# AE4M33RZN, Fuzzy logic: Introduction, Fuzzy operators

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21/10/2013

# Description logics

```
<Ontology ontologyIRI="http://example.com/tea.owl" ...>
  <Prefix name="owl" IRI="http://www.w3.org/2002/07/owl#"/>
  <Declaration>
    <Class IRI="Tea"/>
  </Declaration>
</Ontology>
```

- A *description logic* is a decideable fragment of *first order logic* (FOL).
- Uses *concepts*, *roles* and *individuals* to capture **structured knowledge**.
- An unexpected fact in the $\mathscr{A}$*Box* might lead to a *contradiction*, which is a pain. (See an example in a minute.)

# Graphical probabilistic models

(Photo by ICMA Photos


- GPM is an **efficient representation** of large probability distributions.
- \+ Captures *uncertainty* well.
- \+ Even unlikely events (tossing *head* 100 times in a row) can be processed.
- – Cannot formulate complex statements **explicitly**, such as “**Every** object in the database has **at least** one...”

# Example: Smoking friends (1)

To illustrate the limitations of DL and GPM, consider an example from [Domingos and Lowd, 2009].




### Obervation 1: High-school experience.

People start or stop smoking in groups of friends.

### Obervation 2: Six degrees of separation.

Everyone is on average approximately six steps away, by way of introduction, from any other person in the world, so that **a chain of "a friend of a friend" statements can** be made, on average, **to connect any two people in six steps.** [Wikipedia, 2012]

# Example: Smoking friends (2)

To formalize the example, let's use description logic $\mathscr{ALC}$:

**Obervation 1: High-school experience.**

If you have a friend, who is a smoker, you are a smoker as well:

$$\exists \mathsf{friendOf} \cdot \mathsf{Smoker} \sqsubseteq \mathsf{Smoker}$$

**Obervation 2: Six degrees of separation.**

Joining the $\mathsf{friendOf}$ relation 6 times gives the ***top relation***.

$$\mathsf{friendOf} \circ \ldots \circ \mathsf{friendOf} \sqsubseteq \top$$

**Note:** You have to assume $\mathsf{friendOf}$ is reflexive.

What is wrong with this model?

# Example: Smoking friends (3)

- If there is one smoker, the whole world starts smoking. (Formally, an interpretation $\mathscr{I}$ must satisfy $\mathsf{Smoker}^{\mathscr{I}} = \emptyset$ or $\mathsf{Smoker}^{\mathscr{I}} = \Delta$.)
- We start from **reasonable assumptions** and arrive at **counter-intuitive** conclusion. What's wrong with our reasoning?
- We would like to express something like

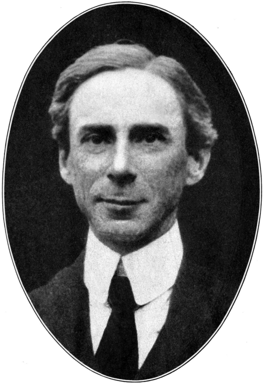
$$(\exists\, \mathsf{friendOf} \cdot \mathsf{Smoker} \sqsubseteq \mathsf{Smoker}) \text{ is “mostly” true.}$$

- Fuzzy logic can do that!

# Conclusion

All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

*Bertrand Russel [Russell, 1923]*



# Crisp sets: Definition

- (Informally:) A *crisp set* („ostrá množina“) $X$ is a collection of *objects* $x \in X$ that can be finite, countable or overcountable.
- We will speak about sets with in relation to a *universe* set („univerzum“).
- The universe is usually denoted as $\Delta$.
- Let $\mathbb{P}(\Delta)$ be the *powerset* (a set of all subsets) of $\Delta$ (the universe). Then any crisp set is an element in the powerset of its universe: $A \in \mathbb{P}(\Delta)$.

