defintion

The *S-implication* is a function obtained from a fuzzy disjunction $\overset{\circ}{\vee}$:

$$\alpha \underset{\circ}{\overset{\circ}{\Rightarrow}} \beta = \underset{\circ}{\neg} \alpha \overset{\circ}{\vee} \beta \quad (\text{SI})$$

Example

Kleene-Dienes implication from $\overset{\circ}{\vee}$

$$\alpha \underset{\circ}{\overset{\circ}{\Rightarrow}} \beta = \max(1 - \alpha, \beta) \quad (6)$$

Generalized fuzzy inclusion

Previously, we used the logical negation \neg to define the set complement, the conjunction \wedge to define the set intersection, etc.

Can we use the implication $\overset{\circ}{\Rightarrow}$ to define the the fuzzy inclusion?

Definition

The *generalized fuzzy inclusion* $\overset{\circ}{\subseteq}$ is a function that assigns a degree to the the inclusion of set $A \in \mathbb{F}(\Delta)$ in set $B \in \mathbb{F}(\Delta)$:

$$A \overset{\circ}{\subseteq} B = \inf\{A(\mathbf{x}) \overset{\circ}{\Rightarrow} B(\mathbf{x}) \mid \mathbf{x} \in \Delta\} \quad (7)$$

Generalized fuzzy inclusion: Example

Fuzzy inclusion (non-generalized)

Definition

The *fuzzy inclusion* \subseteq is a predicate (assigns a true/false value) which hold for two fuzzy sets $A, B \in \mathbb{F}(\Delta)$ iff

$$\mu_A(\mathbf{x}) \leq \mu_B(\mathbf{x}) \text{ for all } \mathbf{x} \in \Delta. \quad (8)$$

Fuzzy inclusion (non-generalized)

In vertical representation, the definition has a straightforward equivalent:

$$\mu_A \leq \mu_B \quad (9)$$

In horizontal representation, there is a theorem:

Theorem 115.

Let $A, B \in \mathbb{F}(\Delta)$ if and only if

$$R_A(\alpha) \subseteq R_B(\alpha) \text{ for all } \alpha \in [0, 1] . \quad (10)$$

Fuzzy inclusion (non-generalized)

Proof of theorem 115.

- \Rightarrow Assume $A \subseteq B$ and $x \in R_A(\alpha)$ for some value α . If $\alpha \leq A(x)$, then $A(x) \leq B(x)$ (from the definition of $A \subseteq B$) and therefore $x \in R_B(\alpha)$ and $R_A(\alpha) \subseteq R_B(\alpha)$.
- \Leftarrow Assume $R_A(\alpha) \subseteq R_B(\alpha)$. Firstly recall the horizontal-vertical translation formula: $\mu_A(x) = \sup\{\alpha \in [0, 1] \mid x \in R_A(\alpha)\}$. Since $\{x \mid x \in R_A(\alpha)\} \subseteq \{x \mid x \in R_B(\alpha)\}$, the inequality $A(x) \leq \sup\{\alpha \mid x \in R_B(\alpha)\} \leq B(x)$ holds.

Cutworthiness

We ended up with 2 equal definitions of set inclusion: using vertical and horizontal representation. Can we generalize this?

Cutworthiness

Let P be a predicate (returns true/false) over fuzzy sets. P is called *cutworthy* („řezově dědičná vlastnost“) if the implication holds:

$$P(A_1, \dots, A_n) \Rightarrow P(R_{A_1}(\alpha), \dots, R_{A_n}(\alpha)) \text{ for all } \alpha \in [0, 1] \quad (11)$$

There is a related notion: We define P as *cut-consistent* („řezově konzistentní“) using the same definition, but replacing \Rightarrow with \Leftrightarrow .

Cutworthiness: Examples

- The theorem 115 can be stated as: “Set inclusion is cut-consistent.”

Brain teasers

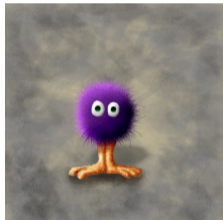
- *Strong normality* of A is defined as $A(x) = 1$ for some $x \in \Delta$.

iff every cut strongly normal. iff every its cut is non-empty
 A is strongly-normal Strong normality is cut-consistent:

- *Being crisp* is

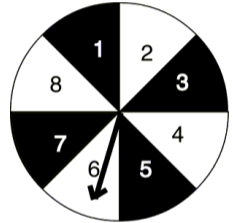
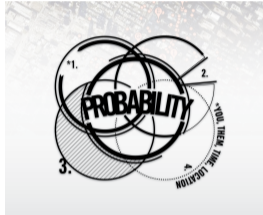
therefore the property is not cut-consistent.
But even non-crisp sets have crisp cuts,
Every cut is crisp by definition, therefore cutworthiness.
cutworthy, but not cut-consistent:

Google: “fuzzy”



Sources: M. Taylor's Weblog, M. Taylor's Weblog, Eddie's Trick Shop.

