Cheat sheet

AE4M33RZN

1 Bayesian networks

1.1 Evaluation functions

Bayesian Information Criterion (BIC)

$$BIC(G:D) = -\frac{K}{2}\log_2 M + \log_2 L(G:D) =$$
$$= -\frac{K}{2}\log_2 M - M\sum_{i=1}^{n} H(P_i \mid \text{parents}(P_i)^G),$$

- $K = \sum_{i=1}^{n} q_i(r_i - 1)$,
- q_i is the number of unique instantiations of P_i parents,
- n is the number of variables,
- r_i is the number of distinct P_i values and
- M the number of observations.

$$H(P_i \mid \text{parents}(P_i)^G) = -\sum_{j=1}^{q_i}\sum_{k=1}^{r_i} \frac{N_{ij}}{M}\frac{N_{ijk}}{N_{ij}}\log_2\frac{N_{ijk}}{N_{ij}} =$$
$$= -\sum_{j=1}^{q_i}\sum_{k=1}^{r_i} \frac{N_{ijk}}{M}\log_2\frac{N_{ijk}}{N_{ij}},$$

- N_{ij} is the number of samples, where parents(P_i) take the j-th instantiation of values,
- N_{ijk} is the number of samples, where P_i takes the k-th value and parents(P_i) the j-th instantiation of values.

Bayesian score

$$\ln Pr(D \mid G) = \ln\prod_{i=1}^{n} g(P_i, \text{parents}(P_i)^G)$$

$$g(P_i, \text{parents}(P_i)) = \prod_{j=1}^{q_i}\frac{(r_i - 1)!}{(N_{ij} + r_i - 1)!}\prod_{k=1}^{r_i} N_{ijk}!$$

1.2 Dynamic Bayesian Networks (DBNs)

The recursive definition of filtering task:

$$Pr(X_{t+1} \mid e_{1:t+1}) = \alpha Pr(e_{t+1} \mid X_{t+1}) Pr(X_{t+1} \mid e_{1:t}) =$$
$$= \alpha Pr(e_{t+1} \mid X_{t+1}) \sum_{x_t} Pr(X_{t+1} \mid x_t) Pr(x_t \mid e_{1:t})$$

$X_t \cdots$ the set of unobservable state variables at time t,
$E_t \cdots$ the set of observable evidence variables at time t.

2 Description Logics

$\rightarrow_\sqcap$ rule

if $(C_1 \sqcap C_2) \in L_G(a)$ a $\{C_1, C_2\} \nsubseteq L_G(a)$ for some $a \in V_G$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, a $L_{G'}(a) = L_G(a) \cup \{C_1, C_2\}$ and otherwise is the same as L_G.

$\rightarrow_\sqcup$ rule

if $(C_1 \sqcup C_2) \in L_G(a)$ a $\{C_1, C_2\} \cap L_G(a) = \emptyset$ for some $a \in V_G$.

then $S' = S \cup \{G_1, G_2\} \setminus \{G\}$, where $G_{(1|2)} = (V_G, E_G, L_{G_{(1|2)}})$ and $L_{G_{(1|2)}}(a) = L_G(a) \cup \{C_{(1|2)}\}$ and otherwise is the same as L_G.

$\rightarrow_\exists$ rule

if $(\exists R \cdot C) \in L_G(a)$ and there exists no $b \in V_G$ such that $R \in L_G(a,b)$ and at the same time $C \in L_G(b)$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G \cup \{b\}, E_G \cup \{\langle a, b\rangle\}, L_{G'})$ and $L_{G'}(b) = \{C\}, L_{G'}(a,b) = \{R\}$ and otherwise is the same as L_G.

$\rightarrow_\forall$ rule

if $(\forall R \cdot C) \in L_G(a)$ and there exists no $b \in V_G$ such that $R \in L_G(a,b)$ and at the same time $C \notin L_G(b)$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, a $L_{G'}(b) = L_G(b) \cup \{D\}$ and otherwise is the same as L_G.

$\rightarrow_\sqsubseteq$ rule

if $\top_C \notin L_G(a)$ for some $a \in V_G$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, a $L_{G'}(a) = L_G(a) \cup \{\top_C\}$ and otherwise is the same as L_G.