# Crisp sets: Example

Equivalent ways of describing a *crisp set* in $\mathbb{N}$:

$$A = \{1, 3, 5\} \tag{1}$$

$$A = \{x \in \mathbb{N} \mid x \leqslant 5 \text{ and } x \text{ is odd}\} \tag{2}$$

$$\mu_A(x) = \begin{cases} 0 & x > 5 \\ 0 & x \text{ is even} \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

$\mu_A$ is called the *membership function* („charakteristická funkce“, „funkce příslušnosti“).

# Membership function

If $\mu_A$ is a function $\Delta \longrightarrow \{0, 1\}$, the *inverse membership function* $\mu_A^{-1}$ returns objects with the given membership degree:

$$\mu_A^{-1}(M) = \{x \in \Delta \mid \mu_A(x) \in M\} \tag{4}$$

## Example

$$\mu_A^{-1}(\{1\}) = \{1, 3, 5\} \tag{5}$$

## Note

$\mu_A^{-1}$ is not an inverse in a strict mathematical sense. The inverse of $\Delta \longrightarrow \{0, 1\}$ should be $\{0, 1\} \longrightarrow \Delta$, but $\mu_A^{-1} : \mathbb{P}(\{0, 1\}) \longrightarrow \mathbb{P}(\Delta)$.

Check your knowledge:

$$\mu_{\varnothing} = ? \quad 0$$

$$\mu_{\Delta} = ? \quad 1$$

$$\mu^{-1}(\{0,1\}) = ? \quad \Delta$$

# Fuzzy set

## Definition

***Fuzzy set*** („fuzzy množina“) is an object $A$ described by a ***generalized membership function*** $\mu_A : \Delta \to [0, 1]$.

For better readability, $A(x) \equiv \mu_A(x)$.

The set of all fuzzy subsets of a crisp universe $\Delta$ will be denoted as $\mathbb{F}(\Delta)$.

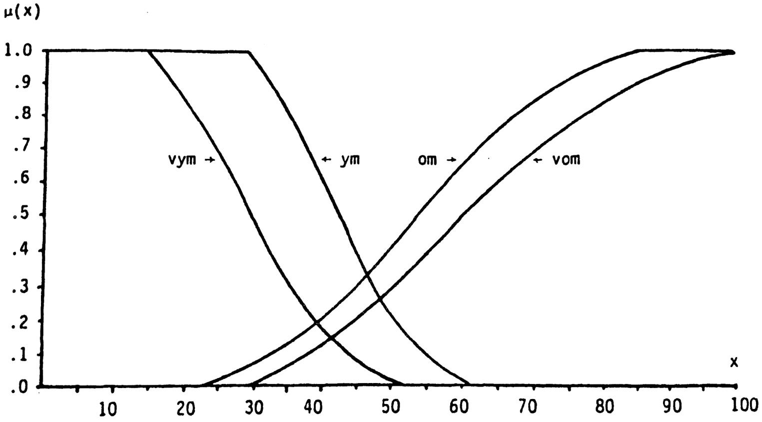



Figure 16–8. Empirical membership functions “Very Young Man,” “Young Man,” “Old Man,” “Very Old Man.”

Source: [Zimmermann, 2001]

# Fuzzy set: Properties (1)

- *Cardinality* is the size of a fuzzy set.

$$|A| = \sum_{x \in \Delta} A(x) \tag{6}$$

- *Height* of a fuzzy set is the highest value of the membership function.

$$\mathrm{Height}(A) = \sup \{\alpha \mid x \in \Delta, A(x) = \alpha\} \tag{7}$$

# Fuzzy set: Properties (2)

- ***Support*** („nosič“) is the set of objects contained in the fuzzy set “at least a bit”.

$$\mathrm{Supp}(A) = \{x \in \Delta \mid A(x) > 0\} = \mu_A^{-1}((0, 1]) \tag{8}$$

- ***Core*** („jádro“) is the set of objects “fully contained” in the fuzzy set.

$$\mathrm{Core}(A) = \{x \in \Delta \mid A(x) = 1\} = \mu_A^{-1}(\{1\}) \tag{9}$$

