AE4M33RZN, Fuzzy logic:
Introduction, Fuzzy operators

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Plan of the lecture

Description logics

```
<Ontology ontologyIRI="http://example.com/tea.owl" ...>
  <Prefix name="owl" IRI="http://www.w3.org/2002/07/owl#"/>
  <Declaration>
    <Class IRI="Tea"/>
  </Declaration>
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- A *description logic* is a decideable fragment of *first order logic* (FOL).

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- A *description logic* is a decideable fragment of *first order logic* (FOL).

+ Uses *concepts*, *roles* and *individuals* to capture **structured knowledge**.

– Does not capture imprecise or vague information. (See an example in a minute.)

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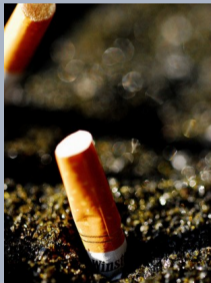- + Even unlikely events (tossing *head* 100 times in a row) can be processed.

(Photo by ICMA Photos
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- GPM is an **efficient representation** of large probability distributions.
+ Captures *uncertainty* well.
+ Even unlikely events (tossing *head* 100 times in a row) can be processed.
– Cannot formulate complex statements **explicitly**, such as "**Every** object in the database has **at least one...**"

To illustrate the limitations of DL and GPM, consider an example from [Domingos and Lowd, 2009].

(Image: Matthew Romack under the CC-BY-SA 2.0.)

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Obervation 1: High-school experience.
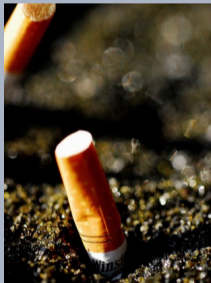
People start or stop smoking in groups of friends.

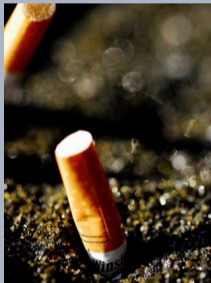To illustrate the limitations of DL and GPM, consider an example from [Domingos and Lowd, 2009].

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Obervation 1: High-school experience.

People start or stop smoking in groups of friends.

Obervation 2: Six degrees of separation.

Everyone is on average approximately six steps away, by way of introduction, from any other person in the world, so that **a chain of "a friend of a friend" statements can** be made, on average, **to connect any two people in six steps**. [Wikipedia, 2012]

To formalize the example, let's use description logic $\mathscr{ALC}$:

Example: Smoking friends (2)

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If you have a friend, who is a smoker, you are a smoker as well:

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Joining the $\mathsf{friendOf}$ relation 1..6 times gives the *top relation*.

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What is wrong with this model?

- If there is one smoker, the whole world starts smoking. (Formally, an interpretation $\mathscr{I}$ must satisfy $\text{Smoker}^{\mathscr{I}} = \varnothing$ or $\text{Smoker}^{\mathscr{I}} = \Delta$.)

Example: Smoking friends (3)

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- We start from **reasonable assumptions** and arrive at **counter-intuitive** conclusion. What's wrong with our reasoning?
- We would like to express something like

$$(\exists\, \text{friendOf} \cdot \text{Smoker} \sqsubseteq \text{Smoker}) \text{ is "mostly" true.}$$

- Fuzzy logic can do that!

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

Bertrand Russel [Russell, 1923]
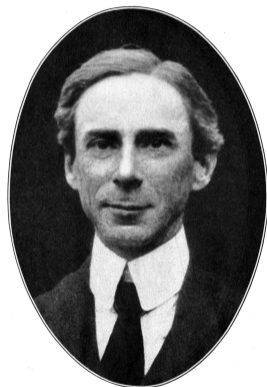
- (Informally:) A *crisp set* („ostrá množina") X is a collection of *objects* $x \in X$ that can be finite, countable or overcountable.

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- We will speak about sets with in relation to a *universe* set („univerzum").