Google: “probability”



Sources: Life123, WhatWeKnowSoFar, Probability Problems.

Fuzzy vs. probability

- *Vagueness vs. uncertainty.*
- Fuzzy logic is *functional*.

Crisp relations

Definition

A *binary crisp relation* R from X onto Y is a subset of the cartesian product $X \times Y$:

$$R \in \mathbb{P}(X \times Y) \quad (12)$$

Definition

The *inverse relation* R^{-1} to R is a relation from Y to X s.t.

$$R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\} \quad (13)$$

Crisp relations: Inverse

Definition

Let X, Y, Z be sets. Then the *compound* of relations $R \subseteq X \times Y, S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(x, z) \in X \times Z \mid (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y\} \quad (14)$$

Crisp relations: Properties

The *identity* relation on Δ is $E = \{(x, x) \mid x \in \Delta\}$.

Then the relation $R \subseteq \Delta \times \Delta$ is called

property	using logical connectives	using set axioms
reflexive	$\forall x. (x, x) \in R$	$E \subseteq R$
symmetric	$(x, y) \in R \Rightarrow (y, x) \in R$	$R = R^{-1}$
anti-symmetric	$(x, y) \in R \wedge (y, z) \in R \Rightarrow y = z$	$R \cap R^{-1} \subseteq E$
transitive	$(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$	$R \circ R \subseteq R$
partial order	reflexive, transitive and anti-symmetric	
equivalence	reflexive, transitive and symmetric	

Fuzzy relations

Definition

A *binary fuzzy relation* R from X onto Y is a fuzzy subset on the universe $X \times Y$.

$$R \in \mathbb{F}(X \times Y) \quad (15)$$

Definition

The *fuzzy inverse* relation $R^{-1} \in \mathbb{F}(Y \times X)$ to $R \in \mathbb{F}(X \times Y)$, s.t.

$$R(\mathbf{y}, \mathbf{x}) = R^{-1}(\mathbf{x}, \mathbf{y}) \quad (16)$$

Projection

Defintion

Let $R \in \mathbb{F}(X \times Y)$ be a fuzzy binary relation. The *first* and second projection of R is

$$R^{(1)}(\mathbf{x}) = \bigvee_{\mathbf{y} \in Y}^S R(\mathbf{x}, \mathbf{y}) \quad (17)$$

$$R^{(2)}(\mathbf{y}) = \bigvee_{\mathbf{x} \in X}^S R(\mathbf{x}, \mathbf{y}) \quad (18)$$

Projection: Example

R	y_1	y_2	y_3	y_4	y_5	y_6	$R^{(1)}(x)$
x_1	0.1	0.2	0.4	0.8	1	0.8	↓
x_2	0.2	0.4	0.8	1	0.8	0.6	↓
x_3	0.4	0.8	1	0.8	0.4	0.2	↓
$R^{(2)}(y)$	0.7	0.8	1	1	1	0.8	

Sometimes there is a *total projection* defined as

$$R^{(T)} = \bigvee_{x \in X} \bigvee_{y \in Y} R(x, y).$$

But we already know this notion as $\text{Height}(R)$.

Cylindrical extension

Can we reconstruct a fuzzy relation from its projections? There is an unique largest relation with prescribed projections:

Definition

Let $A \in \mathbb{F}(X)$ and $B \in \mathbb{F}(Y)$ be fuzzy sets. The *cylindrical extension* („cylindrické rozšíření“, „kartézský součin fuzzy množin“) is defined as

$$A \times B(x, y) = A(x) \underset{S}{\wedge} B(y) \quad (19)$$

Brain teaser

Why can't there be a relation Q bigger than $A \times B$, whose projections are $Q^{(1)} = A$ and $Q^{(2)} = B$?

Cylindrical extension: Drawing

$$A(x) = \begin{cases} x - 1 & x \in [1, 2] \\ 3 - x & x \in [2, 3] \\ 0 & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} x - 3 & x \in [3, 4] \\ 5 - x & x \in [4, 5] \\ 0 & \text{otherwise} \end{cases}$$

Composition of fuzzy relations

Definition

Let X, Y, Z be crisp sets. $R \in \mathbb{F}(X \times Y)$, $S \in \mathbb{F}(Y \times Z)$ and $\underset{\circ}{\wedge}$ some fuzzy conjunction. Then the $\underset{\circ}{\circ}$ -composition („ $\underset{\circ}{\circ}$ -složená relace“) is

$$R \underset{\circ}{\circ} S(x, z) = \bigvee_{y \in Y}^S R(x, y) \underset{\circ}{\wedge} S(y, z) \quad (20)$$

1. For infinite domains, \bigvee^S is computed using the sup instead of max.
2. Instead of the “for some y ” in *crisp relations*, the disjunction “finds such a y ” that maximizes the conjunction.