# Horizontal representation

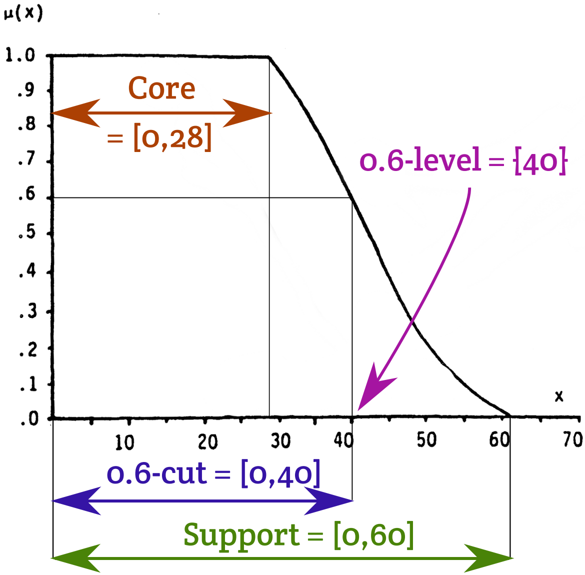
The $\alpha$*-level* („$\alpha$-hladina“) of a fuzzy set $A$ is a crisp set

$$\mu_A^{-1}(M) = \{x \in \Delta \mid A(x) \in M\} \tag{10}$$

The $\alpha$*-cut* („$\alpha$-řez“) of a fuzzy set $A$ is a crisp set

$$\mathrm{R}_A(\alpha) = \{x \in \Delta \mid A(x) \geqslant \alpha\} = \mu_A^{-1}([\alpha, 1]) \tag{11}$$

Sometimes we speak about a *strong* $\alpha$*-cut* („ostrý $\alpha$-řez“), where $\geqslant$ in the definition is replaced by $>$.
For better readability $A^{-1}(x) \equiv \mu_A^{-1}(x)$.




The set “Age of Young Men” with its properties.

Check your knowledge:

$$\mathrm{R}_A(0) = ? \quad \Delta$$

$$\mathrm{Core}(A) = ? \quad \mathrm{R}_A(1)$$

$$\mathrm{Height}(A) = \mathbf{sup}\,\{? \mid \mathrm{R}_A \;\; ?\} \quad \mathbf{sup}\,\{\alpha \in [0,1] \mid \mathrm{R}_A(\alpha) \neq \varnothing\}$$

# Converting vertical and horizontal representation

- Horizontal representation $\sim$ the $\alpha$-cuts $\mathtt{R}$.
- Vertical representation $\sim$ the characteristic function $\mu$.

$1 \Rightarrow 2$ : From the definition on the previous slide.

$2 \Rightarrow 1$ : By taking the “highest” $\alpha$-level containing $x$:

$$A(x) = \max\{\alpha \in [0,1] \mid x \in \mathtt{R}_A(\alpha)\} \tag{12}$$

# Special cases of fuzzy sets

## Definition

***Fuzzy interval*** $A$ is a fuzzy set on $\Delta = \mathbb{R}$ s.t.

- $\mathrm{R}_A(\alpha)$ is a **closed interval** for all $\alpha \in [0, 1]$
- $\mathrm{R}_A(1)$ is **not empty**.
- $\left|\mathrm{Supp}(A)\right|$ is **finite** (has a maximum and a minimum).

## Special cases of fuzzy intervals

- *Fuzzy number* $A$ is a fuzzy interval s.t. $|\mathrm{Core}(A)| = 1$
- *Trapezoidal interval* will be denoted by $\langle a, b, c, d\rangle$.
- *Triangular number* will be denoted by $\langle a, b, c\rangle = \langle a, b, b, c\rangle$.
- A *crisp interval* $[a, b]$ is also $\langle a, a, b, b\rangle$.