- The universe is usually denoted as Δ.
- Let $\mathbb{P}(\Delta)$ be the *powerset* (a set of all subsets) of Δ (the universe). Then any crisp set is an element in the powerset of its universe: $A \in \mathbb{P}(\Delta)$.

Equivalent ways of describing a *crisp set* in $\mathbb{N}$:

$$A = \{1, 3, 5\} \tag{1}$$

$$A = \{x \in \mathbb{N} \mid x \leq 5 \text{ and } x \text{ is odd}\} \tag{2}$$

Equivalent ways of describing a *crisp set* in $\mathbb{N}$:

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$$A = \{x \in \mathbb{N} \mid x \leq 5 \text{ and } x \text{ is odd}\} \tag{2}$$

$$\mu_A(x) = \begin{cases} 0 & x > 5 \\ 0 & x \text{ is even} \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

μ_A is called the *membership function* („charakteristická funkce", „funkce příslušnosti").

If μ_A is a function $\Delta \rightarrow \{0, 1\}$, the *inverse membership function* μ_A^{-1} returns objects with the given membership degree:

$$\mu_A^{-1}(M) = \{x \in \Delta \mid \mu_A(x) \in M\} \tag{4}$$

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Note

μ_A^{-1} is not an inverse in a strict mathematical sense. The inverse of $\Delta \rightarrow \{0, 1\}$ should be $\{0, 1\} \rightarrow \Delta$, but $\mu_A^{-1} : \mathbb{P}(\{0, 1\}) \rightarrow \mathbb{P}(\Delta)$.

Check your knowledge:

$$\mu_\varnothing = \text{?}$$
$$\mu_\Delta = \text{?}$$
$$\mu^{-1}(\{0, 1\}) = \text{?}$$

Check your knowledge:

$$\mu_\varnothing = 0$$
$$\mu_\Delta = 1$$
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Definition

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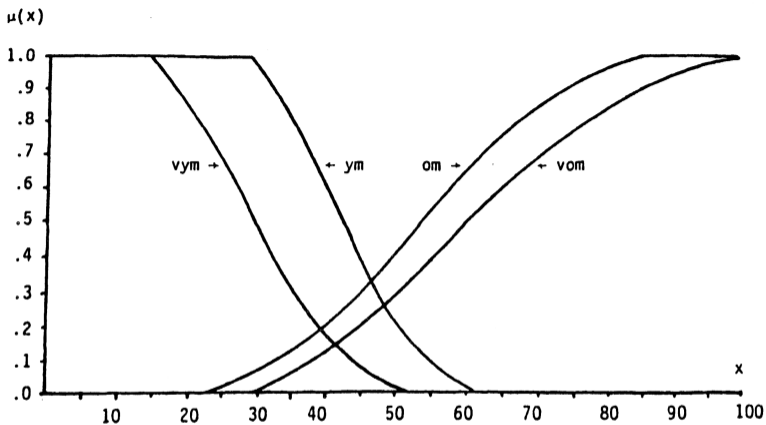The set of all fuzzy subsets of a crisp universe Δ will be denoted as $\mathbb{F}(\Delta)$.

Figure 16–8. Empirical membership functions "Very Young Man," "Young Man," "Old Man," "Very Old Man."

Source: [Zimmermann, 2001]

- *Cardinality* is the size of a fuzzy set.

$$|A| = \sum_{x \in \Delta} A(x) \tag{6}$$

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- *Height* of a fuzzy set is the highest value of the membership function.

$$\mathrm{Height}(A) = \sup \{\alpha \mid x \in \Delta, A(x) = \alpha\} \tag{7}$$

- *Support* („nosič") is the set of objects contained in the fuzzy set "at least a bit".

$$\mathrm{Supp}(A) = \{x \in \Delta \mid A(x) > 0\} = \mu_A^{-1}((0,1]) \tag{8}$$

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- *Core* („jádro") is the set of objects "fully contained" in the fuzzy set.

