Introduction, Description Logics

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Our plan

Course Information

Course Information

- web page:
  http://cw.felk.cvut.cz/doku.php/courses/ae4m33rzn/start

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- three basic topics: description logics, fuzzy (description) logic, probabilistic models

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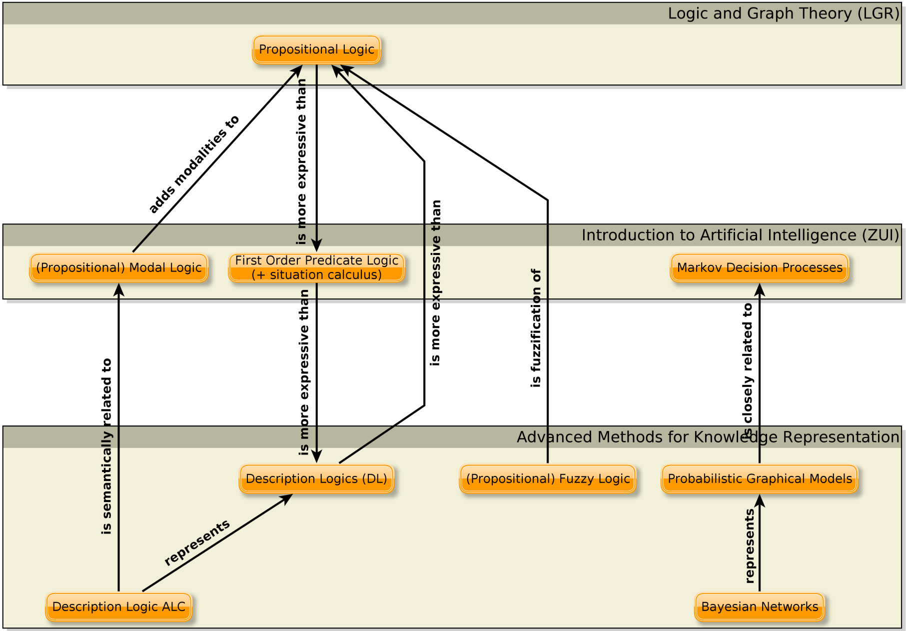
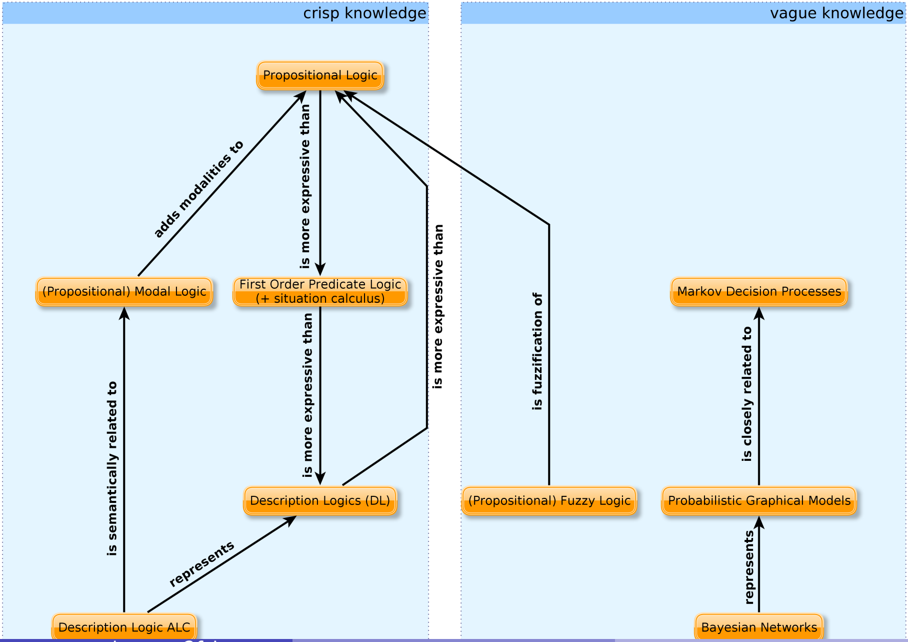
- **Please go through the course web page carefully !!!**

Course Roadmap



Course Roadmap (2)