# Operations on fuzzy sets

| crisp set operation | propositional operation |
|---|---|
| $\overline{\cdot} : \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$ | $\neg \cdot : \{0,1\} \to \{0,1\}$ |
| $\cdot \cap \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$ | $\cdot \wedge \cdot : \{0,1\}^2 \to \{0,1\}$ |
| $\cdot \cup \cdot : \mathbb{P}(\Delta) \times \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$ | $\cdot \vee \cdot : \{0,1\}^2 \to \{0,1\}$ |

We can use the **logical operators** to define the **set operators:**

$$\overline{A} = \{x \in \Delta \mid \neg(x \in A)\} \tag{LS1}$$

$$A \cap B = \{x \in \Delta \mid (x \in A) \wedge (x \in B)\} \tag{LS2}$$

$$A \cup B = \{x \in \Delta \mid (x \in A) \vee (x \in B)\} \tag{LS3}$$

Therefore we will cover the logical negation, conjunction and disjunction. We get the set operations “for free”.

# Fuzzy negation

*Fuzzy negation* is a *non-increasing*, *involutive*, *unary* function $\underset{\circ}{\neg} : [0, 1] \to [0, 1]$ s.t.

$$\text{if } \alpha \leqslant \beta \text{ then } \underset{\circ}{\neg}\beta \leqslant \underset{\circ}{\neg}\alpha \tag{N1}$$

$$\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \alpha \tag{N2}$$

## Example

Standard („standardní“), Łukasiewicz negation

$$\underset{S}{\neg}\alpha = 1 - \alpha \tag{13}$$

The fuzzy set *complement* is a defined using (LS1).

# Fuzzy negation: More examples

- Cosine negation

$$\underset{\cos}{\neg}\alpha = (\cos(\pi\alpha) + 1)/2 \tag{14}$$

- Sugeno negation

$$\underset{S\lambda}{\neg}\alpha = \frac{1-\alpha}{1+\lambda\alpha}, \quad \lambda > -1 \tag{15}$$

- Yager negation

$$\underset{Y\lambda}{\neg}\alpha = (1-\alpha^{\lambda})^{1/\lambda} \tag{16}$$

# Fuzzy negation: Properties

The axioms (N1) and (N2) imply more properties of fuzzy negations:

### Theorem 24

Every fuzzy negation $\underset{\circ}{\neg}$ is a

- continuous
- decreasing
- bijective
- generalization of the propositional negation $\neg$

# Fuzzy negation: Proof of 24

- **Injective ($f(a) = f(b) \Rightarrow a = b$):** Take 2 values, whose negations are equal: $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$. By (N2) $\alpha = \underset{\circ}{\neg}\overbrace{\underset{\circ}{\neg}\alpha}^{\square}$. The $\square$ can be substituted using the assumption: $\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\underset{\circ}{\neg}\beta$. Using (N1) gives $\underset{\circ}{\neg}\underset{\circ}{\neg}\beta = \beta$. Therefore $\alpha = \beta$.
- Every non-increasing function (N1) which is injective, must be **decreasing**. If $\alpha < \beta$ WLOG, then $\underset{\circ}{\neg}\alpha \geqslant \underset{\circ}{\neg}\beta$. Then either $\underset{\circ}{\neg}\alpha > \underset{\circ}{\neg}\beta$ and $\underset{\circ}{\neg}$ is decreasing, or $\underset{\circ}{\neg}\alpha = \underset{\circ}{\neg}\beta$, which contradicts the injectivity.