$$\text{Core}(A) = \{x \in \Delta \mid A(x) = 1\} = \mu_A^{-1}(\{1\}) \tag{9}$$

The inverse membership fn. has the same def. in fuzzy and crisp world:

$$\mu_A^{-1}(M) = \{x \in \Delta \mid A(x) \in M\} \tag{10}$$

If $|M| = 1$, it defines the *α-level* („α-hladina") of a fuzzy set A (it is a crisp set).

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The *α-cut* („*α*-řez") of a fuzzy set A is a crisp set

$$R_A(\alpha) = \{x \in \Delta \mid A(x) \geq \alpha\} = \mu_A^{-1}([\alpha, 1]) \tag{11}$$

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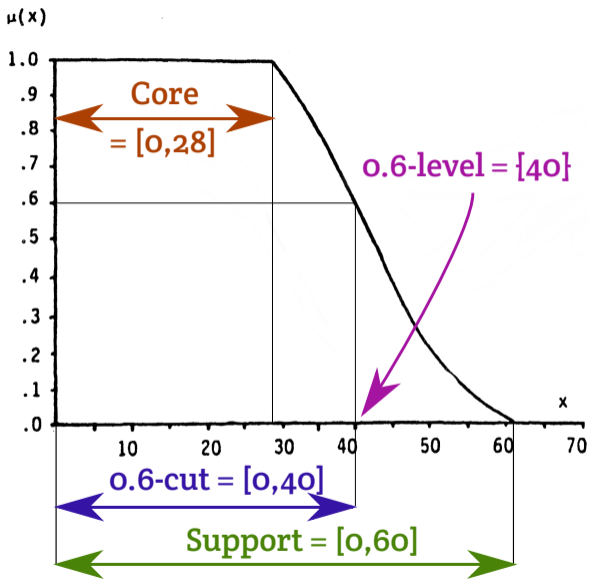$$\mu_A^{-1}(M) = \{x \in \Delta \mid A(x) \in M\} \tag{10}$$

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For better readability $A^{-1}(x) \equiv \mu_A^{-1}(x)$.

The set "Age of Young Men" with its properties.

Check your knowledge:

$$R_A(o) = ?$$
$$\text{Core}(A) = ?$$
$$\text{Height}(A) = \sup \{? \mid R_A \ ?\}$$

Check your knowledge:

$$R_A(0) = \Delta$$
$$\mathrm{Core}(A) = R_A(1)$$
$$\mathrm{Height}(A) = \sup\{\alpha \in [0,1] \mid R_A(\alpha) \neq \varnothing\}$$

- Horizontal representation $\sim$ the α-cuts R.
- Vertical representation $\sim$ the characteristic function μ.

$1 \Rightarrow 2$: From the definition on the previous slide.

$2 \Rightarrow 1$: By taking the "highest" α-level containing x:

$$A(x) = \max\{\alpha \in [0, 1] \mid x \in R_A(\alpha)\} \tag{12}$$

Special cases of fuzzy sets

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Special cases of fuzzy intervals

- *Fuzzy number A* is a fuzzy interval s.t. $|\mathrm{Core}(A)| = 1$
- *Trapezoidal interval* will be denoted by $\langle a, b, c, d \rangle$.
- *Triangular number* will be denoted by $\langle a, b, c \rangle = \langle a, b, b, c \rangle$.
- A *crisp interval* $[a, b]$ is also $\langle a, a, b, b \rangle$.

Operations on fuzzy sets

crisp set operation	propositional operation
$\overline{\cdot} : \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$	$\neg \cdot : [0,1] \to [0,1]$
$\cdot \cap \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$	$\cdot \wedge \cdot : [0,1]^2 \to [0,1]$
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We can use the **logical operators** to define the **set operators**:

$$\overline{A} = \{x \in \Delta \mid \neg(x \in A)\} \tag{LS1}$$

$$A \cap B = \{x \in \Delta \mid (x \in A) \wedge (x \in B)\} \tag{LS2}$$

$$A \cup B = \{x \in \Delta \mid (x \in A) \vee (x \in B)\} \tag{LS3}$$

Therefore we will cover the logical negation, conjunction and disjunction. We get the set operations "for free".

