

# AE4M33RZN, Fuzzy logic: Introduction, Fuzzy operators

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# Plan of the lecture

## Introduction

- AMKR course so far
- Criticism of both approaches

## Basic definitions

- Crisp sets
- Vertical representation
- Horizontal representation
- Special cases of fuzzy sets

## Operations on fuzzy sets

- Negation
- Conjunction
- Disjunction
- Conjunction - disjunction duality
- Criteria for selecting operators

## Bibliography



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<Ontology ontologyIRI="http://example.com/tea.owl" ...>
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  <Declaration>
    <Class IRI="Tea"/>
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- A *description logic* is a decidable fragment of *first order logic* (FOL).
- + Uses *concepts*, *roles* and *individuals* to capture structured knowledge.
- An unexpected fact in the *Box* might lead to a *contradiction*, which is a pain.  
(See an example in a minute.)



(Photo by ICMA Photos  
under the CC-BY-SA 2.0.)

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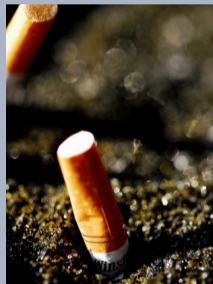


(Photo by ICMA Photos  
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- GPM is an **efficient representation** of large probability distributions.
- + Captures *uncertainty* well.
- + Even unlikely events (tossing *head* 100 times in a row) can be processed.
- Cannot formulate complex statements **explicitly**, such as “**Every** object in the database has **at least one...**”

## Example: Smoking friends (1)

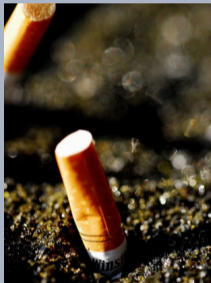
To illustrate the limitations of DL and GPM, consider an example from [?].



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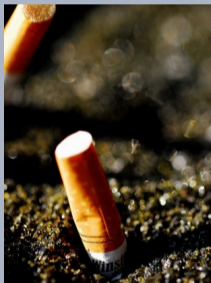
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**Observation 1: High-school experience.**

People start or stop smoking in groups of friends.

## Example: Smoking friends (1)

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(Image: Matthew Romack under the CC-BY-SA 2.0.)

**Observation 1: High-school experience.**

People start or stop smoking in groups of friends.

**Observation 2: Six degrees of separation.**

Everyone is on average approximately six steps away, by way of introduction, from any other person in the world, so that a chain of “a friend of a friend” **statements can be made, on average, to connect any two people in six steps. [?]**

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Joining the **friendOf** relation 6 times  
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What is wrong with this model?



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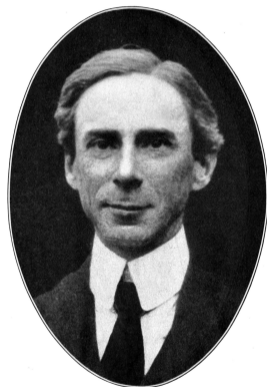
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- We start from **reasonable assumptions** and arrive at **counter-intuitive** conclusion. What's wrong with our reasoning?
- We would like to express something like  
 $(\exists \text{friendOf} \cdot \text{Smoker} \sqsubseteq \text{Smoker})$  is “mostly” true.
- Fuzzy logic can do that!

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

*Bertrand Russel [?]*



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- The universe is usually denoted as  $\Delta$ .
- Let  $\mathbb{P}(\Delta)$  be the *powerset* (a set of all subsets) of  $\Delta$  (the universe). Then any crisp set is an element in the powerset of its universe:  $A \in \mathbb{P}(\Delta)$ .

Equivalent ways of describing a *crisp set* in  $\mathbb{N}$ :

$$A = \{1, 3, 5\} \quad (1)$$

$$A = \{x \in \mathbb{N} \mid x \leq 5 \text{ and } x \text{ is odd}\} \quad (2)$$



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$$\mu_A(x) = \begin{cases} 0 & x > 5 \\ 0 & x \text{ is even} \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

$\mu_A$  is called the *membership function* („charakteristická funkce“, „funkce příslušnosti“).