3 Basics of fuzzy logic

Inverse membership	$\mu_A^{-1}(M) = \{x \in X \mid A(x) \in M\}$
Height	$\text{Height}(A) = \sup\{\alpha \mid x \in \Delta, A(x) = \alpha\}$
Support	$\text{Supp}(A) = \{x \in X \mid A(x) > 0\}$
Core	$\text{Core}(A) = \{x \in X \mid A(x) = 1\}$
Vertical to horizontal	$\text{R}_A(\alpha) = \{x \in X \mid A(x) \geq \alpha\}$
Horizontal to vertical	$\mu_A(x) = \max\{\alpha \in [0,1] \mid x \in \text{R}_A(\alpha)\}$

Fuzzy inclusion	$A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in \Delta$
Cutworthiness	$P(A_1, ..., A_n) \Rightarrow P(\text{R}_{A_1}(\alpha), ..., \text{R}_{A_n}(\alpha))$
Cut-consistency	$P(A_1, ..., A_n) \Leftrightarrow P(\text{R}_{A_1}(\alpha), ..., \text{R}_{A_n}(\alpha))$

3.1 Negation

Fuzzy negation	if $\alpha \leq \beta$ then $\underset{\circ}{\neg}\beta \leq \underset{\circ}{\neg}\alpha$ (N1)
	$\underset{\circ}{\neg}\underset{\circ}{\neg}\alpha = \alpha$ (N2)
Standard negation	$\underset{\text{S}}{\neg}\alpha = 1 - \alpha$
Cosine negation	$\underset{\cos}{\neg}\alpha = (\cos(\pi\alpha) + 1)/2$
Sugeno negation	$\underset{S\lambda}{\neg}\alpha = \frac{1-\alpha}{1+\lambda a}$
Yager negation	$\underset{Y\lambda}{\neg}\alpha = (1 - a^\lambda)^{1/\lambda}$
Non-involutive negation	$\underset{\circ}{\neg}\underset{\circ}{\neg}0 = 1$ and $\underset{\circ}{\neg}\underset{\circ}{\neg}1 = 0$ (N0)
	if $\alpha \leq \beta$ then $\underset{\circ}{\neg}\beta \leq \underset{\circ}{\neg}\alpha$ (N1)
Gödel negation	$\underset{\text{G}}{\neg}\alpha = \begin{cases} 1 & \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$

3.2 Implication

General fuzzy implication	$(x \overset{\circ}{\underset{\circ}{\Rightarrow}} y) = (x \Rightarrow y)$ on $x, y \in \{0, 1\}$
Residue implication	$\alpha \overset{\text{R}}{\underset{\circ}{\Rightarrow}} \beta = \sup\{\gamma \mid \alpha \underset{\circ}{\wedge} \gamma \leq \beta\}$
R-impl. properties	$\alpha \overset{\text{R}}{\underset{\circ}{\Rightarrow}} \beta = 1$ iff $\alpha \leq \beta$ (I1)
	$1 \overset{\text{R}}{\underset{\circ}{\Rightarrow}} \beta = \beta$ (I2)
	$\alpha \overset{\text{R}}{\underset{\circ}{\Rightarrow}} \beta$ is not increasing in α and not decreasing in β (I3)
Standard implication	$\alpha \overset{\text{R}}{\underset{\text{S}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$
Łukasiewicz implication	$\alpha \overset{\text{R}}{\underset{\text{L}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha + \beta & \text{otherwise} \end{cases}$
Algebraic implication	$\alpha \overset{\text{R}}{\underset{\text{A}}{\Rightarrow}} \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{otherwise} \end{cases}$
S-implication	$\alpha \overset{\text{S}}{\underset{\circ}{\Rightarrow}} \beta = \underset{\text{S}}{\neg}\alpha \overset{\circ}{\vee} \beta$
Kleene-Dienes implication	$\alpha \overset{\text{S}}{\underset{\text{S}}{\Rightarrow}} \beta = \max(1 - \alpha, \beta)$
Generalized fuzzy inclusion	$A \overset{\circ}{\underset{\circ}{\subseteq}} B = \inf\{A(x) \overset{\circ}{\underset{\circ}{\Rightarrow}} B(x) \mid x \in \Delta\}$

3.3 Conjunction

Definition	$\alpha \underset{\circ}{\wedge} \beta = \beta \underset{\circ}{\wedge} \alpha$ (T1)
	$\alpha \underset{\circ}{\wedge} (\beta \underset{\circ}{\wedge} \gamma) = (\alpha \underset{\circ}{\wedge} \beta) \underset{\circ}{\wedge} \gamma$ (T2)
	if $\beta \leq \gamma$ then $(\alpha \underset{\circ}{\wedge} \beta) \leq (\alpha \underset{\circ}{\wedge} \gamma)$ (T3)
	$(\alpha \underset{\circ}{\wedge} 1) = \alpha$ (T4)
Standard c.	$\alpha \underset{S}{\wedge} \beta = \min(\alpha, \beta)$
Łukasiewicz c.	$\alpha \underset{L}{\wedge} \beta = \max(\alpha + \beta - 1, 0)$
Algebraic product	$\alpha \underset{A}{\wedge} \beta = \alpha \cdot \beta$
Weak conjunction	$\alpha \underset{W}{\wedge} \beta = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$