Fuzzy negation is a *non-increasing*, *involutive*, *unary* function
$\neg : [0, 1] \to [0, 1]$ s.t.

$$\text{if } \alpha \leq \beta \text{ then } \neg\beta \leq \neg\alpha \tag{N1}$$

$$\neg\,\neg\,\alpha = \alpha \tag{N2}$$

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$\neg_{\circ} : [0, 1] \to [0, 1]$ s.t.

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$$\neg_{\circ} \neg_{\circ} \alpha = \alpha \tag{N2}$$

Example

Standard („standardní"), Łukasiewicz negation

$$\neg_{S} \alpha = 1 - \alpha \tag{13}$$

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$$\underset{\circ}{\neg} \underset{\circ}{\neg} \alpha = \alpha \tag{N2}$$

Example

Standard („standardní"), Łukasiewicz negation

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The fuzzy set *complement* is a defined using (LS1).

Fuzzy negation: More examples

- Cosine negation

$$\underset{\text{cos}}{\neg}\, \alpha = (\cos(\pi\alpha) + 1)/2 \tag{14}$$

- Sugeno negation

$$\underset{S\lambda}{\neg}\, \alpha = \frac{1 - \alpha}{1 + \lambda\alpha}, \;\; \lambda > -1 \tag{15}$$

- Yager negation

$$\underset{Y\lambda}{\neg}\, \alpha = (1 - \alpha^{\lambda})^{1/\lambda} \tag{16}$$

The axioms (N1) and (N2) imply more properties of fuzzy negations:

Theorem 125

Every fuzzy negation $\underset{\circ}{\neg}$ is a

- continuous
- decreasing
- bijective
- generalization of the propositional negation $\neg$

- **Injective ($f(a) = f(b) \Rightarrow a = b$):** Take 2 values, whose negations are equal: $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$. By (N2) $\alpha = \underset{\circ}{\neg}\overset{\square}{\underset{\circ}{\neg}\alpha}$. The $\square$ can be substituted using the assumption: $\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\underset{\circ}{\neg}\beta$. Using (N1) gives $\underset{\circ}{\neg}\underset{\circ}{\neg}\beta = \beta$. Therefore $\alpha = \beta$.

- **Every non-increasing function (N1) which is injective, must be decreasing.** If $\alpha < \beta$ WLOG, then $\underset{\circ}{\neg}\alpha \geq \underset{\circ}{\neg}\beta$. Then either $\underset{\circ}{\neg}\alpha > \underset{\circ}{\neg}\beta$ and $\underset{\circ}{\neg}$ is decreasing, or $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$, which contradicts the injectivity.

- **Surjective** $\forall y \exists x.f(x) = y$: We seek a value of β for each α s.t. $\alpha = \underset{\circ}{\neg} \beta$. Using injectivity, the condition is equivalent to $\underset{\circ}{\neg} \alpha = \underset{\circ}{\neg} \underset{\circ}{\neg} \beta$. Using (N2), we find the value of β for any α: $\beta = \underset{\circ}{\neg} \alpha$.

- **Bijection** is an injective and surjective function (by definition).

- **Continuous:** Every decreasing bijection is continuous.

- **Boundary values:** Let $\underset{\circ}{\neg} 0 = \alpha$ and suppose that $\alpha < 1$. Then from surjectivity, there must be some other $\beta > 0$ s.t. $\underset{\circ}{\neg} \beta = 1$. This contracits monotonicity, because $\underset{\circ}{\neg} 0 < \underset{\circ}{\neg} \beta$. The other boundary value is proven similarly.

Fuzzy t-norm (triangluar norm, conjunction) is a binary, *comutative*, operation $\underset{\circ}{\wedge}$ s.t.

