# **Graphical probabilistic models – introduction**

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#### **GPM** lectures – an overview

- L1: introduction
  - Bayesian networks motivation and definitions,
  - how graphs can help conditional independence,
- L2: inference
  - network applications in predictive tasks,
  - inference engine fundamental algorithms,
- L3: learning networks from data
  - using networks for modelling,
  - networks as tools for understanding of relations among variables,
- L4: extensions
  - time, continuous variables, unoriented graphs,
- L5: simple (restricted) graph models
  - feasible models in expert systems,
  - final exam form, questions.

## Agenda

- Motivation for graphical models
  - general probabilistic model and its curse of dimensionality,
  - general probabilistic model and knowledge?
- conditional independence
  - definition, examples,
  - graph equivalent d-separation,
  - graph equivalence wrt conditional independence,
- essential types of graphical probabilistic models
  - brief categorization,
- Bayesian networks
  - basic idea behind,
  - example family house with a dog,
  - fundamental tasks and their complexity.

#### Notation (binary random variables):

A... random variable,  $a \dots A = True$ ,  $\neg a \dots A = False$ ,  $Pr(A, B) \dots$  joint probability distribution (a table),  $Pr(a, b) = Pr(A = True, B = True) \dots$  prob of a particular event (a single item in table Pr(A, B)).

## Why not a general probabilistic model?

- Ex.: 3 statements about world (people), each statement valid or invalid for a person
  - the world can be captured by joint probability,
- H: The person is higher then 180cm. M: The person is a man. Z: The person is a jockey.
  - women and men are equally frequent, men tend to be tall, a jockey is mostly a short man,

Т	Μ	J	Pr(T,M,J)	$j \Rightarrow \neg t$
F	F	F	0.298	F
F	F	Т	0.002	Т
F	Т	F	0.245	F
F	Т	Т	0.005	Т
Т	F	F	0.199	Т
Т	F	Т	0.001	Т
Т	Т	F	0.248	Т
Т	Т	Т	0.002	Т
			1	

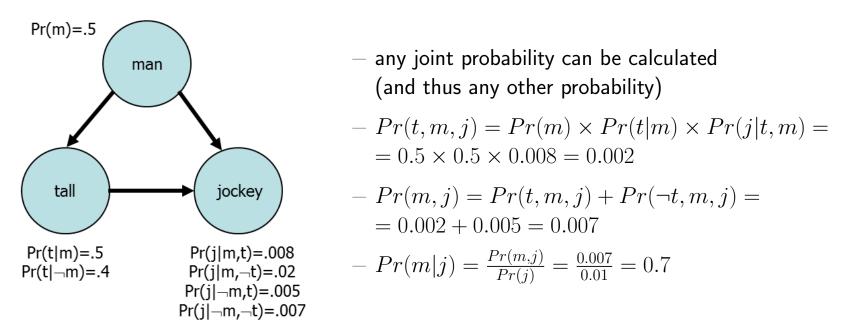
- probability of a formula equals the sum of probability of interpretations that satisfy it
  - Pr(t) = 0.199 + 0.001 + 0.248 + 0.002 = 0.45,45% of population is tall,
  - $Pr(j \Rightarrow \neg t) = 1 0.001 0.002 = 0.997$ , 99.7% of population is not tall or not a jockey,
- arbitrary probabilistic operations can also be applied
  - $Pr(\neg t|j) = \frac{Pr(\neg t,j)}{Pr(j)} = \frac{0.007}{0.01} = 0.7,$ 70% of jockeys are not tall
  - $Pr(m|j) = \frac{Pr(m,j)}{Pr(j)} = \frac{0.007}{0.01} = 0.7,$ 70% of jockeys are men knowing a person is a jockey, in 70% cases it is a man as well

## Why not a general probabilistic model?

- universality makes an asset of this model
  - identical and trivial model structure for all problems,
  - for a **sufficient** sample size its learning converges
    - \* model learning means to estimate (joint) probabilities,
- intractable for real problems
  - $-2^n 1$  probabilities when dealing with n propositions (for discrete variables a different base, for continuous parametric models),
  - infeasible for experts, the same holds for empirical settings based on data,
  - even if probs were known, still exponential in memory and inference time
    - \* obvious for a joint continuous distribution function,
    - \* **curse of dimensionality** the number of observations needed grows exp with the number of variables,
- impenetrable for real tasks
  - model gives no explicit knowledge about the domain,
  - relations among objects remain hidden in a flood of numbers.