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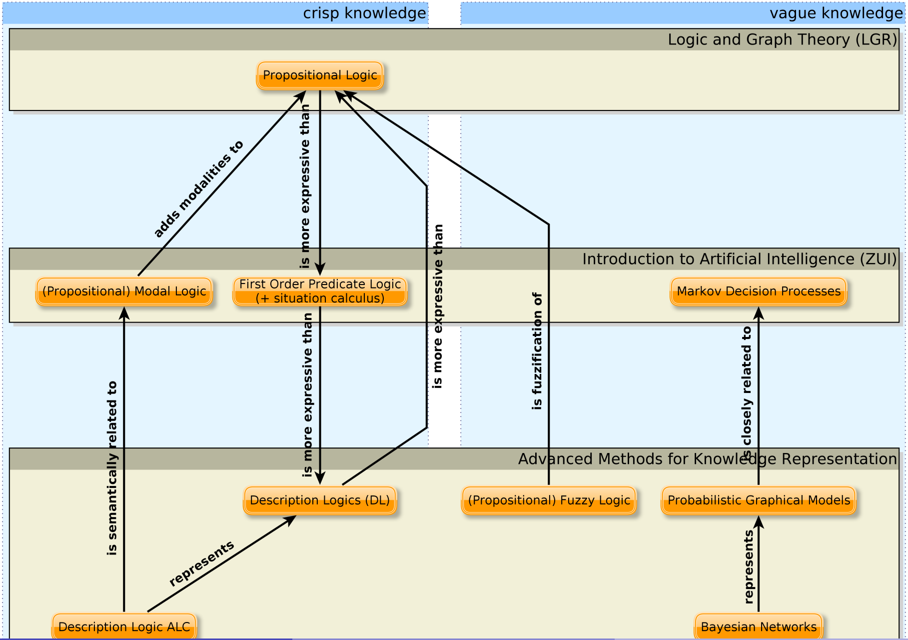
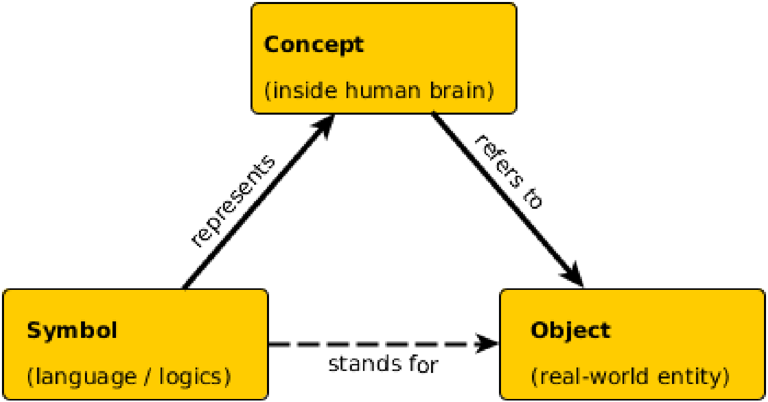


Towards Description Logics

Semiotic Triangle



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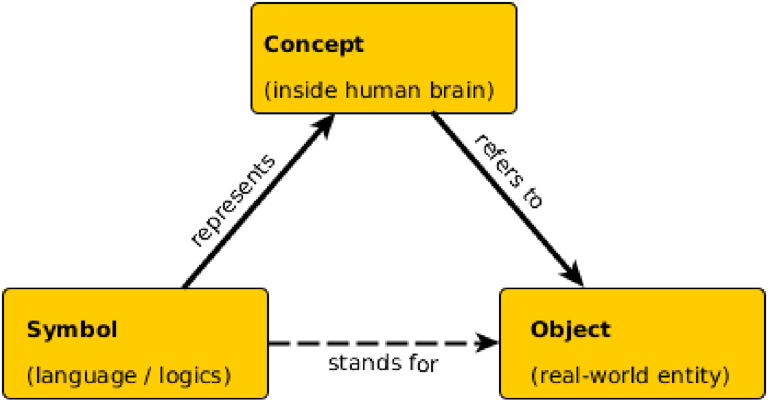

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represents ∼ studied by *formal knowledge representation languages* – **this course**

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- deal with proper representation of conceptual knowledge in a domain

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- Most of them are based on some **logical calculus**.

Logics – a Review

Logics for Knowledge Representation

- propositional logic

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- ... what is the meaning of these formulas ?

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Logics are defined by their

- Syntax – to *represent* concepts

Logics trade-off

A logic calculus is always a trade-off between *expressiveness* and *tractability of reasoning*.