$$\alpha \underset{\circ}{\wedge} \beta = \beta \underset{\circ}{\wedge} \alpha \tag{T1}$$

$$\alpha \underset{\circ}{\wedge} (\beta \underset{\circ}{\wedge} \gamma) = (\alpha \underset{\circ}{\wedge} \beta) \underset{\circ}{\wedge} \gamma \tag{T2}$$

$$\text{if } \beta \leq \gamma \text{ then } (\alpha \underset{\circ}{\wedge} \beta) \leq (\alpha \underset{\circ}{\wedge} \gamma) \tag{T3}$$

$$(\alpha \underset{\circ}{\wedge} \mathbf{1}) = \alpha \tag{T4}$$

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$$(\alpha \underset{\circ}{\wedge} \mathbf{1}) = \alpha \tag{T4}$$

The fuzzy set *intersection* is a defined using (LS2).

Fuzzy conjunctions: Examples

- Standard (Gödel, Zadeh)

$$\alpha \mathop{\wedge}_{S} \beta = \min\left(\alpha, \beta\right) \tag{17}$$

- Łukasiewicz

$$\alpha \mathop{\wedge}_{L} \beta = \max\left(\alpha + \beta - 1, 0\right) \tag{18}$$

- Algebraic product ("součinová")

$$\alpha \mathop{\wedge}_{A} \beta = \alpha \cdot \beta \tag{19}$$

- Weak ("drastická")

$$\alpha \mathop{\wedge}_{W} \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$
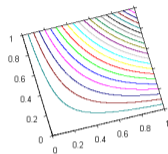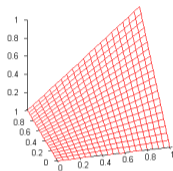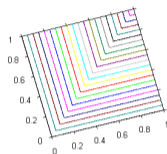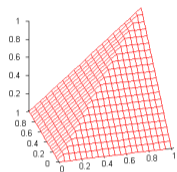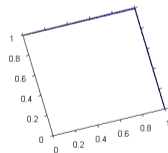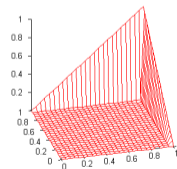
Standard

Algebraic
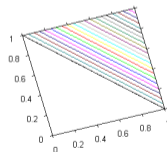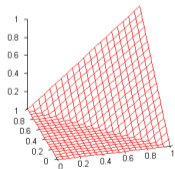
Łukasiewicz

Drastic

Theorem 131

The weak and standard conjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \underset{W}{\wedge} \beta) \leq (\alpha \underset{\circ}{\wedge} \beta) \leq (\alpha \underset{S}{\wedge} \beta) \tag{21}$$

Proof: Assume WLOG $\alpha \leq \beta$.

$\beta = 1$ The condition (T4) gives the same result for all conjunctions.

$\beta < 1$ $\alpha \underset{W}{\wedge} \beta = 0$, which gives the lower bound. The upper bound is rewritten

using the definition of standard conjunction (17): $\alpha \underset{S}{\wedge} \beta = \alpha$. From (T4)

and (T3) follows that $\alpha = \alpha \underset{\circ}{\wedge} 1 \geq \alpha \underset{\circ}{\wedge} \beta$. Together $\alpha \underset{S}{\wedge} \beta = \alpha \geq \alpha \underset{\circ}{\wedge} \beta$.

Theorem 132

The standard conjunction is the only *idempotent* conjunction:

$$\alpha \underset{\circ}{\wedge} \alpha = \alpha \tag{22}$$

Proof: Assume WLOG $\alpha \leq \beta$.

$$\alpha = \alpha \underset{\circ}{\wedge} \alpha \overset{(T3)}{\leq} \alpha \underset{\circ}{\wedge} \beta \overset{(T3)}{\leq} \alpha \underset{\circ}{\wedge} \mathbf{1} \overset{(T4)}{=} \alpha \tag{23}$$

Therefore $\alpha \underset{\circ}{\wedge} \beta = \alpha$. There is only one such conjunction: $\underset{S}{\wedge}$.