## Membership function

If  $\mu_A$  is a function  $\Delta \rightarrow \{0, 1\}$ , the *inverse membership function*  $\mu_A^{-1}$  returns objects with the given membership degree:

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## Note

$\mu_A^{-1}$  is not an inverse in a strict mathematical sense. The inverse of  $\Delta \rightarrow \{0, 1\}$  should be  $\{0, 1\} \rightarrow \Delta$ , but  $\mu_A^{-1} : \mathbb{P}(\{0, 1\}) \rightarrow \mathbb{P}(\Delta)$ .

Check your knowledge:

$$\mu_{\emptyset} = ?$$

$$\mu_{\Delta} = ?$$

$$\mu^{-1}(\{\mathbf{0}, \mathbf{1}\}) = ?$$

Check your knowledge:

$$\mu_{\emptyset} = \mathbf{0}$$

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## Definition

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The set of all fuzzy subsets of a crisp universe  $\Delta$  will be denoted as  $\mathbb{F}(\Delta)$ .

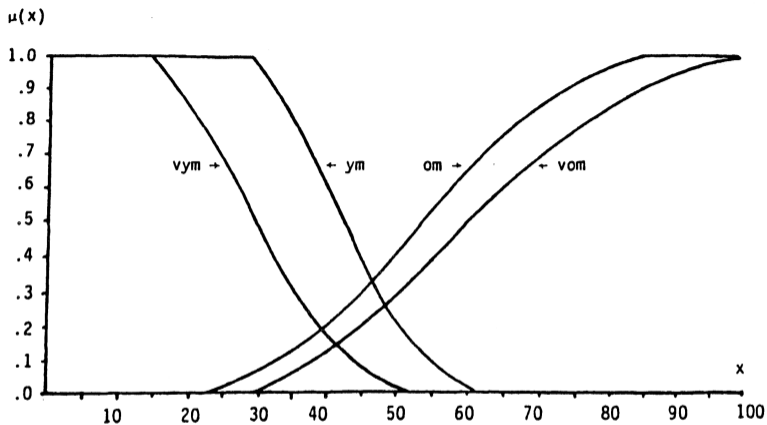


Figure 16–8. Empirical membership functions “Very Young Man,” “Young Man,” “Old Man,” “Very Old Man.”

Source: [?]

- *Cardinality* is the size of a fuzzy set.

$$|A| = \sum_{x \in \Delta} A(x) \quad (6)$$

## Fuzzy set: Properties (1)

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- *Height* of a fuzzy set is the highest value of the membership function.

$$\text{Height}(A) = \sup \{ \alpha \mid x \in \Delta, A(x) = \alpha \} \quad (7)$$

- *Support* („nosič“) is the set of objects contained in the fuzzy set “at least a bit”.

$$\text{Supp}(A) = \{x \in \Delta \mid A(x) > 0\} = \mu_A^{-1}((0, 1]) \quad (8)$$

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- **Core** („jádro“) is the set of objects “fully contained” in the fuzzy set.

$$\text{Core}(A) = \{x \in \Delta \mid A(x) = 1\} = \mu_A^{-1}(\{1\}) \quad (9)$$

The  $\alpha$ -level („ $\alpha$ -hladina“) of a fuzzy set  $A$  is a crisp set

$$\mu_A^{-1}(M) = \{x \in \Delta \mid A(x) \in M\} \quad (10)$$

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$$R_A(\alpha) = \{x \in \Delta \mid A(x) \geq \alpha\} = \mu_A^{-1}([\alpha, 1]) \quad (11)$$

Sometimes we speak about a *strong*  $\alpha$ -cut („ostrý  $\alpha$ -řez“), where  $\geq$  in the definition is replaced by  $>$ .



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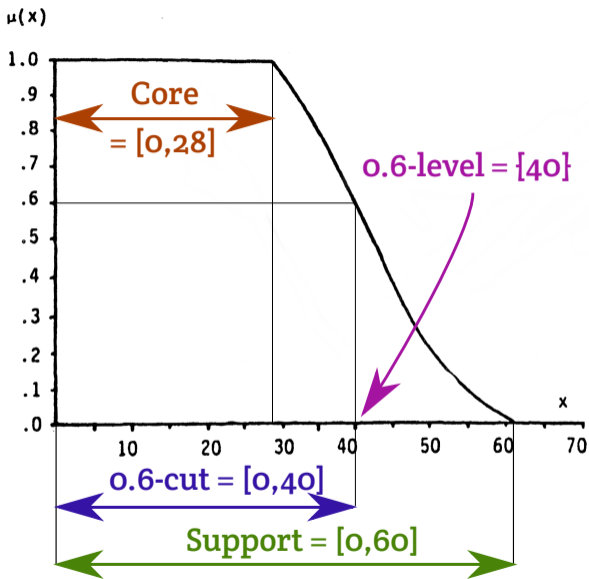
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For better readability  $A^{-1}(x) \equiv \mu_A^{-1}(x)$ .