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- Syntax – to *represent* concepts
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- Proof Theory – to enforce the semantics

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How to check satisfiability of the formula $A \vee (\neg(B \wedge A) \vee B \wedge C)$?

syntax – atomic formulas and $\neg$, $\wedge$, $\vee$, $\Rightarrow$

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complexity – NP-Complete (Cook theorem)

First Order Predicate Logic

Example

What is the meaning of this sentence ?

$$(\forall x_1)((Student(x_1) \wedge (\exists x_2)(GraduateCourse(x_2) \wedge isEnrolledTo(x_1, x_2)))$$
$$\Rightarrow (\forall x_3)(isEnrolledTo(x_1, x_3) \Rightarrow GraduateCourse(x_3)))$$

$$Student \sqcap \exists isEnrolledTo.GraduateCourse \sqsubseteq \forall isEnrolledTo.GraduateCourse$$

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complexity – undecidable (Goedel)

Open World Assumption

OWA

FOPL accepts Open World Assumption, i.e. whatever is not known is not necessarily false.

As a result, FOPL is *monotonic*, i.e.

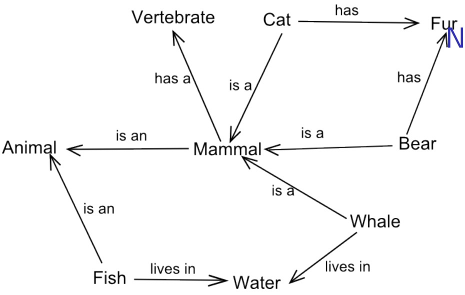
monotonicity

No conclusion can be invalidated by adding extra knowledge.

This is in contrary to relational databases, or Prolog that accept Closed World Assumption.

Semantic Networks and Frames

Semantic Networks



(©wikipedia.org)

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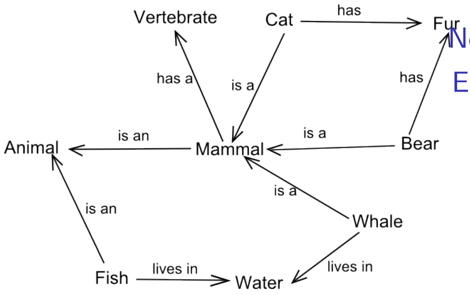
Semantic Networks



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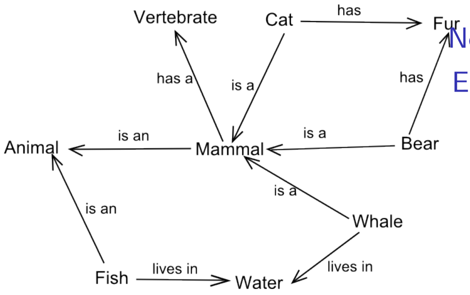

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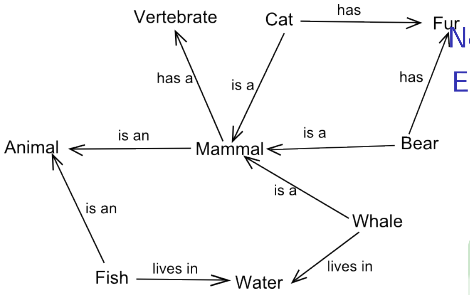

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Example

Each *Cat hasa Vertebrate*, since each *Cat isa Mammal*.

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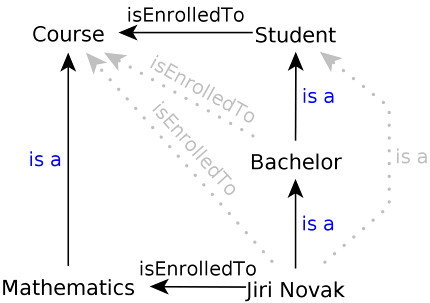

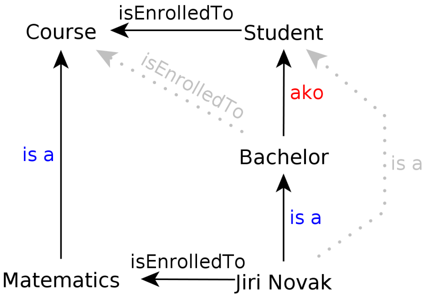

However, this does not allow distinguishing individuals (instances) and groups (classes).