#### The ways to simplify and better organize the model?

- utilize the domain knowledge:
  - is there any relationship between all the random variables?
  - the example: gender influences both height and occupation, height influences occupation.
- let us consider the graph probabilistic representation
  - can relations be posed in terms of graphs?
  - in which way to interpret graphs in probabilistic context?
  - still 7 probability values needed, no simplification, only reformulation,
  - why? edges among all the nodes, no use of (conditional) independence.



# (Conditional) independence

• definition: A and B are conditionally independent given C if:

- $-\Pr(A, B|C) = \Pr(A|C) \times \Pr(B|C), \, \forall A, B, C, \Pr(C) \neq 0$
- denoted as  $A \perp B | C$  (conditional dependence  $A \perp B | C$ )
- (classical independence between A and B:  $Pr(A, B) = Pr(A) \times Pr(B)$ )
- some observations make other observations uninteresting
  - under assumption of conditional independence it holds: Pr(B|C) = Pr(B|A,C) a Pr(A|C) = Pr(A|B,C),
  - observing C, knowledge of A becomes redundant for knowing B,
  - observing C, knowledge of B becomes redundant for knowing A.

# (Conditional) independence

• Example 1:

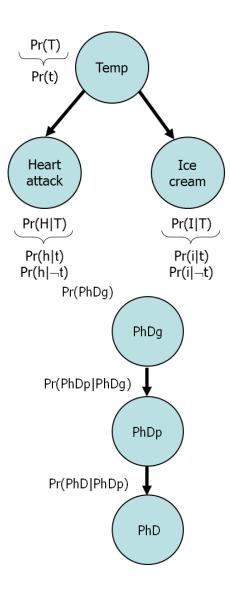
- heart attack rate (H) grows with ice cream sales (I),
- variables H and I are dependent:  $Pr(h|i) > Pr(h) \text{,} \label{eq:prince}$
- both grow only because of temperature (T),
- when conditioned by T, H and I become independent: Pr(H|I,T) = Pr(H|T).

#### • Example 2:

educated grandparents (PhDg) often have educated grandchildren (PhD):

Pr(phd|phdg) > Pr(phd)

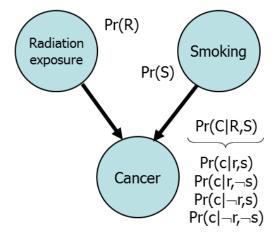
- knowledge of the parents' education level (PhDp) makes grandparents unimportant: Pr(PhD|PhDp, PhDg) = Pr(PhD|PhDp)



# (Conditional) independence

Example 3:

- both radiation (R) and smoking (S) can cause cancer (C)
- R and S are obviously independent variables:  $Pr(R,S) = Pr(R) \times Pr(S)$
- $\text{ concerning C, R and S become seemingly dependent} \\ Pr(r|s,c) < Pr(r|c) \text{ or } Pr(r|s,\not c) < Pr(r|\not c) \\ \end{array}$



#### Summary

- Ad 1 and 2) conditional independence the intermediate variable explains dependency between the ultimate ones,
- Ad 3) independence

the intermediate variable introduces spurious dependency.