The set “Age of Young Men” with its properties.

Check your knowledge:

$$R_A(o) = ?$$

$$\text{Core}(A) = ?$$

$$\text{Height}(A) = \sup \{ ? \mid R_A ? \}$$

Check your knowledge:

$$R_A(\mathbf{0}) = \Delta$$

$$\text{Core}(A) = R_A(\mathbf{1})$$

$$\text{Height}(A) = \sup \{ \alpha \in [0, 1] \mid R_A(\alpha) \neq \emptyset \}$$

# Converting vertical and horizontal representation

- Horizontal representation  $\sim$  the  $\alpha$ -cuts  $R$ .
- Vertical representation  $\sim$  the characteristic function  $\mu$ .

$1 \Rightarrow 2$  : From the definition on the previous slide.

$2 \Rightarrow 1$  : By taking the “highest”  $\alpha$ -level containing  $x$ :

$$A(x) = \max\{\alpha \in [0, 1] \mid x \in R_A(\alpha)\} \quad (12)$$

# Special cases of fuzzy sets

## Definition

*Fuzzy interval*  $A$  is a fuzzy set on  $\Delta = \mathbb{R}$  s.t.

- $R_A(\alpha)$  is a closed interval for all  $\alpha \in [0, 1]$
- $R_A(1)$  is not empty.
- $|\text{Supp}(A)|$  is finite (has a maximum and a minimum).

## Special cases of fuzzy intervals

- *Fuzzy number*  $A$  is a fuzzy interval s.t.  $|\text{Core}(A)| = 1$
- *Trapezoidal interval* will be denoted by  $\langle a, b, c, d \rangle$ .
- *Triangular number* will be denoted by  $\langle a, b, c \rangle = \langle a, b, b, c \rangle$ .
- A *crisp interval*  $[a, b]$  is also  $\langle a, a, b, b \rangle$ .

# Operations on fuzzy sets

## crisp set operation

$$\bar{\cdot} : \mathbb{P}(\Delta) \rightarrow \mathbb{P}(\Delta)$$

$$\cdot \cap \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \rightarrow \mathbb{P}(\Delta)$$

$$\cdot \cup \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \rightarrow \mathbb{P}(\Delta)$$

## propositional operation

$$\neg \cdot : \{0,1\} \rightarrow \{0,1\}$$

$$\cdot \wedge \cdot : \{0,1\}^2 \rightarrow \{0,1\}$$

$$\cdot \vee \cdot : \{0,1\}^2 \rightarrow \{0,1\}$$

# Operations on fuzzy sets

crisp set operation	propositional operation
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$\cdot \cup \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \rightarrow \mathbb{P}(\Delta)$	$\cdot \vee \cdot : \{0,1\}^2 \rightarrow \{0,1\}$

We can use the **logical operators** to define the **set operators**:

$$\bar{A} = \{x \in \Delta \mid \neg(x \in A)\} \quad (\text{LS1})$$

$$A \cap B = \{x \in \Delta \mid (x \in A) \wedge (x \in B)\} \quad (\text{LS2})$$

$$A \cup B = \{x \in \Delta \mid (x \in A) \vee (x \in B)\} \quad (\text{LS3})$$

Therefore we will cover the logical negation, conjunction and disjunction. We get the set operations “for free”.



## Fuzzy negation

*Fuzzy negation is a non-increasing, involutive, unary function*

$\neg : [0,1] \rightarrow [0,1]$  s.t.

$$\text{if } \alpha \leq \beta \text{ then } \neg \beta \leq \neg \alpha \quad (\text{N1})$$

$$\neg \neg \alpha = \alpha \quad (\text{N2})$$

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## Example

Standard („standardní“), Łukasiewicz negation

$$\neg_S \alpha = 1 - \alpha \quad (13)$$

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Standard („standardní“), Łukasiewicz negation

$$\neg_S \alpha = 1 - \alpha \quad (13)$$

The fuzzy set *complement* is defined using (LS1).