Fuzzy s-norm (t-conorm, disjunction) is a binary operation $\overset{\circ}{\vee}$ s.t.

$$\alpha \overset{\circ}{\vee} \beta = \beta \overset{\circ}{\vee} \alpha \tag{S1}$$

$$\alpha \overset{\circ}{\vee} (\beta \overset{\circ}{\vee} \gamma) = (\alpha \overset{\circ}{\vee} \beta) \overset{\circ}{\vee} \gamma \tag{S2}$$

$$\text{if } \beta \leq \gamma \text{ then } (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \gamma) \tag{S3}$$

$$(\alpha \overset{\circ}{\vee} \text{o}) = \alpha \tag{S4}$$

Fuzzy disjunctions (s-norm)

Fuzzy s-norm (t-conorm, disjunction) is a binary operation $\overset{\circ}{\vee}$ s.t.

$$\alpha \overset{\circ}{\vee} \beta = \beta \overset{\circ}{\vee} \alpha \tag{S1}$$

$$\alpha \overset{\circ}{\vee} (\beta \overset{\circ}{\vee} \gamma) = (\alpha \overset{\circ}{\vee} \beta) \overset{\circ}{\vee} \gamma \tag{S2}$$

$$\text{if } \beta \leq \gamma \text{ then } (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \gamma) \tag{S3}$$

$$(\alpha \overset{\circ}{\vee} 0) = \alpha \tag{S4}$$

Union

The fuzzy set *union* is a defined using the disjunction:

$$\mu_{A \cup B}(x) = \mu_A(x) \overset{\circ}{\vee} \mu_B(x) \tag{24}$$

- Standard (Gödel, Zadeh)

$$\alpha \overset{S}{\vee} \beta = \max(\alpha, \beta) \tag{25}$$

- Łukasiewicz

$$\alpha \overset{L}{\vee} \beta = \min(\alpha + \beta, 1) \tag{26}$$

- Algebraic sum („součinová")

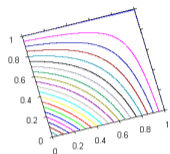$$\alpha \overset{A}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta \tag{27}$$
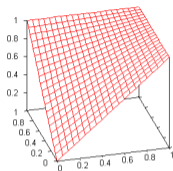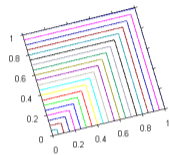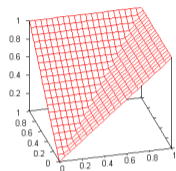
- Weak ("drastická")

$$\alpha \overset{W}{\vee} \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ \beta & \text{if } \alpha = 0 \\ 1 & \text{otherwise} \end{cases} \tag{28}$$
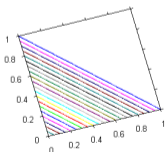
- Einstein

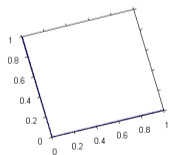$$\alpha \overset{E}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta} \tag{29}$$

Standard

Algebraic
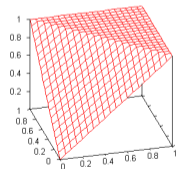
Łukasiewicz

Drastic
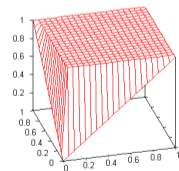
- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{S}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{W}{\vee} \beta) \tag{30}$$

- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{S}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{W}{\vee} \beta) \tag{30}$$

- The standard disjunctions is the only *idempotent* conjunction:

$$\alpha \overset{\circ}{\vee} \alpha = \alpha \tag{31}$$

A If $\underset{\circ}{\wedge}$ is a fuzzy conjunction, then $\alpha \overset{\circ}{\vee} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg} \alpha \underset{\circ}{\wedge} \underset{\circ}{\neg} \beta)$ is a fuzzy disjunction (dual to $\underset{\circ}{\wedge}$ w.r.t. $\underset{\circ}{\neg}$).

