Syntax and Semantics of FuzzyDL

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1. Comments Any line beginning with # or % is considered a comment.

2. Fuzzy operators. $\ominus, \oplus, \ominus$ and $\Rightarrow$ denote a t-norm, t-conorm, negation function and implication function respectively; $\alpha, \beta \in [0, 1]$.

Łukasiewicz negation	$\ominus_{\text{Ł}} \alpha$	$1 - \alpha$
Gödel t-norm	$\alpha \otimes_G \beta$	$\min\{\alpha, \beta\}$
Łukasiewicz t-norm	$\alpha \otimes_{\text{Ł}} \beta$	$\max\{\alpha + \beta - 1, 0\}$
Gödel t-conorm	$\alpha \oplus_G \beta$	$\max\{\alpha, \beta\}$
Łukasiewicz t-conorm	$\alpha \oplus_{\text{Ł}} \beta$	$\min\{\alpha + \beta, 1\}$
Gödel implication	$\alpha \Rightarrow_G \beta$	$\begin{cases} 1, & \text{if } \alpha \leq \beta \\ \beta, & \text{if } \alpha > \beta \end{cases}$
Łukasiewicz implication	$\alpha \Rightarrow_{\text{Ł}} \beta$	$\min\{1, 1 - \alpha + \beta\}$
Kleene-Dienes implication	$\alpha \Rightarrow_{KD} \beta$	$\max\{1 - \alpha, \beta\}$
Zadeh'set inclusion	$\alpha \Rightarrow_Z \beta$	1 iff $\alpha \leq \beta$, 0 otherwise

The reasoner can accept three different semantics, which are used to interpret $\ominus, \oplus, \ominus$ and $\Rightarrow$.

- Zadeh semantics: Łukasiewicz negation, Gödel t-norm, Gödel t-conorm and Kleene-Dienes implication (except in GCIs, where we have that the degree of membership to the subsumed concept should be less or equal than the degree of membership to the subsumer concept). This semantics is included for compatibility with earlier papers about fuzzy description logics.

- Łukasiewicz semantics: Łukasiewicz negation, Łukasiewicz t-norm, Łukasiewicz t-conorm and Łukasiewicz implication.

- Classical semantics: classical (crisp) conjunction, disjunction, negation and implication.

Syntax to define the semantics of the knowledge base:

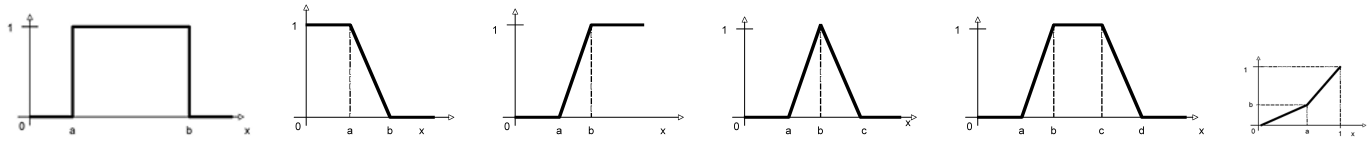
(define-fuzzy-logic [lukasiewicz — zadeh — classical])

3. Truth constants. Truth constants can be defined as follows (and later on, they can be used as the lower bound of a fuzzy axiom): (define-truth-constant constant n), where n is a rational number in $[0, 1]$.

4. Concept modifiers. Modifiers change the membership function of a fuzzy concept.

(define-modifier CM linear-modifier(c))	linear hedge with $c > 0$ (Figure 1 (f))
(define-modifier CM triangular-modifier(a,b,c))	triangular function (Figure 1 (d))

5. Concrete Fuzzy Concepts. Concrete Fuzzy Concepts (CFCs) define a name for a fuzzy set with an explicit fuzzy membership function (we assume $a \leq b \leq c \leq d$).



(a) (b) (c) (d) (e) (f)

Figure 1: (a) Crisp value; (b) L-function; (c) R-function; (d) (b) Triangular function; (e) Trapezoidal function; (f) Linear hedge

(define-fuzzy-concept CFC crisp(k1,k2,a,b))	crisp interval (Figure 1 (a))
(define-fuzzy-concept CFC left-shoulder(k1,k2,a,b))	left-shoulder function (Figure 1 (b))
(define-fuzzy-concept CFC right-shoulder(k1,k2,a,b))	right-shoulder function (Figure 1 (c))
(define-fuzzy-number CFC triangular(k1,k2,a,b,c))	triangular function (Figure 1 (d))
(define-fuzzy-concept CFC trapezoidal(k1,k2,a,b,c,d))	trapezoidal function (Figure 1 (e))
(define-fuzzy-concept CFC linear(k1,k2,a,b))	linear function (Figure 1 (f))
(define-fuzzy-concept CFC modified(mod,F))	modified datatype