## Fuzzy negation: More examples

- Cosine negation

$$\neg_{\cos} \alpha = (\cos(\pi\alpha) + 1)/2 \quad (14)$$

- Sugeno negation

$$\neg_{s\lambda} \alpha = \frac{1 - \alpha}{1 + \lambda\alpha}, \quad \lambda > -1 \quad (15)$$

- Yager negation

$$\neg_{Y\lambda} \alpha = (1 - \alpha^\lambda)^{1/\lambda} \quad (16)$$

The axioms (N1) and (N2) imply more properties of fuzzy negations:

## Theorem 125

Every fuzzy negation  $\neg_{\circ}$  is a

- continuous
- decreasing
- bijective
- generalization of the propositional negation  $\neg$

## Fuzzy negation: Proof of 125

- **Injective ( $f(a) = f(b) \Rightarrow a = b$ ):** Take 2 values, whose negations are equal:  $\neg \alpha = \neg \beta$ . By (N2)  $\alpha = \neg \neg \alpha$ . The  $\square$  can be substituted using the assumption:  $\neg \neg \alpha = \neg \neg \beta$ . Using (N1) gives  $\neg \neg \beta = \beta$ . Therefore  $\alpha = \beta$ .
- **Every non-increasing function (N1) which is injective, must be decreasing.** If  $\alpha < \beta$  WLOG, then  $\neg \alpha \geq \neg \beta$ . Then either  $\neg \alpha > \neg \beta$  and  $\neg$  is decreasing, or  $\neg \alpha = \neg \beta$ , which contradicts the injectivity.

## Fuzzy negation: Proof of 125

- **Surjective**  $\forall y \exists x. f(x) = y$ : We seek a value of  $\beta$  for each  $\alpha$  s.t.  $\alpha = \neg \beta$ .  
Using injectivity, the condition is equivalent to  $\neg \alpha = \neg \neg \beta$ . Using (N2), we find the value of  $\beta$  for any  $\alpha$ :  $\beta = \neg \alpha$ .
- **Bijection** is an injective and surjective function (by definition).
- **Continuous**: Every decreasing bijection is continuous.
- **Boundary values**: Let  $\neg 0 = \alpha$  and suppose that  $\alpha < 1$ . Then from surjectivity, there must be some other  $\beta > 0$  s.t.  $\neg \beta = 1$ . This contradicts monotonicity, because  $\neg 0 < \neg \beta$ . The other boundary value is proven similarly.

## Fuzzy conjunctions (t-norms)

*Fuzzy t-norm* (triangular norm, conjunction) is a binary, *comutative*, operation  $\underset{\circ}{\wedge}$  s.t.

$$\alpha \underset{\circ}{\wedge} \beta = \beta \underset{\circ}{\wedge} \alpha \quad (\text{T1})$$

$$\alpha \underset{\circ}{\wedge} (\beta \underset{\circ}{\wedge} \gamma) = (\alpha \underset{\circ}{\wedge} \beta) \underset{\circ}{\wedge} \gamma \quad (\text{T2})$$

$$\text{if } \beta \leq \gamma \text{ then } (\alpha \underset{\circ}{\wedge} \beta) \leq (\alpha \underset{\circ}{\wedge} \gamma) \quad (\text{T3})$$

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$$(\alpha \underset{\circ}{\wedge} 1) = \alpha \quad (\text{T4})$$

The fuzzy set *intersection* is defined using (LS2).

## Fuzzy conjunctions: Examples

- Standard (Gödel, Zadeh)

$$\alpha \underset{S}{\wedge} \beta = \min(\alpha, \beta) \quad (17)$$

- Łukasiewicz

$$\alpha \underset{L}{\wedge} \beta = \max(\alpha + \beta - 1, 0) \quad (18)$$

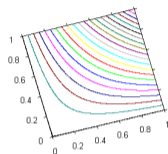
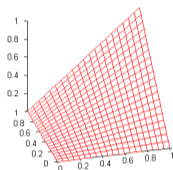
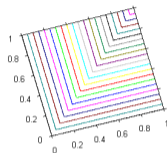
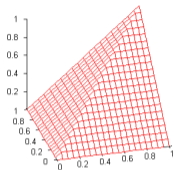
- Algebraic product („součinnová“)

$$\alpha \underset{A}{\wedge} \beta = \alpha \cdot \beta \quad (19)$$

- Weak („drastická“)