## **Connection types**

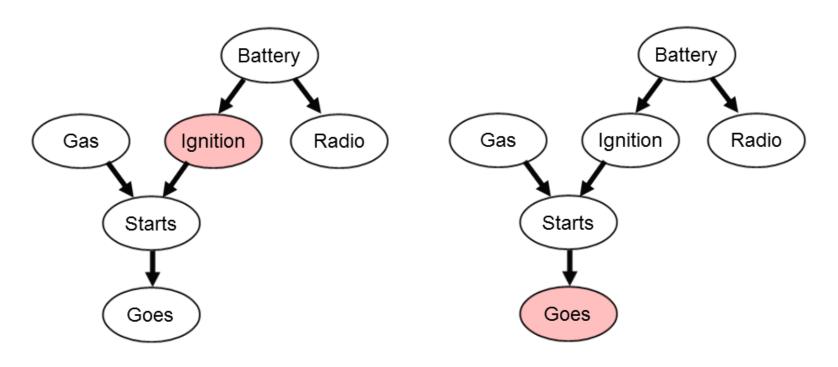
- Nomenclature
  - parent p and child/son c a directed edge from p to c,
  - ancestor a and descendant d a directed path from a to d,
- three connection types
  - diverging
    - \* terminal nodes dependent, dependence disappears when (surely) knowing middle node,
    - \* intermediate variable (daytime) explains dependence,
    - \* crime-rate  $\leftarrow$  daytime  $\rightarrow$  energy consumption (and Ex. 1 heart attacks).
  - linear (serial)
    - \* terminal nodes dependent, dependence disappears when (surely) knowing middle node,
    - \* intermediate variable (branch of study) explains dependence,
    - \* Simpson's paradox: gender  $\rightarrow$  branch of study  $\rightarrow$  admission (and Ex. 2 PhD),
  - converging
    - \* terminal nodes indep., spurious dependence introduced with knowledge of middle node,
    - \* temperature  $\rightarrow$  ice cream sales  $\leftarrow$  salesperson skills (and Ex. 3 radiation exposure),
- analogy e.g. with partial correlations.

#### **D**-separation

- uses connections to determine conditional independence between sets of nodes
  - linear and diverging connection transmit information **not given** middle node,
  - converging connection transmits information **given** middle node or its descendant,

- two node sets X and Y are d-separated by a node set Z iff
  - all undirected paths between arbitrary node pairs  $x \in X$  and  $y \in Y$  are blocked
    - \* there is a linear or diverging node  $z \in Z$  on the path, or
    - \* there is a converging node  $w \notin Z$  (none of its descendants w must not be in Z),
- d-separation is equivalent of conditional independence between X and Y given Z,
- a tool of abstraction
  - generalizes from 3 to multiple nodes when studying information flow through a network.

## D-separation – example, car diagnosis BN [Russel: AIMA]



- $Gas, Start, Go \perp Bat, Rad|Ign$
- $\{Gas, Start, Go\}$  and  $\{Bat, Rad\}$  c.ind
- sets are d-separated
- no open path between any pair of nodes
  - Gas x Battery, Gas x Radio etc.
  - all paths blocked by the middle linear node

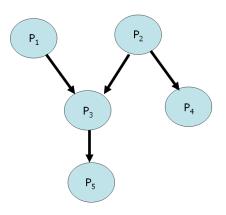
- $Gas \blacksquare Ign, Bat, Rad | Go$
- Gas and  $\{Ign, Bat, Rad\}$  are c.dependent
- sets are not d-separated
- $\hfill \hfill node Goes$  opens one path at least
  - $-\ Gas$  connected with Ignition via Starts
  - observed descendant of converging node

## **Graphical probabilistic models**

- exploit both probability theory and graph theory,
- graph = qualitative part of model
  - nodes represent events / random variables,
  - edges represent dependencies between them,
  - conditional independence can be seen in graph.
- probability = quantitative part of model
  - local information about node and its neighbors,
  - the strength of dependency, way of inference,
- different graph types (directed/undirected edges, constraints), probability encoding and focus
  - Bayesian networks causal and probabilistic processes,
  - Markov networks images, hidden causes,
  - data flows deterministic computations,
  - influence diagrams decision processes.

#### **Bayesian networks**

- Bayesian or Bayes or belief or causal networks (BNs, CNs),
- What is a Bayesian network?
  - directed acyclic graph DAG,
  - nodes represent random variables (typically discrete),
  - edges represent direct dependence,
  - nodes annotated by probabilities (tables, distributions)
    - node probability is conditioned by conjunction of all its parent nodes,
    - \*  $Pr(P_{j+1}|P_1,\ldots,P_j) = Pr(P_{j+1}|parents(P_{j+1}))$
    - \* root nodes annotated by prior distributions,
    - \* internal nodes conditioned by their parent variables,
    - \* other (potential) dependencies are ignored,
- Network interpretation?
  - concised representation of probability distribution given conditional independence relations,
  - qualitative constituent = graph,
  - quantitative constituent = a set of conditional probability tables (CPTs).