# Fuzzy negation: Proof of 24

- **Surjective** $\forall y \exists x . f(x) = y$: We seek a value of $\beta$ for each $\alpha$ s.t. $\alpha = \underset{\circ}{\neg}\,\beta$. Using injectivity, the condition is equivalent to $\underset{\circ}{\neg}\,\alpha = \underset{\circ}{\neg}\,\underset{\circ}{\neg}\,\beta$. Using (N2), we find the value of $\beta$ for any $\alpha$: $\beta = \underset{\circ}{\neg}\,\alpha$.
- **Bijection** is an injective and surjective function (by definition).
- **Continuous:** Every decreasing bijection is continuous.
- **Boundary values:** Let $\underset{\circ}{\neg}\,0 = \alpha$ and suppose that $\alpha < 1$. Then from surjectivity, there must be some other $\beta > 0$ s.t. $\underset{\circ}{\neg}\,\beta = 1$. This contracits monotonicity, because $\underset{\circ}{\neg}\,0 < \underset{\circ}{\neg}\,\beta$. The other boundary value is proven similarly.

# Fuzzy conjunctions (t-norms)

***Fuzzy t-norm*** (triangluar norm, conjunction) is a binary, ***comutative***, operation $\underset{\circ}{\wedge}$ s.t.

$$\alpha \underset{\circ}{\wedge} \beta = \beta \underset{\circ}{\wedge} \alpha \tag{T1}$$

$$\alpha \underset{\circ}{\wedge} (\beta \underset{\circ}{\wedge} \gamma) = (\alpha \underset{\circ}{\wedge} \beta) \underset{\circ}{\wedge} \gamma \tag{T2}$$

$$\text{if } \beta \leqslant \gamma \text{ then } (\alpha \underset{\circ}{\wedge} \beta) \leqslant (\alpha \underset{\circ}{\wedge} \gamma) \tag{T3}$$

$$(\alpha \underset{\circ}{\wedge} 1) = \alpha \tag{T4}$$

The fuzzy set *intersection* is a defined using (LS2).

# Fuzzy conjunctions: Examples

- Standard (Gödel, Zadeh)

$$\alpha \underset{S}{\wedge} \beta = \min(\alpha, \beta) \tag{17}$$

- Łukasiewicz

$$\alpha \underset{L}{\wedge} \beta = \max(\alpha + \beta - 1, 0) \tag{18}$$

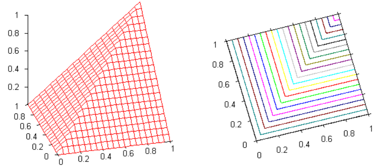
- Algebraic product („součinová“)

$$\alpha \underset{A}{\wedge} \beta = \alpha \cdot \beta \tag{19}$$

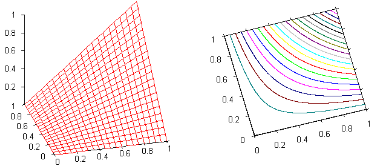
- Weak („drastická“)

$$\alpha \underset{W}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

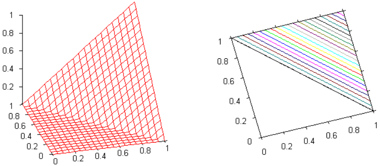
# Fuzzy conjunctions: Visualization [Wikipedia]



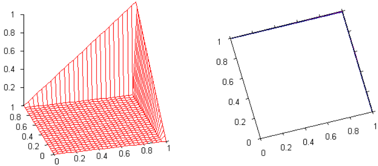

Standard




Algebraic




Łukasiewicz




Drastic

# Fuzzy conjunctions: Properties (1)

## Theorem 30

The weak and standard conjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \underset{W}{\wedge} \beta) \leqslant (\alpha \underset{\circ}{\wedge} \beta) \leqslant (\alpha \underset{S}{\wedge} \beta) \tag{21}$$

**Proof:** Assume WLOG $\alpha \leqslant \beta$.

$\beta = 1$ The condition (T4) gives the same result for all conjunctions.

$\beta < 1$ $\alpha \underset{W}{\wedge} \beta = 0$, which gives the lower bound. The upper bound is rewritten using the definition of standard conjunction (17): $\alpha \underset{S}{\wedge} \beta = \alpha$. From (T4) follows that $\alpha = \alpha \underset{\circ}{\wedge} 1 \geqslant \alpha \underset{\circ}{\wedge} \beta$. Together $\alpha \underset{S}{\wedge} \beta = \alpha \geqslant \alpha \underset{\circ}{\wedge} \beta$.