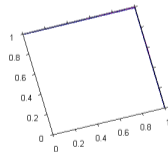
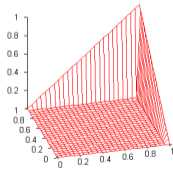
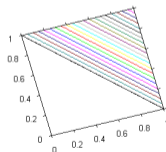
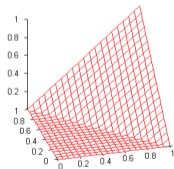
$$\alpha \underset{W}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \quad (20)$$

# Fuzzy conjunctions: Visualization [Wikipedia]



Standard

Algebraic



Łukasiewicz

Drastic

# Fuzzy conjunctions: Properties (1)

## Theorem 131

The weak and standard conjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \underset{W}{\wedge} \beta) \leq (\alpha \underset{\circ}{\wedge} \beta) \leq (\alpha \underset{S}{\wedge} \beta) \quad (21)$$

**Proof:** Assume WLOG  $\alpha \leq \beta$ .

$\beta = 1$  The condition (T4) gives the same result for all conjunctions.

$\beta < 1$   $\alpha \underset{W}{\wedge} \beta = 0$ , which gives the lower bound. The upper bound is rewritten using the definition of standard conjunction (17):  $\alpha \underset{S}{\wedge} \beta = \alpha$ . From (T4) follows that  $\alpha = \alpha \underset{\circ}{\wedge} 1 \geq \alpha \underset{\circ}{\wedge} \beta$ . Together  $\alpha \underset{S}{\wedge} \beta = \alpha \geq \alpha \underset{\circ}{\wedge} \beta$ .

## Fuzzy conjunctions: Properties (2)

### Theorem 132

The standard conjunction is the only *idempotent* conjunction:

$$\alpha \underset{\circ}{\wedge} \alpha = \alpha \quad (22)$$

**Proof:** Assume WLOG  $\alpha \leq \beta$ .

$$\alpha = \alpha \underset{\circ}{\wedge} \alpha \stackrel{(T3)}{\leq} \alpha \underset{\circ}{\wedge} \beta \stackrel{(T3)}{\leq} \alpha \underset{\circ}{\wedge} \mathbf{1} \stackrel{(T4)}{=} \alpha \quad (23)$$

Therefore  $\alpha \underset{\circ}{\wedge} \beta = \alpha$ . There is only one such conjunction:  $\underset{\circ}{\wedge}$ .

## Fuzzy disjunctions (s-norm)

*Fuzzy s-norm* (t-conorm, disjunction) is a binary operation  $\overset{\circ}{\vee}$  s.t.

$$\alpha \overset{\circ}{\vee} \beta = \beta \overset{\circ}{\vee} \alpha \quad (\text{S1})$$

$$\alpha \overset{\circ}{\vee} (\beta \overset{\circ}{\vee} \gamma) = (\alpha \overset{\circ}{\vee} \beta) \overset{\circ}{\vee} \gamma \quad (\text{S2})$$

$$\text{if } \beta \leq \gamma \text{ then } (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \gamma) \quad (\text{S3})$$

$$(\alpha \overset{\circ}{\vee} \mathbf{o}) = \alpha \quad (\text{S4})$$

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$$(\alpha \overset{\circ}{\vee} \mathbf{o}) = \alpha \quad (\text{S4})$$

### Union

The fuzzy set *union* is defined using the disjunction:

$$\mu_{A \cup B}(x) = \mu_A(x) \overset{\circ}{\vee} \mu_B(x) \quad (24)$$

## Fuzzy disjunctions: Examples (1)

- Standard (Gödel, Zadeh)

$$\alpha \overset{S}{\vee} \beta = \max(\alpha, \beta) \quad (25)$$

- Łukasiewicz

$$\alpha \overset{L}{\vee} \beta = \min(\alpha + \beta, 1) \quad (26)$$

- Algebraic sum („součinná“)

$$\alpha \overset{A}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta \quad (27)$$