3.4 Disjunction

Definition	$\alpha \overset{\circ}{\vee} \beta = \beta \overset{\circ}{\vee} \alpha$ (S1)
	$\alpha \overset{\circ}{\vee} (\beta \overset{\circ}{\vee} \gamma) = (\alpha \overset{\circ}{\vee} \beta) \overset{\circ}{\vee} \gamma$ (S2)
	if $\beta \leq \gamma$ then $(\alpha \overset{\circ}{\vee} \beta) \leq (\alpha \overset{\circ}{\vee} \gamma)$ (S3)
	$(\alpha \overset{\circ}{\vee} 0) = \alpha$ (S4)
Standard d.	$\alpha \overset{S}{\vee} \beta = \max(\alpha, \beta)$
Łukasiewicz d.	$\alpha \overset{L}{\vee} \beta = \min(\alpha + \beta, 1)$
Algebraic sum	$\alpha \overset{A}{\vee} \beta = \alpha + \beta - \alpha \cdot \beta$
Weak disjunction	$\alpha \overset{W}{\vee} \beta = \begin{cases} \alpha & \text{if } \beta = 0 \\ \beta & \text{if } \alpha = 0 \\ 1 & \text{otherwise} \end{cases}$

3.5 Relations

Composition	$R \underset{\circ}{\bigcirc} S(x, z) = \sup_{y \in Y} R(x, y) \underset{\circ}{\wedge} S(y, z)$
Comp. properties	$R \underset{\circ}{\bigcirc} (S \underset{\circ}{\bigcirc} T) = (R \underset{\circ}{\bigcirc} S) \underset{\circ}{\bigcirc} T$ (C3)
	$R \underset{\circ}{\bigcirc} E = R,\ E \underset{\circ}{\bigcirc} R = R$ (C1)
	$(R \overset{S}{\cup} S) \underset{\circ}{\bigcirc} T = (R \underset{\circ}{\bigcirc} T) \overset{S}{\cup} (S \underset{\circ}{\bigcirc} T)$ (C4)
	$(R \underset{\circ}{\bigcirc} S)^{-1} = S^{-1} \underset{\circ}{\bigcirc} R^{-1}$ (C2)
	$R \underset{\circ}{\bigcirc} (S \overset{S}{\cup} T) = (R \underset{\circ}{\bigcirc} S) \overset{S}{\cup} (R \underset{\circ}{\bigcirc} T)$ (C5)
First projection	$R^{(1)}(x) = \sup_{y \in Y} R(x, y)$
Second projection	$R^{(2)}(y) = \sup_{x \in X} R(x, y)$
Cylindrical extension	$A \times B(x, y) = A(x) \underset{S}{\wedge} B(y)$
Identity relation	$E = \{(x, x) \mid x \in \Delta\}$
Reflexivity	$E \subseteq R$
Symmetrictity	$R = R^{-1}$
∘-anti-symmetricity	$R \underset{\circ}{\cap} R^{-1} \subseteq E$
∘-transitivity	$R \underset{\circ}{\bigcirc} R \subseteq R$
∘-partial order	reflexive, ∘-transitive and ∘-anti-symmetric
∘-equivalence	reflexive, ∘-transitive and ∘-symmetric

4 Fuzzy description logic

non-atomic c.	interpretation
$\bot$	0
$\top$	1
A	$A^{\mathcal{I}}(x)$
$\neg C$	$\underset{S}{\neg} C^{\mathcal{I}}(x)$
$C \underset{\circ}{\sqcap} D$	$C^{\mathcal{I}}(x) \underset{\circ}{\wedge} D^{\mathcal{I}}(x)$
$C \overset{\circ}{\sqcup} D$	$C^{\mathcal{I}}(x) \overset{\circ}{\vee} D^{\mathcal{I}}(x)$
$C \underset{\circ}{\overset{\circ}{\mapsto}} D$	$C^{\mathcal{I}}(x) \underset{\circ}{\overset{\circ}{\Rightarrow}} D^{\mathcal{I}}(x)$
$\exists R \cdot C$	$\sup_y R^{\mathcal{I}}(x, y) \underset{\circ}{\wedge} C^{\mathcal{I}}(y)$
$\forall R \cdot C$	$\inf_y R^{\mathcal{I}}(x, y) \underset{\circ}{\overset{\circ}{\Rightarrow}} C^{\mathcal{I}}(y)$
$(n\, C)$	$n \cdot C(x)$
$mod(C)$	$mod(C^{\mathcal{I}}(x))$
$w_1 C_1 + \ldots + w_k C_k$	$w_1 C_1^{\mathcal{I}}(x) + \ldots + w_k C_k^{\mathcal{I}}(x)$