6. Fuzzy Numbers. Firstly, if fuzzy numbers are used, one has to define the range $[k_1, k_2] \subseteq \mathbb{R}$ as follows:

`(define-fuzzy-number-range k1 k2)`

Let `fi` be a fuzzy number (a_i, b_i, c_i) $(a \leq b \leq c)$, and $n \in \mathbb{R}$. Valid fuzzy number expressions (see Figure 1 (d)) are:

name	fuzzy number definition	*name*
(a, b, c)	fuzzy number	(a, b, c)
n	real number	(n, n, n)
(f+ f1 f2 … fn)	addition	$(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i)$
(f- f1 f2)	substraction	$(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(f* f1 f2 … fn)	product	$(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i, \prod_{i=1}^{n} c_i)$
(f/ f1 f2)	division	$(a_1/c_2, b_1/b_2, c_1/a_2)$

Fuzzy numbers can be named as:

`(define-fuzzy-number name fuzzyNumberExpression)`

7. Features. Features are functional datatype attributes.

(functional F)	Firstly, the feature is defined. Then we set the range
(range F *integer* k1 k2)	The range is an integer number in $[k_1, k_2]$
(range F *real* k1 k2)	The range is a rational number in $[k_1, k_2]$
(range F *string*)	The range is a string

8. Datatype restrictions.

(>= F var)	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq var)]$
(>= F $f(F_1, \ldots, F_n)$)	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(>= F FN)	at least datatype restriction	$\sup_{b, b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq b') \otimes FN^{\mathcal{I}}(b')]$
(<= F var)	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq var)]$
(<= F $f(F_1, \ldots, F_n)$)	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(<= F FN)	at most datatype restriction	$\sup_{b, b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq b') \otimes FN^{\mathcal{I}}(b')]$
(= F var)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b = var)]$
(= F $f(F_1, \ldots, F_n)$)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b = f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(= F FN)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes FN^{\mathcal{I}}(b)]$

In datatype restrictions, the variable *var* may be replaced with a value (an integer, a real, or a string, depending on the range of the feature F).

Furthermore, is defined as follows:

$$
\begin{array}{ll}
f(F_1, \ldots, F_n) \rightarrow & F \\
& real \\
& (nF) \mid (n * F) \\
& (F_1 - F_2) \\
& (F_1 + F_2 + \ldots + F_n)
\end{array}
$$

9. Constraints. Constraints are of the form (constraints ⟨ constraint-i ⟩ +), where ⟨ constraint-i ⟩ is one of the following (with $OP = \geq \mid \leq \mid =$.):

(a1 * var1 + ... + ak * vark OP number)	linear inequation	$a_1 var_1 + \ldots + a_k * var_k \ OP \ number$
(binary var)	binary variable	$var \in \{0, 1\}$
(free var)	binary variable	$var \in (-\infty, \infty)$

10. Show statements.

(show-concrete-fillers F1 ... Fn)	show value of the fillers of $F_1 \ldots F_n$
(show-concrete-fillers-for a F1 ... Fn)	show value of the fillers of $F_1 \ldots F_n$ for a
(show-concrete-instance-for a F C1 ... Cn)	show degrees of being the F filler of a an instance of C_i
(show-abstract-fillers R1 ... Rn)	show fillers of $R_1 \ldots R_n$ and membership to any concept
(show-abstract-fillers-for a R1 ... Rn)	show fillers of $R_1 \ldots R_n$ for a and membership to any concept
(show-concepts a1 ... an)	show membership of $a_1 \ldots a_n$ to any concept
(show-instances C1 ... Cn)	show value of the instances of the concepts $C_1 \ldots C_n$
(show-variables x1 ... xn)	show value of the variables $x_1 \ldots x_n$
(show-language)	show language of the KB, from $\mathcal{ALC}$ to $\mathcal{SHIF}(D)$

where C_i is the name of a defined concrete fuzzy concept. We assume that an abstract role R appears in at most one statement of the forms show-abstract-fillers? or show-abstract-fillers-for?.