## Fuzzy disjunctions: Examples (2)

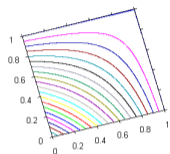
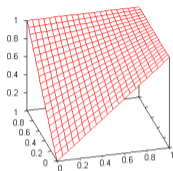
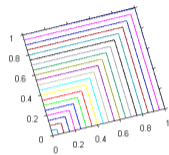
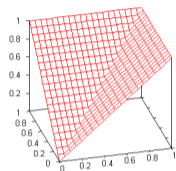
- Weak („drastická“)

$$\alpha \overset{W}{\vee} \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ \beta & \text{if } \alpha = 0 \\ 1 & \text{otherwise} \end{cases} \quad (28)$$

- Einstein

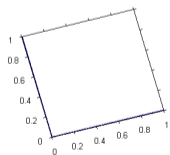
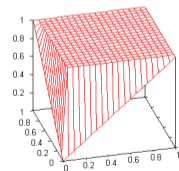
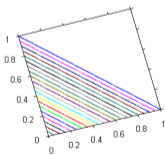
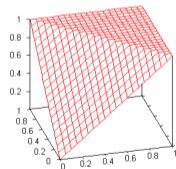
$$\alpha \overset{E}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta} \quad (29)$$

# Fuzzy disjunctions: Visualization [Wikipedia]



Standard

Algebraic



Łukasiewicz

Drastic

- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{S}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{W}{\vee} \beta) \quad (30)$$

- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{S}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{W}{\vee} \beta) \quad (30)$$

- The standard disjunction is the only *idempotent* conjunction:

$$\alpha \overset{\circ}{\vee} \alpha = \alpha \quad (31)$$

## Conjunction - disjunction duality

A If  $\overset{\circ}{\wedge}$  is a fuzzy conjunction, then  $\alpha \overset{\circ}{\vee} \beta = \neg(\neg \alpha \overset{\circ}{\wedge} \neg \beta)$  is a fuzzy disjunction (dual to  $\overset{\circ}{\wedge}$  w.r.t.  $\neg$ ).

## Conjunction - disjunction duality

- A If  $\overset{\circ}{\wedge}$  is a fuzzy conjunction, then  $\alpha \overset{\circ}{\vee} \beta = \neg(\neg \alpha \overset{\circ}{\wedge} \neg \beta)$  is a fuzzy disjunction (dual to  $\overset{\circ}{\wedge}$  w.r.t.  $\neg$ ).
- B If  $\overset{\circ}{\vee}$  is a fuzzy disjunction, then  $\alpha \overset{\circ}{\wedge} \beta = \neg(\neg \alpha \overset{\circ}{\vee} \neg \beta)$  is a fuzzy conjunction (dual to  $\overset{\circ}{\vee}$  w.r.t.  $\neg$ ).

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- B If  $\overset{\circ}{\vee}$  is a fuzzy disjunction, then  $\alpha \overset{\circ}{\wedge} \beta = \neg(\neg \alpha \overset{\circ}{\vee} \neg \beta)$  is a fuzzy conjunction (dual to  $\overset{\circ}{\vee}$  w.r.t.  $\neg$ ).

### Theorems

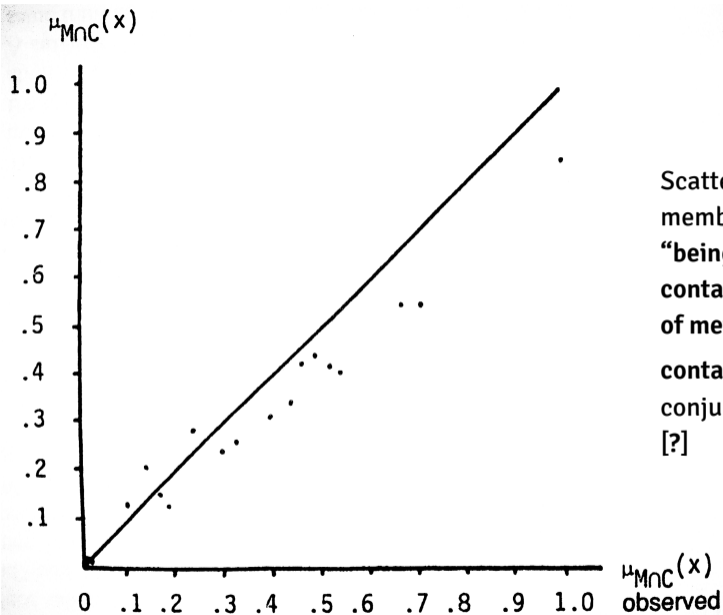
- Łukasiewicz operations  $\overset{L}{\wedge}, \overset{L}{\vee}$  are dual w.r.t. **standard** negation.
- Algebraic operations  $\overset{A}{\wedge}, \overset{A}{\vee}$  are dual w.r.t. **standard** negation.
- Standard operations  $\overset{S}{\wedge}, \overset{S}{\vee}$  are dual w.r.t. **any** negation.
- Weak operations  $\overset{W}{\wedge}, \overset{W}{\vee}$  are dual w.r.t. **any** negation.