To solve this, a new relationship type "is a kind of" **ako** can be introduced and used for inheritance, while **is a** relationships would be restricted to expressing individual-group relationships.

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- Wordnet, Semantic Wiki, aj.

Frames

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slots:
is a: car
has engine: four-stroke engine
has transmission system: manual
has carb: *value*: Jikov
default: Pierburg

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([MvL93])

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- ☺ Facets allow non-monotonic reasoning.
- ☺ *Daemons* are triggers for actions perfomed on facets (read, write, delete). Can be used e.g for consistency checking.

Frames (2)

Example

Typically, `Škoda Favorit` **has carb** of type `Pierburg`, but this particular `Škoda Favorit` **has carb** of type `Jikov`.

- frames can be grouped into *scenarios* that represent typical situations, e.g. going into a restaurant. [MvL93]

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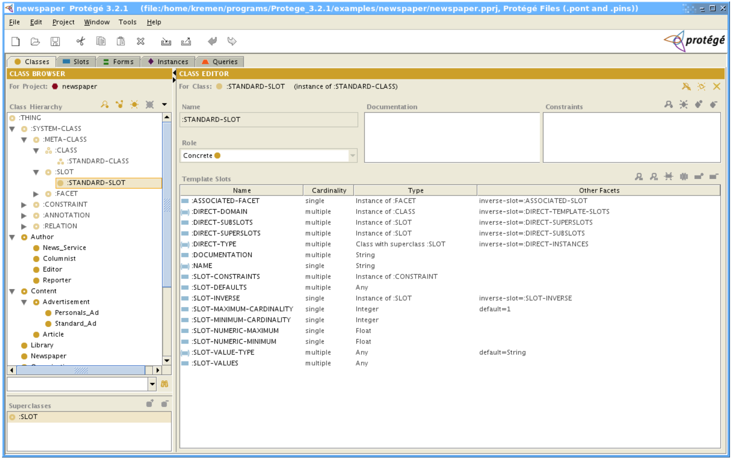
- Protégé - http://protege.stanford.edu/overview/protege-frames.html

Protégé


newspaper Protégé 3.2.1 (file:/home/kremen/programs/Protege_3.2.1/examples/newspaper/newspaper.pprj, Protégé Files (.pont and .pins))
File Edit Project Window Tools Help
protégé
Classes Slots Forms Instances Queries
CLASS BROWSER
For Project: newspaper
Class Hierarchy
:THING
:SYSTEM-CLASS
:META-CLASS
:CLASS
:STANDARD-CLASS
:SLOT
:STANDARD-SLOT
:FACET
:CONSTRAINT
:ANNOTATION
:RELATION
Author
News_Service
Columnist
Editor
Reporter
Content
Advertisement
Personals_Ad
Standard_Ad
Article
Library
Newspaper
Superclasses
:SLOT
CLASS EDITOR
For Class: :STANDARD-SLOT (instance of :STANDARD-CLASS)
Name
:STANDARD-SLOT
Documentation
Constraints
Role
Concrete
Template Slots
Name Cardinality Type Other Facets
:ASSOCIATED-FACET single Instance of :FACET inverse-slot=:ASSOCIATED-SLOT
:DIRECT-DOMAIN multiple Instance of :CLASS inverse-slot=:DIRECT-TEMPLATE-SLOTS
:DIRECT-SUBSLOTS multiple Instance of :SLOT inverse-slot=:DIRECT-SUPERSLOTS
:DIRECT-SUPERSLOTS multiple Instance of :SLOT inverse-slot=:DIRECT-SUBSLOTS
:DIRECT-TYPE multiple Class with superclass :SLOT inverse-slot=:DIRECT-INSTANCES
:DOCUMENTATION multiple String
:NAME single String
:SLOT-CONSTRAINTS multiple Instance of :CONSTRAINT
:SLOT-DEFAULTS multiple Any
:SLOT-INVERSE single Instance of :SLOT inverse-slot=:SLOT-INVERSE
:SLOT-MAXIMUM-CARDINALITY single Integer default=1
:SLOT-MINIMUM-CARDINALITY single Integer
:SLOT-NUMERIC-MAXIMUM single Float
:SLOT-NUMERIC-MINIMUM single Float
:SLOT-VALUE-TYPE multiple Any default=String
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- ☹ ad-hoc reasoning procedures, that complicates (and broadens ambiguity during) translation to First Order Predicate Logic (FOPL),
- ☹ problems – querying, debugging.