#### **Bayesian networks**

- sacrifice accuracy and completeness focus on fundamental relationships,
- reduce complexity of representation and subsequent inference,
- full probability model
  - can be deduced by the gradual decomposition (factorization):

$$Pr(P_1, P_2, \dots, P_n) = Pr(P_1) \times Pr(P_2, \dots, P_n | P_1) =$$
  
=  $Pr(P_1) \times Pr(P_2 | P_1) \times Pr(P_3, \dots, P_n | P_1, P_2) = \dots =$   
=  $Pr(P_1) \times Pr(P_2 | P_1) \times Pr(P_3 | P_1, P_2) \times \dots \times Pr(P_n | P_1, \dots, P_{n-1})$ 

BNs simplify the model:

$$-Pr(P_1,\ldots,P_n) = Pr(P_1|rodice(P_1)) \times \cdots \times Pr(P_n|rodice(P_n))$$

- ie. the other (possible) dependencies are ignored,
- ultimate case is naïve inference assuming variable independence

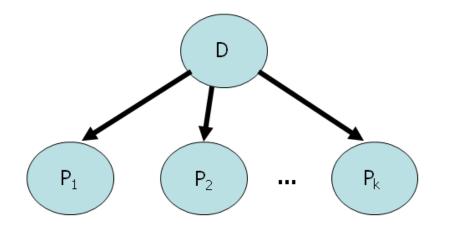
$$-Pr(P_1, P_2, \dots, P_n) = Pr(P_1) \times Pr(P_2) \times \dots \times Pr(P_n)$$

- uses marginal probs only linear complexity in the number of variables,
- used e.g. in classification.

#### Naïve Bayes classifier

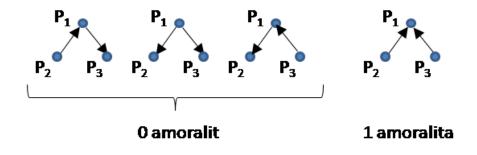
- a special case of Bayesian network
  - based on purely diagnostic reasoning,
  - assumes conditional independence among features  $P_1, \ldots, P_k$  given the diagnosis D,
  - the target variable is determined in advance.

$$Pr(D|P_1, \dots, P_k) = \frac{Pr(P_1, \dots, P_k|D) \times Pr(D)}{Pr(P_1, \dots, P_k)}$$
$$Pr(P_1, \dots, P_k|D) = Pr(P_1|D) \times Pr(P_2|D) \times \dots \times Pr(P_k|D)$$

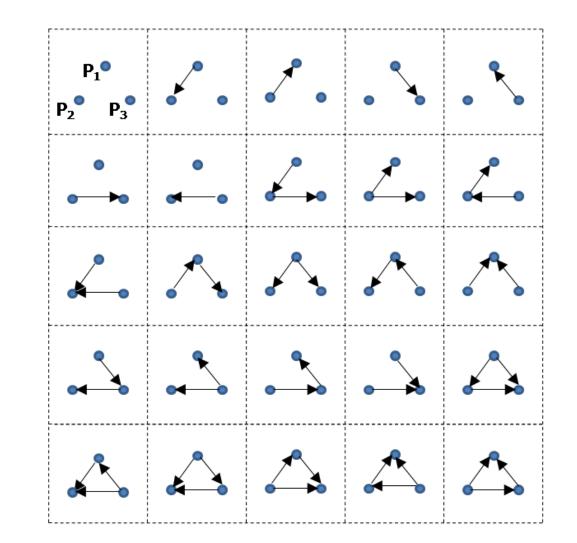


#### Markov equivalence classes

- DAG classes that define identical conditional independence relationships
  - represent identical joint distribution,
- Markov equivalence class is made by directed acyclic graphs which
  - have the identical skeleton
    - \* fully match when edge directions removed,
  - contain the same set of immoralities
    - \* immorality = 3 node subgraph such that:  $X \to Z$  and  $Y \to Z$ , no XY arc,
    - \* ie. the graphs have the same sets of uncoupled parents (without an edge between them),
- when learning from data, graphs from a single class are indistinguishable,
- example: 2 distinct equivalence classes (first  $P_2 \perp \!\!\!\perp P_3 | P_1$ , second  $P_2 \perp \!\!\!\perp P_3 | \emptyset$ ),