Table 16–2. Empirically determined grades of membership.

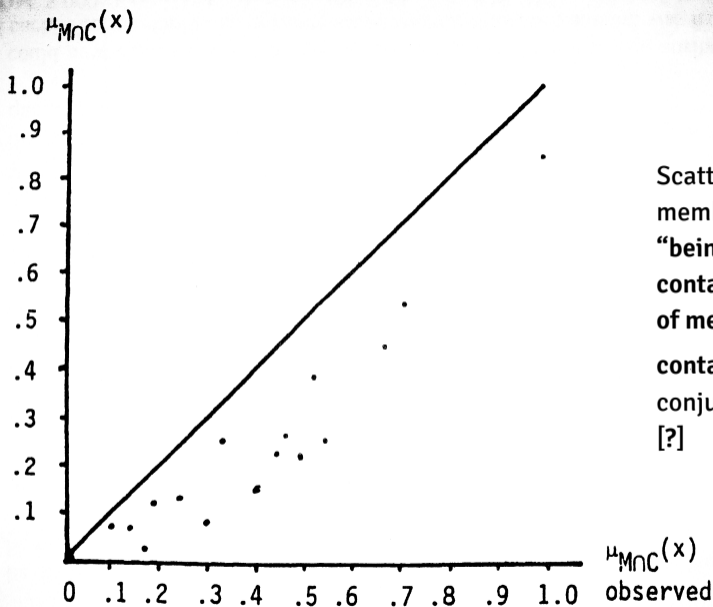
<i>Stimulus x</i>	$\mu_M(x)$	$\mu_C(x)$	$\mu_{MnC}(x)$
1. bag	0.000	0.985	0.007
2. baking tin	0.908	0.419	0.517
3. ballpoint pen	0.215	0.149	0.170
4. bathtub	0.552	0.804	0.674
5. book wrapper	0.023	0.454	0.007
6. car	0.501	0.437	0.493
7. cash register	0.692	0.400	0.537
8. container	0.847	1.000	1.000
9. fridge	0.424	0.623	0.460
10. Hollywood swing	0.318	0.212	0.142
11. kerosene lamp	0.481	0.310	0.401
12. nail	1.000	0.000	0.000
13. parkometer	0.663	0.335	0.437
14. pram	0.283	0.448	0.239
15. press	0.130	0.512	0.101
16. shovel	0.325	0.239	0.301
17. silver spoon	0.969	0.256	0.330
18. sledgehammer	0.480	0.012	0.023
19. water bottle	0.564	0.961	0.714
20. wine barrel	0.127	0.980	0.185

Degree of membership for 20 items into the sets “make of metal”, “being a container” and “being a metallic container”.  
[?]





Scatterplot of membership degree for "being a metallic container" vs. "make of metal"  $\wedge$  "being a container". (standard conjunction).  
[?]



Scatterplot of membership degree for “being a metallic container” vs. “make of metal”  $\hat{\wedge}$  “being a container” (algebraic conjunction).  
[?]

## Criteria for selecting operators (1)

1. **Axiomatic strength:** The set of valid theorems may differ based on the choice of t-norms and s-norms (see tutorials).
2. **Empirical fit:** Using fuzzy theory for a model of the real world, the chosen operator should match the real behavior of the system.
3. **Adaptability:** Operators in a generic system should be able to fit several use cases. One way of increasing adaptability is to use operators with parameters (e.g. Yager and Sugeno negations).

## Criteria for selecting operators (2)

4. **Computational efficiency:** Evaluating e.g. the standard negation is usually faster than the Yager negation, which contains the power.
5. **Aggregating behavior:** When the operators combines a large number of operands, does the value tends to go to 0 (conjunction) or 1 (disjunction). The standard operators behave differently than the algebraic ones.

# Bibliography