A If $\underset{\circ}{\wedge}$ is a fuzzy conjunction, then $\alpha \overset{\circ}{\vee} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg} \alpha \underset{\circ}{\wedge} \underset{\circ}{\neg} \beta)$ is a fuzzy disjunction (dual to $\underset{\circ}{\wedge}$ w.r.t. $\underset{\circ}{\neg}$).

B If $\overset{\circ}{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\circ}{\wedge} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg} \alpha \overset{\circ}{\vee} \underset{\circ}{\neg} \beta)$ is a fuzzy conjunction (dual to $\overset{\circ}{\vee}$ w.r.t. $\underset{\circ}{\neg}$).

Conjunction - disjunction duality

A If $\underset{\circ}{\wedge}$ is a fuzzy conjunction, then $\alpha \overset{\circ}{\vee} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg} \alpha \underset{\circ}{\wedge} \underset{\circ}{\neg} \beta)$ is a fuzzy disjunction (dual to $\underset{\circ}{\wedge}$ w.r.t. $\underset{\circ}{\neg}$).

B If $\overset{\circ}{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\circ}{\wedge} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg} \alpha \overset{\circ}{\vee} \underset{\circ}{\neg} \beta)$ is a fuzzy conjunction (dual to $\overset{\circ}{\vee}$ w.r.t. $\underset{\circ}{\neg}$).

Theorems

- **Łukasiewicz** operations $\underset{L}{\wedge}$, $\overset{L}{\vee}$ are dual w.r.t. **standard** negation.

- **Algebraic** operations $\underset{A}{\wedge}$, $\overset{A}{\vee}$ are dual w.r.t. **standard** negation.

- **Standard** operations $\underset{S}{\wedge}$, $\overset{S}{\vee}$ are dual w.r.t. **any** negation.

- **Weak** operations $\underset{W}{\wedge}$, $\overset{W}{\vee}$ are dual w.r.t. **any** negation.

Table 16–2. Empirically determined grades of membership.

Stimulus x		$\mu_M(x)$	$\mu_C(x)$	$\mu_{M \cap C}(x)$
1.	bag	0.000	0.985	0.007
2.	baking tin	0.908	0.419	0.517
3.	ballpoint pen	0.215	0.149	0.170
4.	bathtub	0.552	0.804	0.674
5.	book wrapper	0.023	0.454	0.007
6.	car	0.501	0.437	0.493
7.	cash register	0.692	0.400	0.537
8.	container	0.847	1.000	1.000
9.	fridge	0.424	0.623	0.460
10.	Hollywood swing	0.318	0.212	0.142
11.	kerosene lamp	0.481	0.310	0.401
12.	nail	1.000	0.000	0.000
13.	parkometer	0.663	0.335	0.437
14.	pram	0.283	0.448	0.239
15.	press	0.130	0.512	0.101
16.	shovel	0.325	0.239	0.301
17.	silver spoon	0.969	0.256	0.330
18.	sledgehammer	0.480	0.012	0.023
19.	water bottle	0.564	0.961	0.714
20.	wine barrel	0.127	0.980	0.185

Degree of membership for 20 items into the sets "make of metal", "being a container" and "being a metalic container".