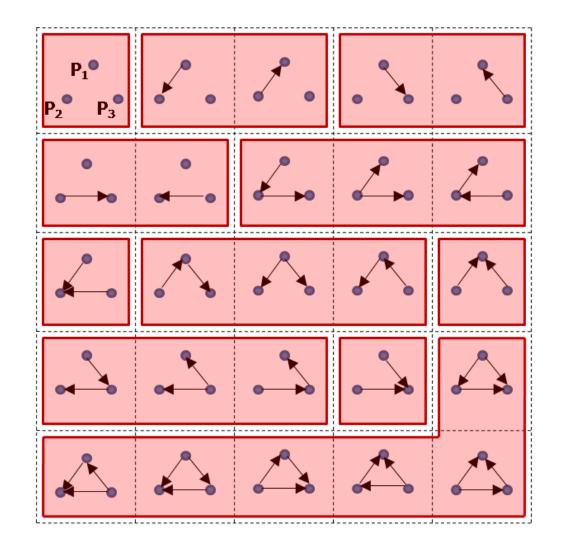
#### Markov equivalence classes



let us consider all 25 directed acyclic graphs with 3 labeled nodes

# Markov equivalence classes

• they make 11 Markov equivalence classes altogether



#### **Characteristics of qualitative model**

#### correctness

- simplification  $Pr(P_{j+1}|P_1,\ldots,P_j) = Pr(P_{j+1}|rodice(P_{j+1}))$  complies with reality,
- each network node is c.ind of its ancestor given its parents,

#### efficiency

- there are no redundant edges,
- actual c.independence relations described by the minimum number of edges,
- extra edges do not violate correctness,
- but slow down computations and make the model difficult to read,

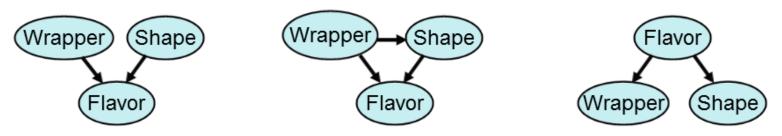
#### causality

- edge directions agree with actual cause-effect relationships,

#### consequences

- graphs lying in the same Markov equivalence class have the same correctness and efficiency,
- complete DAG is always correct, however it is very likely inefficient.

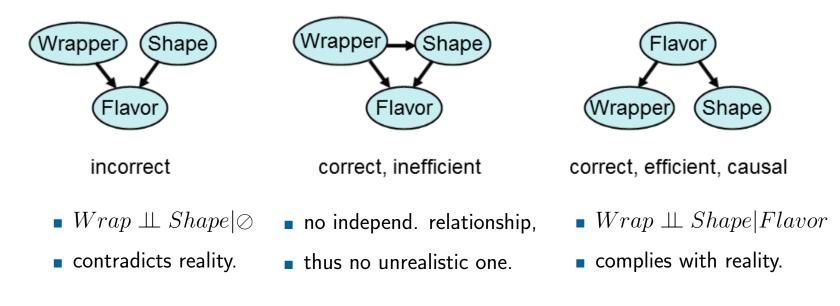
The Surprise Candy Company makes candy in two flavors: 70% are strawberry flavor and 30% are anchovy flavor. Each new piece of candy starts out with a round shape; as it moves along the production line, a machine randomly selects a certain percentage to be trimmed into a square; then, each piece is wrapped in a wrapper whose color is chosen randomly to be red or brown. 80% of the strawberry candies are round and 80% have a red wrapper, while 90% of the anchovy candies are square and 90% have a brown wrapper. All candies are sold individually in sealed, identical, black boxes.



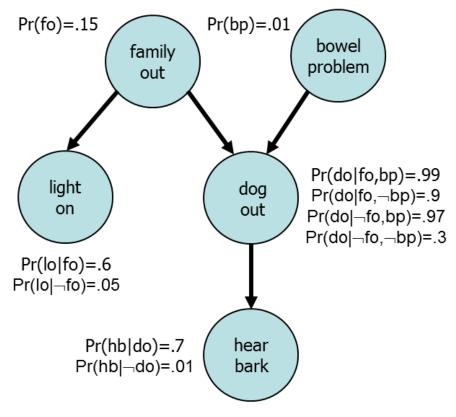
Russell, Norvig: Artificial Intelligence: A Modern Approach.