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  - ☹ Prolog is not an implementation of FOPL – OWA vs. CWA, negation as failure, problems in expressing disjunctive knowledge, etc.

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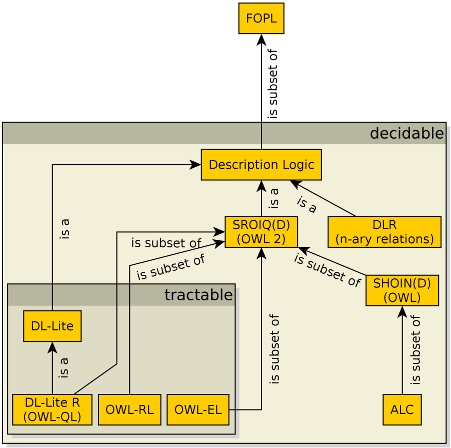
  - What is the semantics of a "slot" in a frame (relation in semantic networks) ? The slot **must/might** be filled **once/multiple times** ?

What are Description Logics ?



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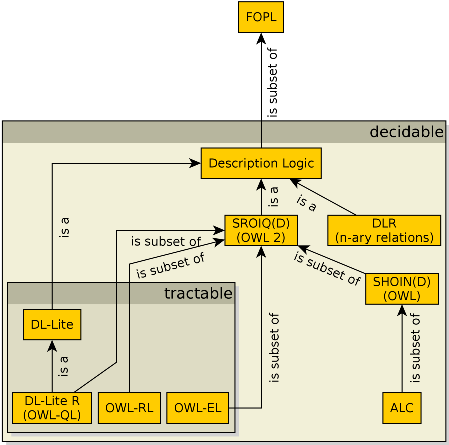

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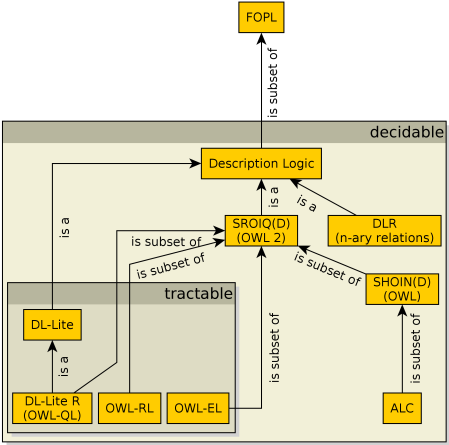

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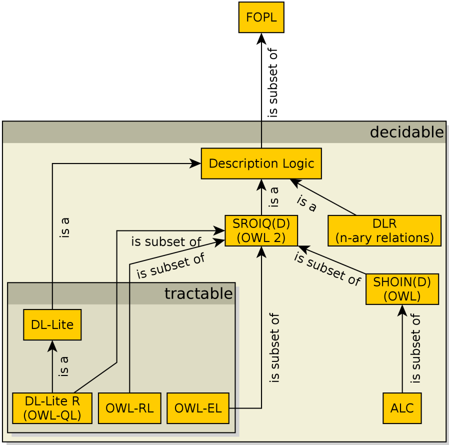

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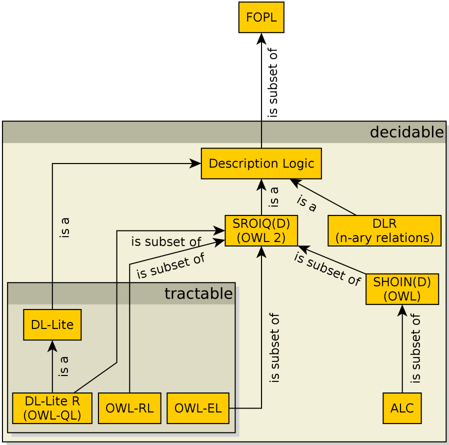

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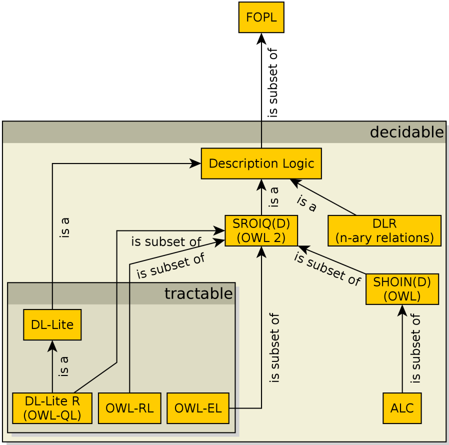

$\mathcal{ALC}$ Language

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- DLs differ in their expressive power (concept/role constructors, axiom types).