# Fuzzy conjunctions: Properties (2)

### Theorem 31

The standard conjunction is the only *idempotent* conjunction:

$$\alpha \underset{\circ}{\wedge} \alpha = \alpha \tag{22}$$

**Proof:** Assume WLOG $\alpha \leqslant \beta$.

$$\alpha = \alpha \underset{\circ}{\wedge} \alpha \overset{(T3)}{\leqslant} \alpha \underset{\circ}{\wedge} \beta \overset{(T3)}{\leqslant} \alpha \underset{\circ}{\wedge} \mathbf{1} \overset{(T4)}{=} \alpha \tag{23}$$

Therefore $\alpha \underset{\circ}{\wedge} \beta = \alpha$. There is only one such conjunction: $\underset{S}{\wedge}$.

# Fuzzy disjunctions (s-norm)

***Fuzzy s-norm*** (t-conorm, disjunction) is a binary operation $\mathring{\vee}$ s.t.

$$\alpha \mathring{\vee} \beta = \beta \mathring{\vee} \alpha \tag{S1}$$

$$\alpha \mathring{\vee} (\beta \mathring{\vee} \gamma) = (\alpha \mathring{\vee} \beta) \mathring{\vee} \gamma \tag{S2}$$

$$\text{if } \beta \leqslant \gamma \text{ then } (\alpha \mathring{\vee} \beta) \leqslant (\alpha \mathring{\vee} \gamma) \tag{S3}$$

$$(\alpha \mathring{\vee} 0) = \alpha \tag{S4}$$

## Union

The fuzzy set ***union*** is a defined using the disjunction:

$$\mu_{A \cup B}(x) = \mu_A(x) \mathring{\vee} \mu_B(x) \tag{24}$$

# Fuzzy disjunctions: Examples (1)

- Standard (Gödel, Zadeh)

$$\alpha \overset{S}{\vee} \beta = \mathbf{max}(\alpha, \beta) \tag{25}$$

- Łukasiewicz

$$\alpha \overset{L}{\vee} \beta = \mathbf{min}(\alpha + \beta, 1) \tag{26}$$

- Algebraic sum („součinová“)

$$\alpha \overset{A}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta \tag{27}$$

# Fuzzy disjunctions: Examples (2)

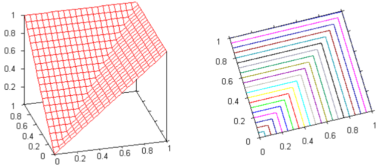
- Weak („drastická“)

$$\alpha \overset{W}{\vee} \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ \beta & \text{if } \alpha = 0 \\ 1 & \text{otherwise} \end{cases} \tag{28}$$

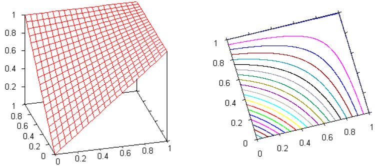
- Einstein

$$\alpha \overset{E}{\vee} \beta = \frac{\alpha + \beta}{1 + \alpha\beta} \tag{29}$$

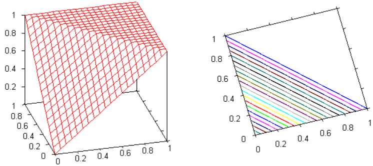
# Fuzzy disjunctions: Visualization [Wikipedia]



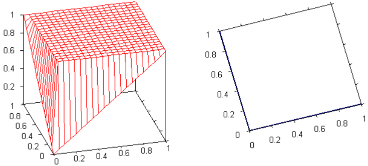
Standard



Algebraic



Łukasiewicz



Drastic

# Fuzzy disjunctions: Properties

- The standard and weak disjunctions provide a lower and upper bound on all possible conjunctions:

$$(\alpha \overset{\mathrm{S}}{\vee} \beta) \leqslant (\alpha \overset{\circ}{\vee} \beta) \leqslant (\alpha \overset{\mathrm{W}}{\vee} \beta) \tag{30}$$