### **Characteristics of qualitative model – example**

The Surprise Candy Company makes candy in two flavors: 70% are strawberry flavor and 30% are anchovy flavor. Each new piece of candy starts out with a round shape; as it moves along the production line, a machine randomly selects a certain percentage to be trimmed into a square; then, each piece is wrapped in a wrapper whose color is chosen randomly to be red or brown. 80% of the strawberry candies are round and 80% have a red wrapper, while 90% of the anchovy candies are square and 90% have a brown wrapper. All candies are sold individually in sealed, identical, black boxes.

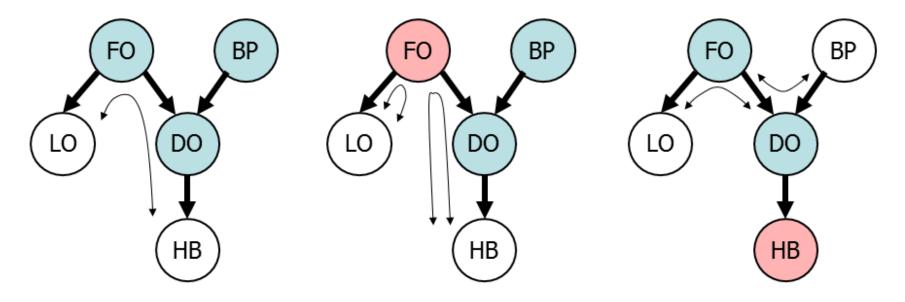


- Family house and events in it:
  - family sometimes goes out,
  - door light can be on or off,
  - family owns a dog, rarely ill,
  - dog can stay in or out,
  - $\log \operatorname{can} \operatorname{bark}$ .
- Relationships between events:
  - often switching the light on when leaving,
  - dog is rather out when leaving,
  - dog is out when ill (bowel problem),
  - dog is barking when out,
  - dog can hardly be heard when in.



©Charniak: Bayesian Networks withou Tears.

#### **D**-separation – examples



- $LOTTHB| \oslash$
- open path from LO to HB,
- LO and HB not d-separated, it also holds
- LO and HB are dependent.

- $LO \perp HB|FO$
- observed FO blocks path
- LO and HB c. independent, LO and BP c. dependent,
- - $-LO \perp HB | DO$
  - -LOTTHB|BP|

- $LO \top BP | HB$
- observed HB opens path
- it also holds
  - $-LO \perp \!\!\!\perp BP | \oslash$
  - -LO T BP | DO

#### **Bayesian networks – fundamental tasks**

- inference reasoning, deduction
  - from observed events assumes on probability of other events,
  - observations (E evidence variables),
  - target variables (Q query variables),
  - $-\ Pr(Q|E)$  , resp.  $Pr(q \in Q|E)$  to be found,
  - network is known (both graph and CPTs),
- learning network parameters from data
  - network structure (graph) is given,
  - "only" quantitative parameters (CPTs) to be optimized,
- learning network structure from data
  - propose an optimal network structure
    - \* which edges of the complete graph shall be employed?,
  - too many arcs  $\rightarrow$  complicated model,
  - too few arcs  $\rightarrow$  inaccurate model.

## **Summary**

probability

- a rigorous tool for uncertainty modeling,
- each atomic event is described by the joint probability distribution,
- queries answered by enumeration of atomic events
  - \* (summing, sometimes with final division),
- needs to be simplified in non-trivial domains
  - reason: curse of dimensionality,
  - solution: independence and conditional independence
  - tool: GPM = graph (quality) + conditional probability tables/functions (quantity).

- Russell, Norvig: AI: A Modern Approach, Uncertain Knowledge and Reasoning (Part V)
  - zejména neurčitost (kap. 14) a probabilistic usuzování (kap. 15),
  - online on Google books: http://books.google.com/books?id=8jZBksh-bUMC,
- Charniak: Bayesian Networks without Tears
  - http://ntu.csie.org/~piaip/docs/BayesianNetworksWithoutTears.pdf,
- Murphy: A Brief Introduction to Graphical Models and Bayesian Networks.
  - http://www.cs.ubc.ca/~murphyk/Bayes/bayes.html,
- Mooney: CS 391L: Machine Learning: Bayesian Learning: Beyond Naive Bayes.
  - http://www.cs.utexas.edu/~mooney/cs391L/slides/bayes2.pdf,
- Bishop: Pattern Recognition and Machine Learning.
  - Chapter 8: Graphical models,
  - http://research.microsoft.com/%7Ecmbishop/PRML/Bishop-PRML-sample.pdf.