[Zimmermann, 2001]
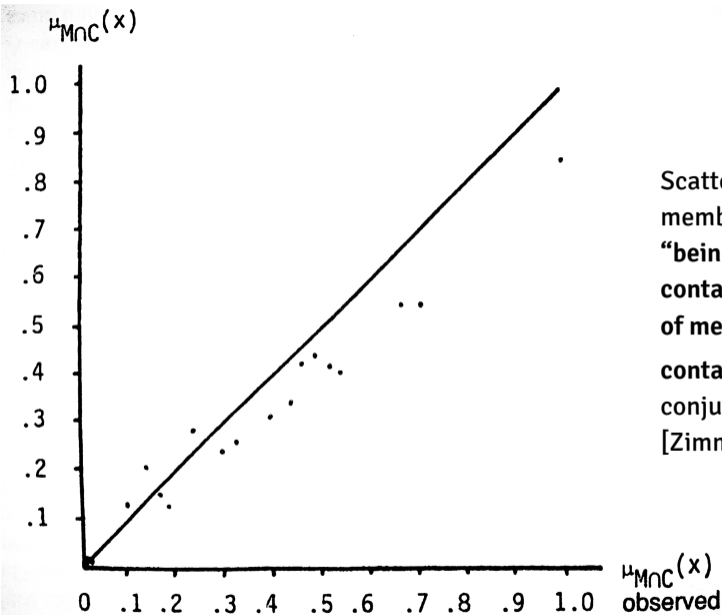
Scatterplot of membership degree for **"being a metalic container"** vs. **"make of metal"** $\wedge$ **"being a container"**. (standard conjunction). [Zimmermann, 2001]
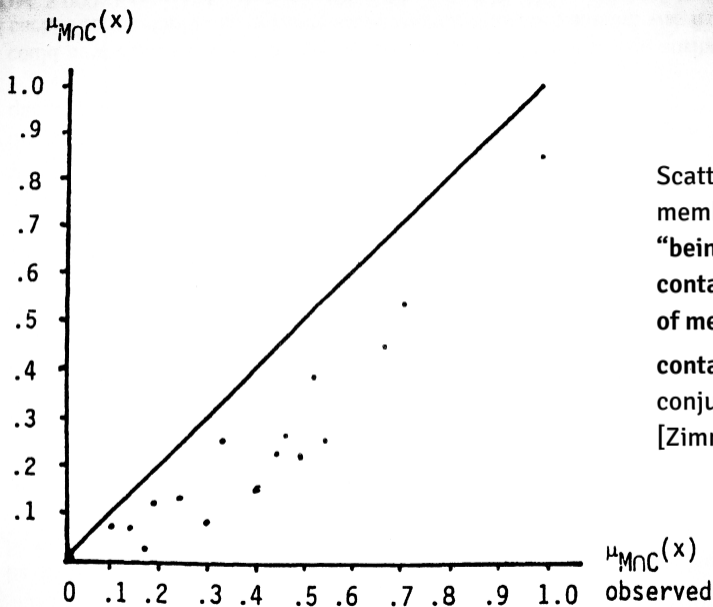
Scatterplot of membership degree for **"being a metalic container"** vs. **"make of metal"** $\wedge$ **"being a container"** (algebraic conjunction). [Zimmermann, 2001]

Criteria for selecting operators (1)

1. **Axiomatic strength:** The set of valid theorems may differ based on the choice of t-norms and s-norms (see tutorials).
2. **Empirical fit:** Using fuzzy theory for a model of the real world, the chosen operator should match the real behavior of the system.
3. **Adaptability:** Operators in a generic system should be able to fit several use cases. One way of increasing adaptibility is to use operators with parameters (e.g. Yager and Sugeno negations).

4. **Computational efficiency:** Evaluating e.g. the standard negation is usually faster than the Yager negation, which contains the power.

5. **Aggregating behavior:** When the operators combines a large number of operands, does the value tends to go to 0 (conjunction) or 1 (disjunction). The standard operators behave differently than the algebraic ones.

Bibliography

📄 Domingos, P. and Lowd, D. (2009).
Markov Logic: An Interface Layer for Artificial Intelligence.
Morgan & Claypool.

📄 Russell, B. (1923).
Vagueness.
In Australasian Journal of Psychology and Philosophy, page 84–92.

📄 Wikipedia (2012).
Six degrees of separation — Wikipedia, the free encyclopedia.
[Online; accessed 15-October-2012].

📄 Zimmermann, H.-J. (2001).
Fuzzy Set Theory and its Applications.
Springer.