- The standard disjunctions is the only *idempotent* conjunction:

$$\alpha \overset{\circ}{\vee} \alpha = \alpha \tag{31}$$

# Conjunction - disjunction duality

A If $\underset{\circ}{\wedge}$ is a fuzzy conjunction, then $\alpha \overset{\circ}{\vee} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg}\, \alpha \underset{\circ}{\wedge} \underset{\circ}{\neg}\, \beta)$ is a fuzzy disjunction (dual to $\underset{\circ}{\wedge}$ w.r.t. $\underset{\circ}{\neg}$).

B If $\overset{\circ}{\vee}$ is a fuzzy disjunction, then $\alpha \underset{\circ}{\wedge} \beta = \underset{\circ}{\neg}(\underset{\circ}{\neg}\, \alpha \overset{\circ}{\vee} \underset{\circ}{\neg}\, \beta)$ is a fuzzy conjunction (dual to $\overset{\circ}{\vee}$ w.r.t. $\underset{\circ}{\neg}$).

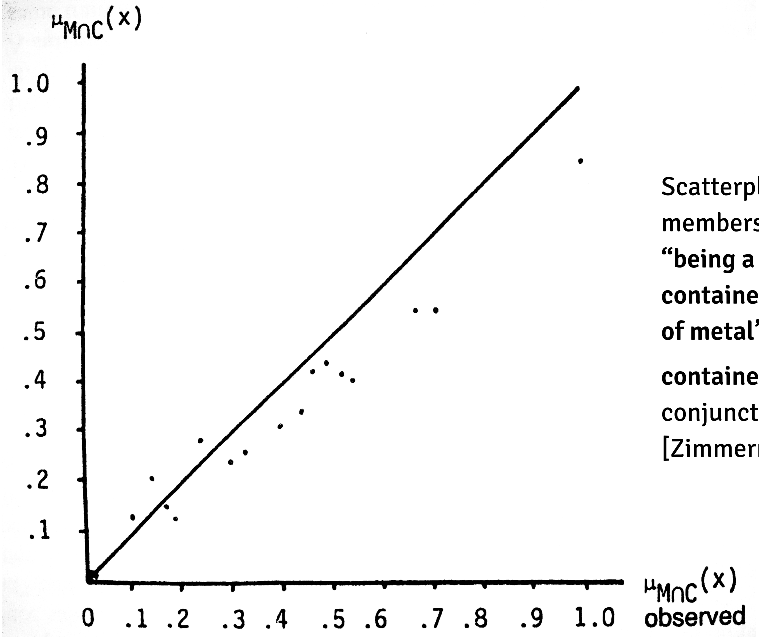
## Theorems

- **Łukasiewicz** operations $\underset{L}{\wedge}, \overset{L}{\vee}$ are dual w.r.t. **standard** negation.
- **Algebraic** operations $\underset{A}{\wedge}, \overset{A}{\vee}$ are dual w.r.t. **standard** negation.
- **Standard** operations $\underset{S}{\wedge}, \overset{S}{\vee}$ are dual w.r.t. **any** negation.
- **Weak** operations $\underset{W}{\wedge}, \overset{W}{\vee}$ are dual w.r.t. **any** negation.