11. Crisp declarations.

(crisp-concept C1 ... Cn)	concepts $C_1 \ldots C_n$ are crisp
(crisp-role R1 ... Rn)	roles $R_1 \ldots R_n$ are crisp

12. Concept expressions.

top	top concept	1
bottom*	bottom concept	0
A	atomic concept	$A^{\mathcal{I}}(x)$
(and C1 C2)	concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes {C_2}^{\mathcal{I}}(x)$
(g-and C1 C2)	Gödel conjunction	${C_1}^{\mathcal{I}}(x) \otimes_G {C_2}^{\mathcal{I}}(x)$
(l-and C1 C2)	Łukasiewicz conjunction	${C_1}^{\mathcal{I}}(x) \otimes_{Ł} {C_2}^{\mathcal{I}}(x)$
(or C1 C2)	concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus {C_2}^{\mathcal{I}}(x)$
(g-or C1 C2)	Gödel disjunction	${C_1}^{\mathcal{I}}(x) \oplus_G {C_2}^{\mathcal{I}}(x)$
(l-or C1 C2)	Łukasiewicz disjunction	${C_1}^{\mathcal{I}}(x) \oplus_{Ł} {C_2}^{\mathcal{I}}(x)$
(not C1)	concept negation	$\ominus_{Ł} {C_1}^{\mathcal{I}}(x)$
(implies C1 C2)	concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x)$
(g-implies C1 C2)	Gödel implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x)$
(l-implies C1 C2)	Łukasiewicz implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x)$
(kd-implies C1 C2)	Kleene-Dienes implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x)$
(all R C1)	universal role restriction	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow {C_1}^{\mathcal{I}}(y)$
(some R C1)	existential role restriction	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \otimes {C_1}^{\mathcal{I}}(y)$
(ua s C1)	upper approximation	$\sup_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$
(la s C1)	lower approximation	$\inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$
(tua s C1)	tight upper approximation	$\inf_{z \in X} \{ s_i^{\mathcal{I}}(x,z) \Rightarrow \sup_{y \in \Delta^{\mathcal{I}}} \{ s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y) \} \}$
(lua s C1)	loose upper approximation	$\sup_{z \in X} \{ s_i^{\mathcal{I}}(x,z) \otimes \sup_{y \in \Delta^{\mathcal{I}}} \{ s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y) \} \}$
(tla s C1)	tight lower approximation	$\inf_{z \in X} \{ s_i^{\mathcal{I}}(x,z) \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}} \{ s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y) \} \}$
(lla s C1)	loose lower approximation	$\sup_{z \in X} \{ s_i^{\mathcal{I}}(x,z) \otimes \inf_{y \in \Delta^{\mathcal{I}}} \{ s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y) \} \}$
(self S)	local reflexivity concept	$S^{\mathcal{I}}(x)(x,x)$
(CM C1)	modifier applied to concept	$f_{\mathtt{m}}({C_1}^{\mathcal{I}}(x))$
(CFC)	concrete fuzzy concept	$CFC^{\mathcal{I}}(x)$
(FN)	fuzzy number	$FN^{\mathcal{I}}(x)$
([>= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \geq w \\ 0, & \text{otherwise} \end{cases}$
([<= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \leq w \\ 0, & \text{otherwise} \end{cases}$
(n C1)	weighted concept	$n{C_1}^{\mathcal{I}}(x)$
(w-sum (n1 C1) ... (nk Ck))	weighted sum	$n_1{C_1}^{\mathcal{I}}(x) + \ldots + n_k{C_k}^{\mathcal{I}}(x)$
(w-max (v1 C1) ... (vk Ck))	weighted maximum	$\max_{i=1}^{k} \min\{v_i, x_i\}$
(w-min (v1 C1) ... (vk Ck))	weighted minimum	$\min_{i=1}^{k} \max\{1 - v_i, x_i\}$
(w-sum-zero (n1 C1) ... (nk Ck))	weighted sum-zero	$\begin{cases} 0 & \text{if } C_i^{\mathcal{I}}(x) = 0 \text{ for some } i \in \{1, \ldots, n\} \\ \text{w-sum} & \text{otherwise} \end{cases}$
(owa (w1 ... wn) (C1 ... Cn))	OWA aggregation operator	$\sum_{i=1}^{n} w_i y_i$
(q-owa q C1 ... Cn)	quantifier-guided OWA	Same as OWA taking $w_i = q(i/n) - q((i-1)/n)$
(choquet (w1 ... wn) (C1 ... Cn))	Choquet integral	$y_1 \cdot w_1 + \sum_{i=2}^{n} (y_i - y_{i-1}) w_i$
(sugeno (v1 ... vn) (C1 ... Cn))	Sugeno integral	$\max_{i=1}^{n} \{\min\{y_i, mu_i\}\}$
(q-sugeno (v1 ... vn) (C1 ... Cn))	Quasi-Sugeno integral	$\max_{i=1}^{n} \{y_i \otimes_{Ł} mu_i\}$
(DR)	datatype restriction	$DR^{\mathcal{I}}(x)$

where:

- $n_1, \ldots, n_k \in [0,1]$ with $\sum_{i=1}^{k} n_i \leq 1$,
- $w_1, \ldots, w_k \in [0,1]$ with $\sum_{i=1}^{k} w_i = 1$,
- $v_1, \ldots, v_k \in [0,1]$ with $\max_{i=1}^{k} k_i = 1$,
- q is a quantifier (defined as a right-shoulder or a linear function),
- w is a variable or a real number in $[0,1]$,