Example of a fuzzy relation

$$R(x, y) = \begin{cases} x + y & x, y \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

$$S(x, y) = \begin{cases} x \cdot y & x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Properties of fuzzy relations

Then the relation $R \subseteq \Delta \times \Delta$ is called

property	using set axioms
reflexive	$E \subseteq R$
symmetric	$R = R^{-1}$
\circ -anti-symmetric	$R \cap_{\circ} R^{-1} \subseteq E$
\circ -transitive	$R \circ_{\circ} R \subseteq R$
\circ -partial order	reflexive, \circ -transitive and \circ -anti-symmetric
\circ -equivalence	reflexive, \circ -transitive and \circ -symmetric

Properties in a finite domain

If the universe Δ is finite, the relation can be written as a matrix. Their properties are reflected in the relation's matrix:

- **Reflexivity:** Cells on the main diagonal are 1.
- **Symmetry:** Cells symmetric over the main diagonal are equal.
- **Anti-symmetry:** Cells symmetric over the main diagonal have conjunction equal to zero.
 - For S- and A-anti-symmetry, one of the elements must be zero.
 - For L-anti-symmetry, their sum must be less or equal to 1.
- **Transitivity:** More difficult (see example on the next slide).

Example on fuzzy relation properties

Let $\Delta = \{A, B, C, D\}$ and $R \in \mathbb{F}(\Delta \times \Delta)$.

R	A	B	C	D
A		0.5		0.1
B			0.2	
C				
D		0.2		

Fill the missing cells in the table to make R

- a) S-equivalence
- b) A-equivalence

Theorem 6.

Let R , S and T be relations (defined over sets that “make sense”) The following equations hold:

$$R \circ E = R, \quad E \circ R = R \quad (21)$$

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1} \quad (22)$$

$$R \circ (S \circ T) = (R \circ S) \circ T \quad (23)$$

$$(R \overset{S}{\cup} S) \circ T = (R \circ T) \overset{S}{\cup} (S \circ T) \quad (24)$$

$$R \circ (S \overset{S}{\cup} T) = (R \circ S) \overset{S}{\cup} (R \circ T) \quad (25)$$

(21) describes the *identity element*, (22) the *inverse of composition*, (23) is the *asociativity*, (24) and (25) the *right-* and *left-distributivity*.

Proof of 6.

Proving (21) and (22) is trivial.

$$"R \circ (S \circ T)"(x, w) = \bigvee_y^s R(x, y) \wedge "S \circ T"(y, w) \quad (26)$$

$$= \bigvee_y^s R(x, y) \wedge \left(\bigvee_z^s S(y, z) \wedge T(z, w) \right) \quad (27)$$

$$= \bigvee_y^s \bigvee_z^s R(x, y) \wedge S(y, z) \wedge T(z, w) \quad (28)$$

$$= \bigvee_z^s \bigvee_y^s R(x, y) \wedge S(y, z) \wedge T(z, w) \quad (29)$$

Proof of 6 (contd.).

$$= \bigvee_z^s \bigvee_y^s R(x, y) \underset{\circ}{\wedge} S(y, z) \underset{\circ}{\wedge} T(z, w) \quad (30)$$

$$= \bigvee_z^s \left(\bigvee_y^s R(x, y) \underset{\circ}{\wedge} S(y, z) \right) \underset{\circ}{\wedge} T(z, w) \quad (31)$$

$$= \bigvee_z^s "R \underset{\circ}{\wedge} S"(x, z) \underset{\circ}{\wedge} T(z, w) \quad (32)$$

$$= "R \underset{\circ}{\wedge} S \underset{\circ}{\wedge} T"(x, w) \quad (33)$$

Proof of (24) and (25) is similar (uses the distributivity law), only shorter. See [Navara and Olšák, 2001] for details.

Extensions: Sometimes it is useful to consider...

- ...a *ε -reflective* relation

$$R(\mathbf{x}, \mathbf{x}) \geq \varepsilon \quad (34)$$

- ...a *weakly reflexive* relation

$$R(\mathbf{x}, \mathbf{y}) \leq R(\mathbf{x}, \mathbf{x}) \text{ and } R(\mathbf{y}, \mathbf{x}) \leq R(\mathbf{x}, \mathbf{x}) \text{ for all } \mathbf{x}, \mathbf{y} \quad (35)$$

- Relation is 1-reflective iff reflexive.
- If a relation is reflexive, then it is weakly reflexive.

Extensions: Sometimes it is useful to consider...

- ...a *non-involutive negation* by refusing (N2)

$$\underset{\circ}{\neg} \underset{\circ}{\neg} \alpha \neq \alpha$$

and adopting a weaker axiom

$$\underset{\circ}{\neg} \underset{\circ}{\neg} \mathbf{0} = \mathbf{1} \text{ and } \underset{\circ}{\neg} \underset{\circ}{\neg} \mathbf{1} = \mathbf{0} \quad (\text{N0})$$

Example

Gödel negation

$$\underset{\text{G}}{\neg} \alpha = \begin{cases} \mathbf{1} & \alpha = \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases} \quad (36)$$

Bibliography



Navara, M. and Olšák, P. (2001).

Základy fuzzy množin.

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