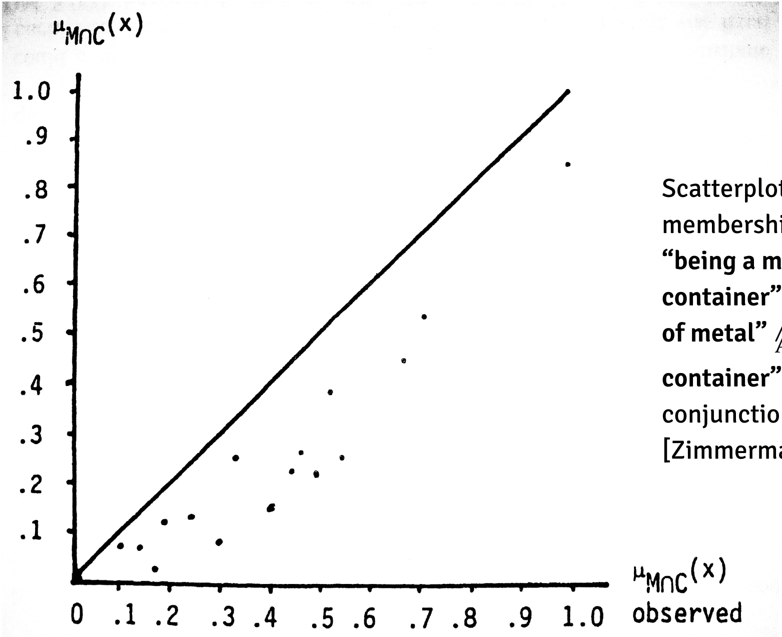
Table 16–2. Empirically determined grades of membership.

| *Stimulus x* | $\mu_M(x)$ | $\mu_C(x)$ | $\mu_{M\cap C}(x)$ |
|---|---|---|---|
| 1. bag | 0.000 | 0.985 | 0.007 |
| 2. baking tin | 0.908 | 0.419 | 0.517 |
| 3. ballpoint pen | 0.215 | 0.149 | 0.170 |
| 4. bathtub | 0.552 | 0.804 | 0.674 |
| 5. book wrapper | 0.023 | 0.454 | 0.007 |
| 6. car | 0.501 | 0.437 | 0.493 |
| 7. cash register | 0.692 | 0.400 | 0.537 |
| 8. container | 0.847 | 1.000 | 1.000 |
| 9. fridge | 0.424 | 0.623 | 0.460 |
| 10. Hollywood swing | 0.318 | 0.212 | 0.142 |
| 11. kerosene lamp | 0.481 | 0.310 | 0.401 |
| 12. nail | 1.000 | 0.000 | 0.000 |
| 13. parkometer | 0.663 | 0.335 | 0.437 |
| 14. pram | 0.283 | 0.448 | 0.239 |
| 15. press | 0.130 | 0.512 | 0.101 |
| 16. shovel | 0.325 | 0.239 | 0.301 |
| 17. silver spoon | 0.969 | 0.256 | 0.330 |
| 18. sledgehammer | 0.480 | 0.012 | 0.023 |
| 19. water bottle | 0.564 | 0.961 | 0.714 |
| 20. wine barrel | 0.127 | 0.980 | 0.185 |

Degree of membership for 20 items into the sets **"make of metal"**, **"being a container"** and **"being a metalic container"**. [Zimmermann, 2001]




Scatterplot of membership degree for **“being a metalic container”** vs. **“make of metal”** $\underset{S}{\wedge}$ **“being a container”**. (standard conjunction). [Zimmermann, 2001]




Scatterplot of membership degree for **"being a metalic container"** vs. **"make of metal"** $\hat{\wedge}$ **"being a container"** (algebraic conjunction). [Zimmermann, 2001]

# Criteria for selecting operators (1)

1. **Axiomatic strength:** The set of valid theorems may differ based on the choice of t-norms and s-norms (see tutorials).
2. **Empirical fit:** Using fuzzy theory for a model of the real world, the chosen operator should match the real behavior of the system.
3. **Adaptability:** Operators in a generic system should be able to fit several use cases. One way of increasing adaptibility is to use operators with parameters (e.g. Yager and Sugeno negations).

# Criteria for selecting operators (2)

4. **Computational efficiency:** Evaluating e.g. the standard negation is usually faster than the Yager negation, which contains the power.
5. **Aggregating behavior:** When the operators combines a large number of operands, does the value tends to go to 0 (conjunction) or 1 (disjunction). The standard operators behave differently than the algebraic ones.

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