- y_i is the i-largest of the ${C_i}^{\mathcal{I}}(x)$,
- mu_i is defined as follows: $mu_1 = ow_1$, $mu_i = ow_i \oplus mu_{i-1}$ for $i \in \{2, \ldots, n\}$,
- ow_i is the weight v_i of the i-largest of the ${C_i}^{\mathcal{I}}(x)$.
- Fuzzy numbers can only appear in existential, universal and datatype restrictions.
- In threshold concepts var may be replaced with w.
- Fuzzy relations s should be previously defined as fuzzy similarity relation or a fuzzy equivalence relation as `(define-fuzzy-similarity s)` or `(define-fuzzy-equivalence s)`, respectively.

Important note: The reasoner restricts the calculus to witnessed models.

13. Axioms.

(instance a C1 [d])	concept assertion	${C_1}^{\mathcal{I}}(a^{\mathcal{I}}) \geq d$
(related a b R [d])	role assertion	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq d$
(implies C1 C2 [d])	GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x) \geq d$
(g-implies C1 C2 [d])	Gödel GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x) \geq d$
(kd-implies C1 C2 [d])	Kleene-Dienes GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x) \geq d$
(l-implies C1 C2 [d])	Łukasiewicz GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x) \geq d$
(z-implies C1 C2 [d])	Zadeh's set inclusion GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_Z {C_2}^{\mathcal{I}}(x) \geq d$
(define-concept A C)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) = C^{\mathcal{I}}(x)$
(define-primitive-concept A C)	concept subsumption	$\inf_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$
(equivalent-concepts C1 C2)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) = {C_2}^{\mathcal{I}}(x)$
(disjoint C1 … Ck)	concept disjointness	$\forall_{i,j \in \{1,\ldots,k\}, i<j}$ (implies (g-and Ci Cj) *bottom*)
(disjoint-union C1 … Ck)	disjoint union	(disjoint C2 … Ck) and C1 = (or C2 … Ck)
(range R C1)	range restriction	(implies *top* (all RN C))
(domain R C1)	fomain restriction	(implies (some RN *top*) C)
(functional R)	functional role	$R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(a,c) \rightarrow b = c$
(inverse-functional R)	inverse functional role	$R^{\mathcal{I}}(b,a) = R^{\mathcal{I}}(c,a) \rightarrow b = c$
(reflexive R)	reflexive role	$\forall a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,a) = 1.$
(symmetric R)	symmetric role	$\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a,b) = R^{\mathcal{I}}(b,a).$
(transitive R)	transitive role	$\forall_{a,b \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a,b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a,c) \otimes R^{\mathcal{I}}(c,b).$
(implies-role R1 R2 [d])	RIA	$\inf_{x,y \in \Delta^{\mathcal{I}}} {R_1}^{\mathcal{I}}(x,y) \Rightarrow_{Ł} {R_2}^{\mathcal{I}}(x,y) \geq d$
(inverse R1 R2)	inverse role	${R_1}^{\mathcal{I}} \equiv ({R_2}^{\mathcal{I}})^{-}$

where d is the degree and can be: *(i)* a variable, *(ii)* an already defined truth constant, *(iii)* a rational number in $[0,1]$, *(iv)* a linear expression.
Notes: Transitive roles cannot be functional. In Zadeh logic, $\Rightarrow$ is Zadeh's set inclusion.

14. Queries.

(sat?)	Is $\mathcal{K}$ consistent?
(max-instance? a C)	$\sup\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(min-instance? a C)	$\inf\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(all-instances? C)	(min-instance? a C) for every individual of $\mathcal{K}$
(max-related? a b R)	$\sup\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(min-related? a b R)	$\inf\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(g-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(g-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(l-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(l-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(kd-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(kd-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(max-sat? C [a])	$\sup_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(min-sat? C [a])	$\inf_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(max-var? var)	$\sup\{\text{var} \mid \mathcal{K}$ is consistent$\}$
(min-var? var)	$\inf\{\text{var} \mid \mathcal{K}$ is consistent$\}$
(defuzzify-lom? C_m a F)	Defuzzify the value of F using largest of the maxima
(defuzzify-mom? C_m a F)	Defuzzify the value of F using middle of the maxima
(defuzzify-som? C_m a F)	Defuzzify the value of F using smallest of the maxima
(bnp? f)	Computes the Best Non-Fuzzy Performance (BNP) of fuzzy number f

where concept C_m represents several Mamdani/Rules IF-THEN fuzzy rules expressing how to obtain the value of concrete feature F.