axiom	satisfied if
$\langle i : C \mid \alpha \rangle$	$C^{\mathcal{I}}(i^{\mathcal{I}}) \geq \alpha$
$\langle (i, j) : R \mid \alpha \rangle$	$R^{\mathcal{I}}(i^{\mathcal{I}}, j^{\mathcal{I}}) \geq \alpha$
$\langle C \sqsubseteq D \mid \alpha \rangle$	$C \underset{\circ}{\overset{\circ}{\subseteq}} D \geq \alpha$
$\langle R_1 \sqsubseteq R_2 \rangle$	$R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$
$\langle$transitive $R\rangle$	R is ∘-transitive
$\left\langle R_1 = R_2^{-1} \right\rangle$	$R_1^{\mathcal{I}} = (R_2^{\mathcal{I}})^{-1}$

5 Fuzzy DL algorithm

The reasoner applies each of the following rules sequentially:

- A If a vertex v is labeled $\langle \mathsf{C}, l\rangle$, add $(x_{v:\mathsf{C}} \geq l)$ into $\mathscr{C}$.
- $\bar{\mathsf{A}}$ If a vertex v is labeled $\langle \neg\, \mathsf{C}, l\rangle$, add $(x_{v:\mathsf{C}} \leq 1 - l)$ into $\mathscr{C}$.
- R If an edge (v, w) is labeled $\langle \mathsf{R}, l\rangle$, add $(x_{(v,w):\mathsf{R}} \geq l)$ into $\mathscr{C}$.
- $\bot$ If a vertex v is labeled $\langle \bot, l\rangle$, add $(l = 0)$ into $\mathscr{C}$.
- $\sqcap$ If a vertex v is labeled $\langle \mathsf{C} \sqcap \mathsf{D}, l\rangle$, append labels $\langle \mathsf{C}, x_1\rangle$, $\langle \mathsf{D}, x_2\rangle$ to v and add the following constraints into $\mathscr{C}$ (with fresh x_1, x_2, y):

$$\begin{aligned} y &\leq 1 - l \\ x_1 &\leq 1 - y \\ x_2 &\leq 1 - y \\ x_1 + x_2 &= l + 1 - y \end{aligned}$$

- $\sqcup$ If a vertex v is labeled $\langle \mathsf{C} \sqcup \mathsf{D}, l\rangle$, append labels $\langle \mathsf{C}, x_1\rangle$, $\langle \mathsf{C}, x_2\rangle$ to v and add $(x_1 + x_2 = l)$ into $\mathscr{C}$ (with fresh x_1, x_2, y).
- $\forall$ If a vertex v is labeled $\langle \forall\, \mathsf{R} \cdot \mathsf{C}, l_1\rangle$, an edge (v, w) is labeled $\langle R, l_2\rangle$ and the rule has not been applied to this pair, then append the label $\langle \mathsf{C}, x\rangle$ to w and add the following constraints into $\mathscr{C}$ (with fresh x, y):

$$l_1 + l_2 - 1 \leq x \leq y \leq l_1 + l_2$$

- $\sqsubseteq$ If $\langle \mathsf{C} \sqsubseteq \mathsf{D} \,|\, n\rangle \in \mathcal{K}$, and the rule has not been applied to a node v, then append labels $\langle \mathrm{nnf}(\neg\, \mathsf{C}), 1 - x_1\rangle$, $\langle \mathsf{D}, x_2\rangle$ to v and add $(x_1 \leq x_2 + 1 - n)$ to $\mathscr{C}$.
- $\exists$ If a vertex v is labeled $\langle \exists\, \mathsf{R} \cdot \mathsf{C}, l\rangle$ and it is not blocked, add a new vertex w and an edge (v, w), add labels $\langle C, x_2\rangle$ to w, and $\langle R, x_1\rangle$ to (v, w) and the following constraints into $\mathscr{C}$ (with fresh x_1, x_2 and y):

$$\begin{aligned} y &\leq 1 - l \\ x_1 &\leq 1 - y \\ x_2 &\leq 1 - y \\ x_1 + x_2 &= l + 1 - y \end{aligned}$$