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- Having *atomic* concept A, *atomic* role R and individual a, then

$$\begin{aligned} A^{\mathcal{I}} &\subseteq \Delta^{\mathcal{I}} \\ R^{\mathcal{I}} &\subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \\ a^{\mathcal{I}} &\in \Delta^{\mathcal{I}} \end{aligned}$$

$\mathcal{ALC}$ (= attributive language with complements)

Having concepts C, D, atomic concept A and atomic role R, then for interpretation $\mathcal{I}$:

concept	*concept*$^{\mathcal{I}}$	*description*
$\top$	$\Delta^{\mathcal{I}}$	(universal concept)
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ABOX (UNA = unique name assumption[1])

axiom	$\mathcal{I} \models$ axiom iff	*description*
$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$	(concept assertion)
$R(a_1, a_2)$	$({a_1}^{\mathcal{I}}, {a_2}^{\mathcal{I}}) \in R^{\mathcal{I}}$	(role assertion)

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- S is consistent, if S has at least one model

$\mathcal{ALC}$ – Example

Example

Consider an information system for genealogical data. Information integration from various sources is crucial – databases, information systems with *different data models*. As an integration layer, let's use a description logic theory. Let's have atomic concepts *Person*, *Man*, *GrandParent* and atomic role *hasChild*.

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- How to define concept *GrandParent* ?
  - $GrandParent \equiv Person \sqcap \exists hasChild \cdot \exists hasChild \cdot \top$
- How does the previous axiom look like in FOPL ?

$$\forall x\, (GrandParent(x) \equiv (Person(x) \wedge \exists y\, (hasChild(x, y) \\ \wedge \exists z\, (hasChild(y, z)))))$$

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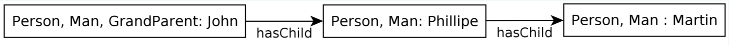
  - $JOHN^{\mathcal{I}_1} = \{John\}$
- this model is finite and has the form of a tree with the root in the node *John* :



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In particular (generalized) TMP is a characteristics that is shared by most DLs and significantly reduces their computational complexity.

Example

Example

primitive concept

defined concept

$$
\begin{aligned}
Woman &\equiv Person \sqcap Female \\
Man &\equiv Person \sqcap \neg Woman \\
Mother &\equiv Woman \sqcap \exists hasChild \cdot Person \\
Father &\equiv Man \sqcap \exists hasChild \cdot Person \\
Parent &\equiv Father \sqcup Mother \\
Grandmother &\equiv Mother \sqcap \exists hasChild \cdot Parent \\
MotherWithoutDaughter &\equiv Mother \sqcap \forall hasChild \cdot \neg Woman \\
Wife &\equiv Woman \sqcap \exists hasHusband \cdot Man
\end{aligned}
$$

Example – CWA $\times$ OWA

Example

ABOX

$hasChild(JOCASTA, OEDIPUS)$
$hasChild(OEDIPUS, POLYNEIKES)$
$Patricide(OEDIPUS)$

$hasChild(JOCASTA, POLYNEIKES)$
$hasChild(POLYNEIKES, THERSANDROS)$
$\neg Patricide(THERSANDROS)$

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Edges represent role assertions of *hasChild*; red/green colors distinguish concepts instances – *Patricide* a ¬*Patricide*

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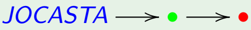
Q1 $(\exists hasChild \cdot (Patricide \sqcap \exists hasChild \cdot \neg Patricide))(JOCASTA)$,



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JOCASTA ⟶ • ⟶ •

Q2 Find individuals x such that $\mathcal{K} \models C(x)$, where C is

$$\neg Patricide \sqcap \exists hasChild^{-} \cdot (Patricide \sqcap \exists hasChild^{-}) \cdot \{JOCASTA\}$$

What is the difference, when considering CWA ?

JOCASTA ⟶ • ⟶ x

* Vladimír Mařík, Olga Štěpánková, and Jiří Lažanský.
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